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The stability of Couette flow in a toroidal magnetic field

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Abstract

The stability of the hydromagnetic Couette flow is investigated when a constant current is applied along the axis of the cylinders. It is shown that if the resulting toroidal magnetic field depends only on this current, no linear instability to axisymmetric disturbances is possible.

Keywords: Magnetic fields - magnetohydrodynamics - methods: analytical

1. Introduction

Magnetorotational instability (MRI) is important to theoretical astrophysics because it is the only linear instability known to grow robustly under the conditions prevailing in most accretion disks: an electrically conducting fluid, rotating with local angular velocity $\Omega(r)$; a positive gradient of specific angular momentum, $\partial(r^2\Omega)^2/\partial r > 0$ (stable by the Rayleigh criterion); and a negative gradient of angular velocity, $\partial\Omega^2/\partial r < 0$. It was revealed by astrophysicists Balbus and Hawley in the early 1990's. Developments in astrophysics, too large to survey in this letter, have grown up around this problem [1]. In the ongoing investigation of the best way to demonstrate MRI in the laboratory, several configurations have been examined. Generally, they involve hydromagnetic Couette flow between rotating cylinders along the axis of which a magnetic field is applied. On the other hand, if an electric current is applied along the axis, a toroidal component to the magnetic field results. Some of the earliest and most relevant theoretical results were those of Velikhov [2] and Chandrasekhar [3], though both these researchers considered ideal Taylor–Couette flow. Elsewhere, Chandrasekhar [4] included dissipation but by dropping one term, assumed to be small, the instability envisioned here cannot arise [5,6]. Shortly after Chandrasekhar's results, Gotoh [7] and then DiPrima and Pan [8] investigated theoretically the effect of a purely toroidal magnetic field on the stability of Couette flow in the Rayleigh-unstable dynamical regime.

More recently researchers have begun to focus on the case where the magnetic field has a toroidal component as well as an axial component, while the basic flow is in the Rayleigh-stable regime dynamically; the regime in which MRI is found. For instance, Rüdiger et al. [9] have reported on ongoing work of this type. The purpose of the present note is to show that without the axial magnetic field, with only the toroidal field due to an axial current,

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the flow is strictly stable to linear axisymmetric disturbances. This is demonstrated using a method of quadratic functionals popularized by Chandrasekhar [4], originally used by Synge [10]. Along with this, an operator notation is introduced, which aids in keeping track of the functionals involved. To the extent that most current work on the subject uses asymptotic and computational methods, this approach is different. The first author has been working with the astrophysicist Goodman to bring these techniques to bear on MRI problems [6].

2. The governing equations and derivation of stability

The basic flow is $\mathbf{v} = r\Omega\mathbf{e}_\theta$ (in cylindrical coordinates r, θ, z) between two cylinders of radii r_1, r_2 and angular velocities Ω_1, Ω_2:

$$\Omega(r) = a + \frac{b}{r^2}, \quad a = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2}, \quad b = \frac{\Omega_1 - \Omega_2}{r_2^{-2} - r_1^{-2}}. \tag{1}$$

A toroidal magnetic field $\mathbf{B} = \frac{2J}{r}\mathbf{e}_\theta \equiv H(r)\mathbf{e}_\theta$ permeates the fluid from a constant line current J along the axis of the cylinders, in which ρ is the density of the fluid and ν is the kinematic viscosity, σ is the electrical conductivity, and μ is the magnetic permeability. The magnetic diffusivity is defined as $\eta = (4\pi\mu\sigma)^{-1}$. The flow regime also admits the possibility of a contribution to the toroidal component of the base magnetic field from an electric current in the fluid in the axial direction, but that is ignored here. Nondimensional parameters for the flow might be magnetic Prandtl number $P_m = \nu/\mu$, and Reynolds number $R = \Omega_1 r_1^2/\nu$. The results to be derived are independent of P_m and R.

