VLASOV-MAXWELL DESCRIPTION OF AXISYMMETRIC TWO-STREAM INSTABILITIES IN HIGH-INTENSITY PARTICLE BEAMS*

Han S. Uhm[†], Ajou University, Suwon 442-749, Korea
Ronald C. Davidson and Igor Kaganovich, Plasma Physics Laboratory, Princeton
NJ 08543, USA.

Abstract

Two-stream instabilities in high-intensity particle beams are investigated by making use of the Vlasov-Maxwell equations in the smooth-focusing approximation. Stability properties for the axisymmetric sausage and hollowing modes, characterized by radial mode numbers $n = 1$ and $n = 2$, respectively, are investigated, and the dispersion relations are obtained for the complex eigenfrequency in terms of the axial wavenumber k and other system parameters. Detailed stability properties are calculated over a wide range of normalized beam intensity s_b and fractional charge neutralization f. The growth rates of the sausage and hollowing modes are found to be of the same order of magnitude as that of the dipole-mode two-stream instability. Therefore, it is concluded that the axisymmetric sausage and hollowing instabilities may also be deleterious to intense ion beam propagation when a background component of electrons is present.

1 INTRODUCTION

Charged particle beams are subject to various collective instabilities that can deteriorate the beam quality. For example, intense charged particle beams can develop a halo structure during propagation. This halo structure may be caused by collective excitations, such as axisymmetric hollowing instabilities. Background electrons are often present at the high beam currents and charge densities of practical interest in many ion beam applications. It has been recognized[1-6] for many years that the relative streaming motion of the high-intensity beam particles through a background charge species can provide the free energy to drive the classical two-stream instability. For example, for electrons interacting with an intense proton beam, as in the Proton Storage Ring (PSR) experiment, or the Spallation Neutron Source (SNS), this instability is referred to as the electron-proton (e-p) two-stream instability[3]. Theoretical treatments of the two-stream instability can be based on either a kinetic model[6] that makes use of the Vlasov-Maxwell equations to describe the self-consistent interaction of the ion and electron distribution functions with the applied field and the self-generated electric and magnetic fields, or on rigid-beam models[1] that analyze the transverse motion of the center-of-mass of the ion and electron charge distributions. In the present analysis, we investigate two-stream

instability properties for axisymmetric perturbations ($\partial/\partial\theta = 0$) about an intense ion beam propagating through background electrons by making use of the Vlasov-Maxwell equations. Therefore, the present work is complementary to a previous study[6] of the two-stream instability carried out for non-axisymmetric perturbations ($\partial/\partial\theta \neq 0$).

2 THEORETICAL MODEL

The equilibrium configuration consists of an intense ion beam with radius r_b that propagates in the z-direction with directed kinetic energy $(\gamma_b-1)m_bc^2$ through a perfectly conducting cylinder with wall radius r_w. The ion beam propagates through background (stationary) electrons with characteristic directed axial momentum $\gamma_b m_b \beta_b c$ in the z-direction, where $V_b = \beta_b c = const.$ is the average axial velocity, and $\gamma_b = (1 - \beta_b^2)^{-1/2}$ is the relativistic mass factor. In order to simplify the analysis, it is assumed that the background column of electrons has radius r_b. In the context of the smooth-focusing approximation, the beam ions are radially confined by the applied transverse focusing force $\boldsymbol{F}_{foc}{}^b$. As for the background electrons, to the extent that the beam ion density exceeds the background electron density, the space-charge force on an electron provides transverse confinement of the background electrons by the electrostatic space-charge potential $\phi(\boldsymbol{x},t)$. It is further assumed that the ion motion in the beam frame is nonrelativistic, and that the transverse momentum components of a beam ion, p_x and p_y, and the characteristic spread in axial momentum, $\delta p_z = p_z - \gamma_b m_b \beta_b c$, are small in comparison with the directed axial momentum $\gamma_b m_b \beta_b c$.

Under the equilibrium assumption that the distribution function for the beam ions and background electrons are axisymmetric and spatially uniform in the axial direction, we recognize that the total transverse energies and axial momentum of the beam ions and background electrons are approximate constants of the motion in the equilibrium fields[6]. For present purposes, the equilibrium distribution functions for the beam ions and the background electrons are taken to be

$$F_b^0 = (n_b / 2\pi\gamma_b m_b)\delta(H_{\perp b} - T_{\perp b})G_b(p_z)$$
$$F_e^0 = (n_e / 2\pi m_e)\delta(H_{\perp e} - T_{\perp e})G_e(p_z). \tag{1}$$

Here, n_b and n_e are the on-axis ion and electron number densities, respectively, $T_{\perp b}$ and $T_{\perp e}$ are positive constants, and $H_{\perp b}$ and $H_{\perp e}$ are the single-particle Hamiltonians for the transverse ion and electron motions.

