Note on the parallel nonlinearity in gyrokinetics

G. W. Hammett

April 20, 2004

The ordering for the conventional gyrokinetic equations is usually written as:

$$\epsilon \sim \frac{\omega}{\Omega} \sim \frac{k_\parallel v_t}{\Omega} \sim \frac{\rho}{L} \sim \frac{e\Phi}{T} \sim \frac{F_1}{F_0} \sim \frac{k_\parallel}{k_\perp} \ll 1$$

and

$$k_\perp \rho \sim 1$$

There is some ongoing discussion on whether all of these are really necessary or whether some of these are overly restrictive and can be relaxed for edge turbulence, for example.

Note that all though perturbations are assumed small, $F_1 \ll F_0$, is is assumed that gradients of perturbations can be just as strong as gradients of the background, $k_\perp F_1 \sim F_0/L$, so that ExB nonlinearities are included in the gyrokinetic equations. For example, comparing the ExB nonlinearity to the linear ExB term gives:

$$\frac{v_{ExB} \cdot \nabla F_1}{v_{ExB} \cdot \nabla F_0} \sim 1$$

The linear $E_\parallel$ term is also the same order:

$$\frac{\frac{e}{m} E_\parallel \frac{\partial f_0}{\partial v_\parallel}}{v_{ExB} \cdot \nabla F_0} \sim \frac{\frac{e}{m} k_\parallel \Phi \frac{1}{v_t}}{\frac{c}{B} k_\perp \Phi \frac{1}{L}} \sim \frac{k_\parallel L}{k_\perp \rho} \sim 1$$

Most present day nonlinear gyrokinetic calculations drop the parallel nonlinearity. If I recall correctly, the original Frieman and Chen gyrokinetic derivation also dropped the parallel nonlinearity, and retain only the ExB nonlinearity, as was done in earlier drift wave studies (Hasegawa-Mima, etc.). This is justified if $\partial F_1/\partial v_\parallel \sim F_1/v_t$ holds. In some special cases, this might break down and one might get sharp derivatives $\partial F_1/\partial v_\parallel \sim (1/\epsilon) F_1/v_t$ so that the parallel nonlinearity should be kept. But in strong turbulence regimes where the nonlinear frequency spread $\Delta\omega_{NL} \sim \omega \sim k_\parallel v_t$, then the parallel nonlinearity should be unimportant. To see this, estimate f_1 from

$$f_1 \sim \frac{(e/m) E_\parallel \partial f_0 / v_\parallel}{i(\omega - k_\parallel v_\parallel) + \Delta\omega_{NL}}$$

We can use this to estimate

$$\frac{\partial f_1}{\partial v_\parallel} \sim \frac{f_1}{v_t} \left(1 + \frac{k_\parallel v_t}{\Delta\omega_{NL}} \right)$$

near resonance ($\omega = k_\parallel - v_t$).

In a very weak turbulence limit, where the nonlinear broadening is extremely weak, $\Delta\omega_{NL} \sim \epsilon k_\parallel v_t$, then one might need to worry about the parallel nonlinearity because $\partial f_1/\partial v_\parallel$ is very large. But for a fixed value of $\Delta\omega_{NL}/k_\parallel v_t$, then one should always be able to make the gyrokinetic expansion parameter ϵ sufficiently small that the parallel nonlinearity is negligible even if $\Delta\omega_{NL}/\omega$ is sufficiently small to be in a weak turbulence regime.

The parallel nonlinearity might first become important for electron dynamics, because often $\Delta\omega_{NL} \sim k_\parallel v_{ti} \sim k_\parallel v_{te}/60$, so that the electron parallel nonlinearity might look like it becomes important if the gyrokinetic expansion parameter is as large as $\epsilon \sim \rho_* \sim 1/60$. But the slab limit is probably misleading, and one would find the parallel nonlinearity is less important in real toroidal geometry where particle trapping is important and the mirror force alters the parallel dynamics. Also, even a very small amount of electron collisions can be very important in such a case, because the electron collision operator has a piece that looks like $\nu_e v_{te}^2 \partial f_1^2/\partial v_\parallel^2$, which becomes very important if there are sharp velocity gradients in f_1.

In W.W. Lee's early gyrokinetic simulations, the δf simulation method hadn't yet been invented and computers weren't very powerful and he had to run with a very large value of $\rho/L_T \sim 1/5$ in order to make the fluctuation amplitudes large enough to beat particle noise. Because of this, some of the early gyrokinetic derivations would go to second order in the gyrokinetic expansion parameter ϵ, leading to additional nonlinearities in the Poisson equation as well as keeping the $E_\parallel$ nonlinearity. In Dimits derivation of the δf algorithm for his thesis (Kotschenreuther independently did it also), he originally developed what was called a "partially linearized" δf algorithm, which just meant that the parallel nonlinearity was dropped, and I believe he gave some arguments as to why this was usually justified in the small ρ_* limit. Later versions of the δf algorithm where able to keep the nonlinearity, but by that time every one was working with very small ρ_* where the parallel nonlinearities aren't very important. I believe Frank Jenko has some papers exploring the importance of parallel nonlinearities for edge drift-wave turbulence regimes, where the effective $\rho_* = \rho/L$ isn't terribly tiny. However, I believe he neglected particle trapping and collisions in those studies, which might reduce the importance of the parallel nonlinearity.