2.1. The governing equations

Following DiPrima and Pan [8], linear perturbations are taken to be independent of θ and proportional to e^{st+ikz}:

$$\mathbf{v}' = \begin{pmatrix} v_r' \\ v_\theta' \\ v_z' \end{pmatrix} = \begin{pmatrix} \varphi_r(r) \\ \varphi_\theta(r) \\ \varphi_z(r) \end{pmatrix} \mathrm{e}^{st+ikz},$$

$$\mathbf{H}' = \begin{pmatrix} H_r' \\ H_\theta' \\ H_z' \end{pmatrix} = \begin{pmatrix} \beta_r(r) \\ \beta_\theta(r) \\ \beta_z(r) \end{pmatrix} \mathrm{e}^{st+ikz}.$$

The axisymmetric linearized equations of motion become

$$s\beta_\theta = \eta(DD_* - k^2)\beta_\theta + \left(\frac{\mathrm{d}H}{\mathrm{d}r} - \frac{H}{r}\right)\varphi_r - r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\beta_r, \tag{2}$$

$$s\varphi_\theta = \nu(DD_* - k^2)\varphi_\theta - \frac{\mu}{4\pi\rho}\frac{1}{r}(D_* H)\beta_r - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\Omega)\varphi_r, \tag{3}$$

$$s\beta_r = \eta(DD_* - k^2)\beta_r, \tag{4}$$

$$s(DD_* - k^2)\varphi_r = \nu(DD_* - k^2)^2\varphi_r - k^2\frac{\mu}{2\pi\rho}\frac{H(r)}{r}\beta_\theta - 2\Omega k^2\varphi_\theta. \tag{5}$$

For perturbations, we follow Chandrasekhar's notation $Df \equiv \mathrm{d}f/\mathrm{d}r$, $D_* f \equiv r^{-1}D(rf)$. The vertical components φ_z and β_z have been eliminated from Eqs. (2)–(5) using $\nabla\cdot\mathbf{v}' = \nabla\cdot\mathbf{H}' = 0$.

The boundaries are impenetrable and "no-slip", so that

$$\varphi_r = D\varphi_r = 0, \tag{6}$$

$$\varphi_\theta = 0, \quad \text{at } r = r_1, r_2. \tag{7}$$

The condition on $D\varphi_r$ derives from the continuity equation $D_*\varphi_r = -k\varphi_z$ since $\varphi_z = 0$ on the walls. We take the cylinders to be partially conducting as was considered by Roberts [11]; requiring that the magnetic perturbations match onto exterior solutions satisfying $\nabla \times \mathbf{H}' = 0$, which are well-behaved as $r \to 0$ and as $r \to \infty$. The boundary conditions on the radial component are therefore

$$D_*\beta_r = \beta_r \frac{[kI_0(kr)]}{I_1(kr)} \quad \text{at } r = r_1, \tag{8a}$$

$$D_*\beta_r = -\beta_r \frac{[kK_0(kr)]}{K_1(kr)} \quad \text{at } r = r_2, \tag{8b}$$

while those on the toroidal component are

$$\frac{\sigma'}{\sigma} D_*\beta_\theta = \beta_\theta \frac{[kI_0(kr)]}{I_1(kr)} \quad \text{at } r = r_1, \tag{9a}$$

$$\frac{\sigma'}{\sigma} D_*\beta_\theta = -\beta_\theta \frac{[kK_0(kr)]}{K_1(kr)} \quad \text{at } r = r_2, \tag{9b}$$

where $I_n(kr)$ and $K_n(kr)$ are the modified Bessel functions (of orders $n = 0, 1$ in this work), σ' is the conductivity of the walls. So when $\sigma' \to 0$, the insulating boundary conditions are recovered. And when $\sigma' \to \infty$, perfectly conducting boundary conditions result. Eqs. (6)–(9b) impose ten boundary conditions on the tenth-order differential system (2)–(5).

2.1.1. *The radial magnetic perturbation*

It becomes clear that (4) uncouples from the rest of the system. It is important to show that with (8a) and (8b), the only solution of (4) which may occur if $\mathrm{Re}(s) \geq 0$, that is, for neutral or amplified disturbances is $\beta_r = 0$. Such a conclusion was drawn by Gotoh [7] some time ago, though with different boundary conditions. This is accomplished by multiplying (4) by $r\bar{\beta}_r$ and integrating from $r = r_1$ to $r = r_2$ to obtain