We now make use of the linearized Vlasov-Maxwell equations[6] to develop a theoretical model of the two-stream instability for perturbations about the equilibrium described by Eq. (1). In the subsequent analysis, we adopt a normal mode approach in which all perturbed quantities are assumed to vary with θ, z, and t according to

$$\delta\xi(r,\theta,z,t) = \xi_1(r)\exp[i(kz - \omega t)],\qquad(2)$$

for axisymmetric perturbations with $\partial/\partial\theta = 0$. Here, ω and k are the complex eigenfrequency and axial wavenumber of the perturbation, with $Im\,\omega > 0$ corresponding to temporal growth. We also consider axial wavelengths that are long and frequencies that are low compared with quantities that characterize the beam radius. The perturbed potential amplitudes, $\psi_1(r)$ and $\phi_1(r)$, for the beam ions and background electrons occurring in the linearized Vlasov equations are determined self-consistently in terms of the perturbed particle number densities. We obtain

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r\,\psi_1(r) = -4\pi e\,[(Z_b/\gamma_b^2)n_{b1}(r) - n_{e1}(r)],$$

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r\,\phi_1(r) = -4\pi e\,[Z_b\,n_{b1}(r) - n_{e1}(r)],\qquad(3)$$

where $\psi_1(r) = \phi_1(r) - \beta_b A_{z1}(r)$, and $n_{b1}(r)$ and $n_{e1}(r)$ are the perturbed number densities of the beam ions and background electrons, respectively. The perturbed densities can be obtained from the linearized Vlasov equations for δF_b and δF_e. For example, the perturbed ion beam density $n_{b1}(r)$ is calculated from

$$n_{b1}(r) = \int d^3p\,\delta F_b.\qquad(4)$$

In Eq. (4), δF_b is the perturbed ion beam distribution function calculated by the method of the characteristics which can be expressed as[6]

$$\delta F_b(x,p,t) = Z_b e G_b(p_z)\frac{\partial}{\partial H_{\perp b}}F_{b0}(H_{\perp b})$$

$$\times\int_{-\infty}^{t}dt'\frac{p_\perp'}{\gamma_b m_b}\cdot\nabla_\perp\delta\psi(x',t'),\qquad(5)$$

where long wavelength and low frequency perturbations are assumed. Here, $x'(t')$ and $p'(t')$ are the particle trajectories in the equilibrium field configuration that pass through the phase space point (x, p) at time $t' = t$.

The self-consistent eigenfunctions $\psi_b(r)$ and $\psi_e(r)$ for radial mode number n are given by[7]

$$\psi_j(r) = \sum_{\alpha=0}^{n}a_{j\alpha}(\frac{r}{r_b})^{2\alpha},\quad 0\le r < r_b,\qquad(6a)$$

and

$$\psi_j(r) = \frac{\ln(r/r_w)}{\ln(r_b/r_w)}\sum_{\alpha=0}^{n}a_{j\alpha},\quad r_b < r \le r_w,\qquad(6b)$$

where r_w is the radius of the conducting cylinder, and $a_{j\alpha}$ are expansion coefficients.

We outline the solution to the coupled eigenvalue equations in Eq. (3) for the case of axisymmetric modes with radial mode numbers $n = 1$ and $n = 2$, which have the functional form in Eqs. (6a) and (6b). We first substitute Eqs. (6a) and (6b) into Eq. (5), and evaluate the perturbed

distribution function for the ion beam and the perturbed ion beam density in Eq. (4). Similarly, we can also calculate the perturbed distribution function and density of the electrons, thereby obtaining a closed form for the coupled eigenvalue equations in Eq. (3). Next we solve these coupled eigenvalue equations, applying the appropriate boundary conditions at $r = r_b$, which are determined by multiplying Eq. (3) by r and integrating over the interval $r_b - \varepsilon < r < r_b + \varepsilon$, with $\varepsilon \to 0_+$. The result is a matrix dispersion equation, which provides the dispersion relation for axisymmetric perturbations for different radial mode numbers.

