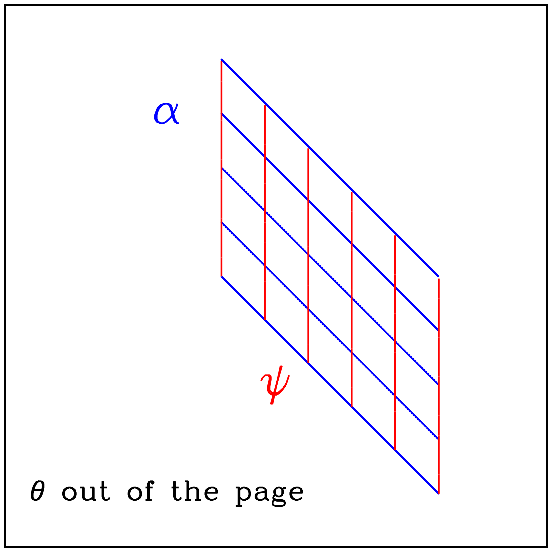
The Gyrokinetic Regime
Geometry
Velocity Space
Linear How-To

Coordinates

- Given a magnetic field in Clebsch representation

$$\mathbf{B} = \nabla\alpha \times \nabla\psi$$



- Natural perpendicular coordinates are ψ and α; distance along the field line is θ

- In general, (ψ, α, θ) is non-orthogonal

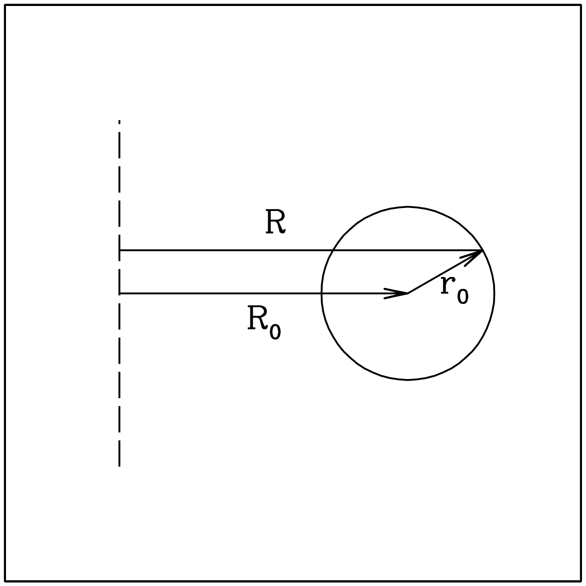
- Nor are they automatically single-valued.

Simplest Toroidal Limit

- Concentric circular flux surfaces:

$$\alpha = \phi - q(\theta - \frac{r}{R_0}\sin\theta)$$

$$\psi = \psi_0 + (r - r_0)\frac{B_0 r_0}{q}$$



(Concentric circles)

- Note θ is the distance along the field line.

Flux Tube Limit

- Simulation domain is some number of gyroradii in ψ and α directions, and $\sim 2\pi q R$ along field line

- Take advantage of ordering to write

$$h = \widehat{h}(\theta) \exp\left(iS\right)$$

(Fluctuations vary slowly along field line, rapidly across)

- Make contact with ballooning theory, defining

$$S = n_0 \left[\alpha + q(\Psi)\theta_0\right]$$

where n_0 is the integer toroidal mode number.

Theta Dependencies of ∇B Drift

- Work out θ dependencies in various terms

- ∇B drift:

$$\frac{v_\perp^2}{2}\frac{1}{\Omega B^2}\mathbf{B}\times\nabla B\cdot\nabla S=\left(\frac{k_\theta\rho_i}{2}\right)\frac{v_\perp^2}{2}\left[\omega_{\nabla B}+\omega_{\nabla B}^{(0)}\theta_0\right],$$

where

$$\omega_{\nabla B}=\frac{2}{B^2}\frac{d\Psi}{d\rho}\hat{\mathbf{b}}\times\nabla B\cdot\nabla\alpha,\qquad \omega_{\nabla B}^{(0)}=\frac{2}{B^2}\frac{d\Psi}{d\rho}\hat{\mathbf{b}}\times\nabla B\cdot\nabla q.$$

- In the *high aspect ratio* concentric circles limit:

$$\omega_{\nabla B}=\frac{2a}{R_0}(\cos\theta+\hat{s}\theta\sin\theta)\qquad \omega_{\nabla B}^{(0)}=-\frac{2a}{R_0}(\hat{s}\sin\theta)$$

Definitions

- To get the previous expressions, define the normalized flux surface coordinate $\rho = r/a$ and also k_θ:

$$k_\theta \equiv \frac{n_0}{a} \frac{d\rho}{d\Psi_N} = \frac{n_0 q_0}{r_0}$$

- Normalized poloidal flux $\Psi_N \equiv \Psi/(a^2 B_a)$

- Here, a is half the diameter of the LCFS at the elevation of the magnetic axis.