$$\begin{aligned}
s\int_{r_1}^{r_2} r|\beta_r|^2 \mathrm{d}r &= \int_{r_1}^{r_2} (DD_* - k^2)\beta_r\bar{\beta}_r r\,\mathrm{d}r \\
&= \int_{r_1}^{r_2} \left\{ r\bar{\beta}_r \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\beta_r)\right] - rk^2|\beta_r|^2 \right\} \mathrm{d}r \\
&= [r\bar{\beta}_r D_*\beta_r]_{r=r_1}^{r=r_2} - \int_{r_1}^{r_2} r(|D_*\beta_r|^2 + k^2|\beta_r|^2)\mathrm{d}r \\
&= -\frac{kK_0(kr_2)}{K_1(kr_2)} r_2|\beta_r(r_2)|^2 - \frac{kI_0(kr_1)}{I_1(kr_1)} r_1|\beta_r(r_1)|^2 \\
&\quad - \int_{r_1}^{r_2} (r|D_*\beta_r|^2 + k^2 r|\beta_r|^2)\mathrm{d}r,
\end{aligned} \tag{10}$$

which follows after integrating by parts and applying the boundary conditions (8a) and (8b). From the real parts of the extremes of (10) we conclude that $\mathrm{Re}(s) < 0$. Henceforth then, we will set $\beta_r = 0$ in the system.

2.2. *Operator formulation*

In order to simplify the derivation of the stability criterion, we make a reformulation. An operator notation is introduced, which clarifies the nature of the analysis. The system thereby becomes

$$-sM\varphi_r = \nu M^* M\varphi_r - k^2 \frac{\mu}{\pi\rho}\frac{J}{r^2}\beta_\theta - 2\Omega k^2\varphi_\theta. \tag{11}$$

$$s\beta_\theta = -\eta M_{\sigma'}\beta_\theta - \frac{4J}{r^2}\varphi_r, \tag{12}$$

$$s\varphi_\theta = -\nu M_0\varphi_\theta - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\Omega)\varphi_r. \tag{13}$$

In this notation, M, M^*, M_0, and $M_{\sigma'}$ all denote $-DD_* + k^2$, but are considered different operators because of the distinct boundary conditions satisfied by the functions on which they act [12]. For this reason, M^*M denotes $(-DD_* + k^2)^2$. That is, M acts on functions that have the same boundary conditions as φ_r [Eq. (6)], M^* assumes that the functions satisfy no particular boundary condition, whereas M_0 uses the boundary conditions of φ_θ [Eq. (7)] and $M_{\sigma'}$, the boundary conditions of β_θ [Eqs. (9a) and (9b)]. Consequently, when $\sigma' \to 0$, the case of insulating boundary conditions, $M_{\sigma'} \to M_0$. The case of perfectly conducting boundary conditions, that is, $\sigma' \to \infty$ also has meaning as will be shown.

Introduce an inner product,

$$\langle f, g\rangle = \int_{r_1}^{r_2} r f(r)\bar{g}(r)\mathrm{d}r, \tag{14}$$

in which the overbar denotes complex conjugation. The differential operators M, M^*M, and $M_{\sigma'}$ all have the property of being *positive definite* in this inner product. For example, let us show that $\langle M_{\sigma'}\beta_\theta, \beta_\theta\rangle > 0$. This follows quite readily by a calculation similar to (10)

$$\langle M_{\sigma'}\beta_\theta, \beta_\theta\rangle = \int_{r_1}^{r_2} r\bar{\beta}_\theta(r)(-DD_* + k^2)\beta_\theta(r)\mathrm{d}r$$

(and integrating by parts to obtain)

$$\begin{aligned} &= -\,[r\bar{\beta}_\theta D_*\beta_\theta]_{r=r_1}^{r=r_2} + \int_{r_1}^{r_2} r(|D_*\beta_\theta|^2 + k^2|\beta_\theta|^2)\mathrm{d}r \\ &= \frac{\sigma}{\sigma'}\frac{kK_0(kr_2)}{K_1(kr_2)}r_2|\beta_\theta(r_2)|^2 + \frac{\sigma}{\sigma'}\frac{kI_0(kr_1)}{I_1(kr_1)}r_1|\beta_\theta(r_1)|^2 \\ &\quad + \int_{r_1}^{r_2} r(|D_*\beta_\theta|^2 + k^2|\beta_\theta|^2)\mathrm{d}r \\ &> 0. \end{aligned} \tag{15}$$

The boundary conditions (9a) and (9b) were applied. The case where $\sigma' \to \infty$, is dealt with by noticing that the boundary conditions terms vanish leading to

$$\langle M_\infty\beta_\theta, \beta_\theta\rangle = \int_{r_1}^{r_2} r(|D_*\beta_\theta|^2 + k^2|\beta_\theta|^2)\mathrm{d}r.$$

Thus, for all wall conductivities σ', $0 \le \sigma' \le \infty$, $\langle M_{\sigma'}\beta_\theta, \beta_\theta\rangle > 0$.