3 SAUSAGE MODE

Axisymmetric perturbations with radial mode number $n = 1$ are characterized by the so-called sausage instability. The dispersion relation for the $n = 1$ mode is obtained from the matrix dispersion equation. Carrying out some straightforward algebraic manipulation, the dispersion relation for the $n = 1$ mode can be expressed as

$$[(\omega - k\beta_b c)^2 - \omega_b^2](\omega^2 - \omega_e^2) = \omega_f^4,\qquad(7)$$

where $\omega_b = [4\omega_{\beta b}^2 - \omega_{pb}^2(1/\gamma_b^2 - 2f)]^{1/2}$, and $\omega_e = [(2-f)\eta\omega_{pb}^2]^{1/2}$ and the coupling term on the right-hand side of Eq. (8) is defined by $\omega_f^4 = \eta f\omega_{pb}^4$. Here $\omega_{\beta b} = const.$ is the effective betatron frequency for transverse ion motion in the applied focusing field, $\omega_{pb} = (4pn_b e_b^2/\gamma_b m_b)^{1/2}$ is the on-axis relativistic beam plasma frequency, the parameter $\eta = \gamma_b m_b/Z_b m_e$ is the mass ratio and $e_b = Z_b e$.

The dispersion relation in Eq. (7) is similar in general form to previous results[6] obtained for the dipole-mode instability. In the absence of background electrons ($f = 0$), the dispersion relation in Eq. (8) gives purely oscillatory beam-mode sideband oscillations with frequency $\omega - k\beta_b c = \pm\,\omega_b$. For $f \ne 0$, however, it follows that $\omega_f \ne 0$, and the right-hand side of Eq. (7) causes an unstable coupling of the electron oscillations, $\omega = \pm\,\omega_e$, and the ion oscillations, $\omega - k\beta_b c = \pm\,\omega_b$, at least for a certain range of the axial wavenumber k. Specifically, for the positive-frequency electron branch in Eq. (7) with $\omega \approx +\,\omega_e$, it can be shown that the dispersion relation in Eq. (7) supports one unstable solution with $Im\,\omega > 0$ for oscillation frequency and wavenumber (ω, k) in the vicinity of (ω_0, k_0) defined by $\omega_0 = \omega_e$ and $k_0\beta_b c = \omega_e + \omega_b$. Indeed, the positive-frequency electron branch ($\omega \approx +\,\omega_e$) couples unstably with the downshifted ion branch ($\omega - k\beta_b c \approx -\omega_b$).

Note that the parameter η occurring in Eq. (7) is much larger than unity for protons and more massive ions. In parameter regimes of practical interest, ω_e in Eq. (7) is much larger than ω_b and ω_f, and therefore $|\delta\omega| = |\omega - \omega_0| << 2\omega_e$. If further, $|\delta\omega|$, $|\beta_b c\delta k| << 2\omega_b$, then Eq. (7) can be approximated by the simple quadratic form

$$\delta\omega(\delta\omega - \beta_b c\,\delta k) = -\Gamma_0^2 \equiv -\frac{\omega_f^4}{4\,\omega_e\omega_b},\qquad(8)$$

which has a maximum growth rate $(Im\,\omega)_{max} = \Gamma_0$ when $\delta k = k - k_0 = 0$. The maximum growth rate Γ_0 in Eq. (8) increases as the fractional charge neutralization increases.

We also note from Eq. (8) that the unstable range of the axial wavenumber k is given by $-2\Gamma_0 < \beta_b c \delta k < 2\Gamma_0$.

The quadratic approximation to the dispersion relation given in Eq. (8) is valid for moderate beam intensities satisfying $s_b = \omega_{pb}^2/2\gamma_b^2\omega_{\beta b}^2 \leq 0.2$. This is the case of interest for proton linacs and storage rings. For heavy ion fusion applications, however, the beam emittance is very low and the normalized beam intensity is such that s_b can approach unity in the absence of background electrons ($f = 0$). At such high beam intensities, it follows that it is necessary to solve the full quartic dispersion relation (7) for the complex oscillation frequency ω. The dispersion relation in Eq. (7) has been solved for singly-charged cesium ions with mass number $A = m_b/m_p = 137$ and for $(\gamma_b - 1)m_b c^2 = 2.5$ GeV, $f = 0.1$. Typical results obtained from Eq. (7) are illustrated in Fig. 1, where the normalized growth rate $u_i = (Im\omega)/\omega_{\beta b}$ is plotted versus the shifted axial wavenumber $\zeta = (k - k_0)\beta_b c/\omega_{\beta b}$. At very high beam intensity with $s_b \to 1$, say, it is evident from Fig. 1 that the normalized growth rate $u_i = (Im\omega)/\omega_{\beta b}$ has a large bandwidth and becomes significantly skewed about $k = k_0$. It is also striking from Fig. 1 that the instability growth rate can be large ($u_i \approx 3$) for the very high beam intensities ($s_b \to 1$) of interest for heavy ion fusion. The normalized real frequency $Re\omega$ can also be obtained numerically from Eq. (7). Profiles of the normalized real frequency for the sausage instability are qualitatively similar to those of the dipole mode[6].
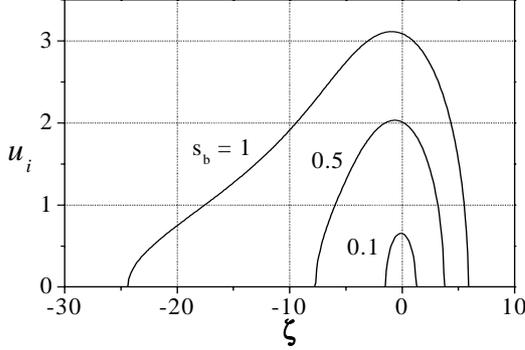