- The field B_a is normalized to the toroidal field at R_a, where R_a is the avg of the min and max R on the LCFS.

Theta Dependencies of Curvature Drift

- Curvature drift:

$$\omega_\kappa = \omega_{\nabla B} + \frac{8\pi}{B^2}\frac{dp}{d\rho}.$$

where $B = B(\theta)$ and p is the total pressure.

- In the *high aspect ratio* concentric circles limit $\omega_{\nabla B} = \omega_\kappa$

- In general, the curvature drift is always *bad* on the outboard midplane of a tokamak, but the ∇B drift can be reversed.

θ Dependencies of Parallel Derivatives

- Parallel derivatives:

$$\hat{\mathbf{b}} \cdot \nabla = \frac{\mathbf{B}_0 \cdot \nabla}{B_0}$$

- In the *high aspect ratio* concentric circles limit $\hat{\mathbf{b}} \cdot \nabla = a/(qR_0)$

- Freedom in the definition of θ can be exploited to remove the θ dependence from this operator (`equal_arc = T`)

θ Dependencies of Perp Gradients

- Bessel functions have argument $\gamma = |\nabla S| v_\perp / \Omega$; always enters as square (γ^2).

- General expression:

$$|\nabla S|^2 = \frac{n_0^2}{a^2} \left|\nabla\left(\alpha + q\theta_0\right)\right|^2$$

$$= k_\theta^2 \left(\frac{d\Psi}{d\rho}\right)^2 \left|\nabla\alpha\cdot\nabla\alpha + 2\theta_0 \nabla\alpha\cdot\nabla q + \theta_0^2 \nabla q\cdot\nabla q\right|.$$

- In the *high aspect ratio* concentric circles limit

$$|\nabla S|^2 = k_\theta^2 \left|1 + \hat{s}^2\theta^2 - 2\theta\theta_0\hat{s}^2 + \theta_0^2\hat{s}^2\right|$$

Flux-Surface Averages

- Given Jacobian J, flux-surface average of a quantity Γ is

$$\langle \Gamma \rangle = \lim_{\Delta\rho\to 0} \frac{\int \Gamma J \, d\theta \, d\alpha \, d\rho}{\int J \, d\theta \, d\alpha \, d\rho}.$$

- Flux-surface avg of a radially directed quantity (a flux) is

$$Q_{\mathsf{sim}} = \frac{\langle \mathbf{Q} \cdot \nabla\rho \rangle}{\langle \nabla\rho \rangle}$$

which will appear in the transport equation as

$$\frac{3}{2}\frac{d}{dt}\langle nT \rangle + \frac{1}{V'}\frac{d}{d\rho} A Q_{\mathsf{sim}} + \ldots = 0,$$

where the surface area $A = 2\pi \langle \, |\nabla\rho| \, \rangle \int J d\theta$.

Loose Ends

- Note the limit in this expression:

$$\langle \Gamma \rangle = \lim_{\Delta\rho \to 0} \frac{\int \Gamma J \, d\theta \, d\alpha \, d\rho}{\int J \, d\theta \, d\alpha \, d\rho}.$$

- In a flux-tube simulation, since the perpendicular box size is measured in gyroradii and $\rho_* \ll 1$, it is appropriate to take the integral over the entire simulation domain.

- In other words, a flux-tube simulation naturally calculates the flux-surface average of the various fluxes.

Critical Elements for Axisymmetric Users

- Tokamak user has three choices:

  1. Assume shifted circles & specify r/a, a/R, q, $\widehat{s}$, and α
  2. Specify the shape of the flux surface and $B_p(\theta)$
  3. Read in numerical specification from MHD equilibrium solver

- In each case, the user can use the two free functions of the Grad-Shafranov equation to vary the magnetic shear and the pressure gradient freely: details controlled by `bishop`

Shifted Circles (`theta_grid_parameters`)

- Magnetic shear: `shat` $= \hat{s}$
- Aspect ratio: `epsl` $= 2a/R_0$
- Safety factor: `pk` $= 2a/(qR_0)$; *i.e.*, $q =$`epsl/pk`
- Minor radius: `eps` $= r/R$
- Shafranov shift: `shift` $= \alpha = -2Rq^2\, d\beta/dr$
  (not the coordinate α!)

Local Equilibrium

- Surface of constant Ψ specified by:

$$R_N(\theta) = R_{0N}(\rho) + \rho \cos\left[\theta + \delta(\rho)\sin\theta\right],$$

$$Z_N(\theta) = \kappa(\rho)\rho\sin\theta.$$

(More general expressions easily implemented.)