The calculations for $\langle M\varphi_r, \varphi_r\rangle$ and $\langle M_0\varphi_\theta, \varphi_\theta\rangle$ are similar to those for $\langle M_\infty\beta_\theta, \beta_\theta\rangle$, since the boundary terms do not arise when the boundary conditions (6) and (7) are applied. Likewise, it may be shown that $\langle M^*M\varphi_r, \varphi_r\rangle = \langle M\varphi_r, M\varphi_r\rangle \equiv \|M\varphi_r\|^2 > 0$.

2.3. Derivation of the stability criterion

We want to derive the following criterion. *MRI is suppressed, in fact no instability at all occurs, for the system (2)–(5). We also conclude that there are no marginal modes.*

The derivation proceeds as follows. Make use of the inner product (14) as in (15), forming that of (11) with φ_r to obtain

$$\langle(\nu M^*M + sM)\varphi_r, \varphi_r\rangle = \frac{k^2\mu}{\pi\rho}\langle Jr^{-2}\beta_\theta, \varphi_r\rangle + \langle 2\Omega k^2\varphi_\theta, \varphi_r\rangle. \tag{16}$$

From (12) and (13), this may be re-written as

$$\langle(\nu M^*M + sM)\varphi_r, \varphi_r\rangle = -\frac{k^2\mu}{4\pi\rho}\langle\beta_\theta, (\eta M_{\sigma'} + s)\beta_\theta\rangle - \frac{1}{a}\langle\Omega k^2\varphi_\theta, (\nu M_0 + s)\varphi_\theta\rangle, \tag{17}$$

since $\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\Omega) = 2a$ is a positive constant. It was established by Synge [10] and by Chandrasekhar [4] that for the azimuthal velocity function φ_θ, the following is true:

$$\mathrm{Re}\langle(-DD_* + k^2)\varphi_\theta, \Omega k^2\varphi_\theta\rangle = \mathrm{Re}\langle\Omega k^2\varphi_\theta, (-DD_* + k^2)\varphi_\theta\rangle > 0, \tag{18}$$

by virtue of (7). Hence, taking the real part of (17), combining terms having the growth rate s, the result is

$$\begin{aligned}&\mathrm{Re}(s)\left\{\langle M\varphi_r,\varphi_r\rangle + a^{-1}\langle \Omega k^2\varphi_\theta,\varphi_\theta\rangle + \frac{\mu k^2}{4\pi\rho}\langle\beta_\theta,\beta_\theta\rangle\right\}\\ &\quad= -\nu\|M\varphi_r\|^2 - \frac{\mu\eta k^2}{4\pi\rho}\langle M_{\sigma'}\beta_\theta,\beta_\theta\rangle - a^{-1}\nu\mathrm{Re}(\langle M_0\varphi_\theta,\Omega k^2\varphi_\theta\rangle)\\ &\quad< 0. \end{aligned} \tag{19}$$

An immediate consequence of this is $\mathrm{Re}(s) < 0$, and hence stability. Since $\mathrm{Re}(s)$ *is* strictly negative, there are no marginal modes.

3. Conclusions

The desire to test for MRI in the laboratory inspires several configurations. Rüdiger et al. [9] showed that imposing both azimuthal and axial magnetic fields together reduces the critical Reynolds number to obtain MRI. They go on to conclude that incorporating an axial current is the most promising design for obtaining MRI in a laboratory experiment. It is of interest then how these results compare. One might therefore study the set-up in which both an axial current and an axial magnetic field occur. If a measure of the ratio of the toroidal to axial magnetic field B_0 is given by

$$\gamma = \frac{2J}{B_0 r_1},$$

where J is the current and r_1 is the radius of the inner cylinder, then the prediction here is that as $B_0 \to 0, \gamma \to \infty$, and in this limit all of the eigenvalues of the linearized axisymmetric stability equations will exhibit $\mathrm{Re}(s) < 0$. Our analytical results suggest that the axial current is ultimately stabilizing. The inclusion of some axial component to the magnetic field appears to be necessary in order to see axisymmetric modes of instability.

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