Figure 1: Normalized growth rate u_i versus the shifted axial wavenumber ζ.

4 HOLLOWING MODE

Axisymmetric perturbations with radial mode number $n = 2$ are characterized by the so-called hollowing instability. The dispersion relation for the $n = 2$ mode is obtained from the matrix dispersion equation. Carrying out some straightforward algebraic manipulation, the dispersion relation for the $n = 2$ mode is given by

$$[(\Omega_b^2 - 16\,v_b^2)(\Omega_b^2 - 4\,v_b^2) - (\omega_{pb}^2/\gamma_b^2)(\Omega_b^2 + 2\,v_b^2)]$$

$$\times [(\omega^2 - 16\,v_e^2)(\omega^2 - 4\,v_e^2) - \eta\,f\,\omega_{pb}^2(\omega^2 + 2\,v_e^2)] \quad (9)$$

$$= f\,\eta\,\omega_{pb}^4(\Omega_b^2 + 2\,v_b^2)(\omega^2 + 2\,v_e^2),$$

where the Doppler-shifted frequency Ω_b is defined by $\Omega_b = \omega - k\beta_b c$, and the (depressed) betatron frequencies, v_b and v_e, are defined by $v_b^2 = \omega_{\beta b}^2 - (\omega_{pb}^2/2)[(1/\gamma_b^2)-f]$ and $v_e^2 = (\omega_{pb}^2/2)\eta(1-f)$.

For high intensity beams with $s_b = \omega_{pb}^2/2\gamma_b^2\omega_{\beta b}^2$ approaching unity, it is necessary to solve the full dispersion relation in Eq. (9) for the complex oscillation frequency ω. Typical numerical results obtained from Eq. (9) are similar qualitatively to those for the dipole and sausage modes. We also find from analysis of Eq. (9) that the growth rate of the hollowing instability ($n = 2$) is comparable in magnitude to that of the sausage instability ($n = 1$) in Fig. 1. In this context, we conclude that the axisymmetric hollowing instability may also be deleterious to intense ion beam propagation through a background population of electrons.

5 CONCLUSIONS

Stability properties of the sausage mode characterized by radial mode number $n = 1$ have been investigated. The dispersion relation for the sausage mode was expressed in quadratic form, similar to the dispersion relation for the hose instability (dipole-mode)[6]. The eigenfunction obtained self-consistently for the sausage mode indicates that the perturbations exist only inside the beam[7]. Therefore, the presence of the grounded conducting wall does not affect the stability behavior. Stability properties of the hollowing instability, characterized by radial mode number $n = 2$, were also investigated. The full dispersion relation for the hollowing mode was obtained, which predicts instability in several ranges of axial wavenumber k. The growth rates of the sausage and hollowing instabilities are of the same order of magnitude as that of the dipole-mode hose instability[6]. In this regard, we emphasize that the axisymmetric sausage and hollowing instabilities may also be deleterious to intense ion beam propagation through background electrons.

6 REFERENCES

[1] R. C. Davidson, and H. S. Uhm, J. Appl. Phys. **51**, 885 (1980).

[2] R. C. Davidson and H. S. Uhm, Phys. Fluids **21**, 60 (1978).

[3] D. Neuffer and C. Ohmori, Nuclear Instruments and Methods in Physics Research **A343**, 390 (1994).

[4] M. Izawa, Y. Sato, and T. Toyomasu, Phys. Rev. Lett. **74**, 5044 (1995).

[5] J. Byrd, A. Chao, S. Heifets, M. Minty, T. O. Roubenheimer, J. Seeman, G. Stupakov, J. Thomson and F. Zimmerman, Phys. Rev. Lett. **79**, (1997).

[6] R. C. Davidson, P. H. Stoltz, and T. S. Wang, Phys. Rev. Special Topics, Accel. And Beams **2**, 054401 (1999).

[7] H. S. Uhm, and R. C. Davidson, Phys. Fluids **23**, 1586 (1980).