- Poloidal field determined from

$$B_p(\theta) = \frac{|\nabla\Psi|}{R} = \frac{d\Psi}{d\rho}\frac{|\nabla\rho|}{R}$$

- Conventions: $R_N = R/a$, *etc.*,

$R_{0N}(\rho) = R_{0N}(\rho_c) + R'_{0N}\,d\rho$, $\quad \delta(\rho) = \delta(\rho_c) + \delta'\,d\rho$, $\quad \kappa(\rho) = \kappa(\rho_c) + \kappa'\,d\rho$

Local Equilibrium: Details I

- Input q, note that $\oint \alpha\, d\theta = -2\pi q$ and then determine $d\Psi/d\rho$ from

$$\frac{d\Psi}{d\rho} = \frac{I}{2\pi q}\oint \frac{d\theta}{R^2}\left(\nabla\theta \times \nabla\rho \cdot \nabla\phi\right)^{-1} = \frac{I}{2\pi q}\oint \frac{J\, d\theta}{R^2}$$

- Define three integrals:

$$A(\theta) = \int J\, d\theta \left[\frac{1}{R^2} + \left(\frac{I}{B_p R^2}\right)^2\right], \qquad B(\theta) = I \int J\, d\theta \left[\frac{1}{(B_p R)^2}\right],$$

$$C(\theta) = I \int J\, d\theta \left[\frac{\sin u + R/R_c}{B_p R^4}\right]$$

Local Equilibrium: Definitions

- $u(\theta)$: angle between the horizontal and the tangent to the magnetic surface in the poloidal plane

- $R_c(\theta)$: local radius of curvature of surface in the poloidal plane.

- From papers by Mercier and Luc, used later by C. Bishop and R. Miller

Local Equilibrium: Details II

- Upon defining $\bar{A} = \oint d\theta \cdots$, one can show

$$\hat{s} = \frac{\rho}{q}\frac{dq}{d\rho} = \frac{\rho}{2\pi q}\frac{d\Psi}{d\rho}\left(\bar{A}I' + \bar{B}p' + 2\bar{C}\right)$$

- Primes are derivatives w.r.t. Ψ

- User thus may specify any two of p', I', and $\hat{s}$. Several input options determined by `bishop`

Local Eq Inputs (`theta_grid_parameters`)

- Magnetic shear: `shat` $= \hat{s}$ (for `bishop` $= 1$)
- Minor radius: `rhoc` $= \rho_c$ (Normalized by a!)
- Safety factor: `qinp` $= q$
- Elongation: `akappa` $= \kappa$
- Deriv of elongation: `akappri` $= d\kappa/d\rho$

Local Eq Inputs (`theta_grid_parameters`)

- Triangularity: `tri` $= \delta$
- Deriv of triangularity: `tripri` $= d\delta/d\rho$
- Center of flux surf: `Rmaj` $= R_{0N}$
- Local shift: `shift` $= dR_{0N}/d\rho < 0$
- Center of LCFS: `R_geo` $= R_{\mathrm{geoN}}$ (for normalization only)

Eq Inputs (`theta_grid_eik_knobs`)

- These parameters may be set whether the equilibrium information comes from a local model *or* from numerical data (a file)

- Magnetic shear: `s_hat_input` $= \hat{s}$
  (for `bishop` > 1)

- Pressure gradient: `beta_prime_input` $= d\beta/d\rho < 0$
  (for `bishop` $= 4$)

Eq Selections (theta_grid_eik_knobs)

- Set at most one to true:
  ppl_eq, transp_eq, efit_eq, gen_eq, vmom_eq, local_eq

- Set iflux $= 1$ to use numerical equilibrium, 0 otherwise

- Choose definition of ρ (flux surface label):
  irho $= 1 \ldots (\rho = \sqrt{\Phi/\Phi_a})$
  irho $= 2 \ldots (\rho = d/D)$
  irho $= 3 \ldots (\rho = \Psi/\Psi_a)$

- Likely never change: itor $= 1$

Eq Recommendations (`theta_grid_eik_knobs`)

- Watch out for the units, esp normalizations by a
- Set `writelots = T` (it's essentially free information)
- Set `isym = 0` (allow up-down asymmetry)
- Set `equal_arc = F` (and avoid a rare bug)
- Set `del_rho = 1.e-3` (nearly always adequate)

Numerical Equilibria (`theta_grid_eik_knobs`)

- If given the choice, use inverse solutions $R(\Psi, \theta), Z(\Psi, \theta)$ for accuracy

- Specify file with data using `eqfile = '...'`
  No blanks allowed, but other characters okay.

- Don't forget to set bishop. Use 1 to get the data from the file, > 1 to alter it.

References

- M. D. Kruskal and R. M. Kulsrud, Phys. Fluids, 1:265, 1958
- M. A. Beer and S. C. Cowley and G. W. Hammett, Phys. Plasmas, 2:2687, 1995
- R. L. Miller, *et al.*, Phys. Plasmas, 5:973, 1998

- Notes for this talk: `http://gk.umd.edu/g_short.pdf`