Chapter 21

Drift waves and instabilities*

We have now considered two types of instabilities that can arise in the fluid plasma model: the first, the ideal MHD flute instability (the pressure-driven version of the Rayleigh–Taylor instability), which draws upon the *thermal* energy of the plasma as it expands unstably across a curved (concave toward the plasma) magnetic field, and the second, the resistive tearing instability, which draws upon the energy of the magnetic field in the plasma as it rearranges itself toward a configuration of lower *magnetic* energy. There is yet a third important class of instability of a fluid plasma, the so-called 'drift-wave instability', which requires neither a curved magnetic field nor a magnetic configuration for which lower magnetic-energy states exist. Indeed, drift-wave instabilities occur in the simplest and most 'universal' of configurations, namely a plasma of non-uniform density maintained in equilibrium by a strong and essentially straight magnetic field. Because of the pervasiveness of this situation, instabilities of this type have sometimes been called 'universal instabilities'. Like flute instabilities, drift-wave instabilities draw upon the thermal energy of the plasma as it expands across a magnetic field. Unlike flute instabilities, however, they have finite wavelengths along the field, and the plasma motion is decoupled, to a significant extent, from that of the magnetic field, so as to avoid energetically unfavorable bending of the field lines. Because of the difficulty of drawing upon the thermal energy of expansion in this way, drift-wave instabilities tend to have rather small growth rates—certainly smaller than those characteristic of flute instabilities.

Unlike Rayleigh–Taylor, flute and resistive-tearing instabilities, drift-wave instabilities are not purely growing, but have complex frequencies ω, with the imaginary part, denoted by γ (the growth rate), usually much smaller than the real part. Of course, any such mode of perturbation can be made purely growing by transforming to a moving frame in which the wave is at rest, but in such a frame the plasma itself will acquire a non-zero velocity. Normally, we choose to work in the 'laboratory frame' in which the plasma is assumed to be at rest (more precisely, the mass velocity $\mathbf{u}$ is taken to be zero) in the unperturbed equilibrium state. In such a frame, the drift-wave instabilities have complex frequencies ω, i.e. they are partly travelling waves and partly growing waves.

Drift waves require non-zero plasma resistivity, or (as we will see in Chapter 26) other forms of dissipation, to be unstable. However, the waves themselves (i.e. without instability) can exist and propagate in any non-uniform plasma. Moreover, as we will see, except at relatively high values of the plasma β (but still $\beta \ll 1$), drift waves do not produce a significant perturbation of the magnetic field. Rather they involve a self-consistent wave-like pattern of density perturbations and flow velocities that propagates partly along and partly across a fixed, approximately uniform, straight magnetic field.

21.1 THE PLANE PLASMA SLAB

We will analyze drift waves in the simplest possible configuration involving a non-uniform plasma, the so-called 'plane plasma slab'. In this configuration, there is a plasma with non-uniform density $n(x)$ and pressure $p(x)$, maintained in equilibrium by a strong magnetic field, B_z. There is no variation of the equilibrium in the y or z directions. The plasma is at rest in the equilibrium configuration, i.e. $\mathbf{u} = 0$, but there is, of course, a non-zero current density $j_y(x)$ needed to provide equilibrium, i.e. to provide a $\mathbf{j} \times \mathbf{B}$ force that balances the pressure gradient ∇p. The magnetic field B_z will be modified (and will acquire a variation with x) as a result of the plasma currents, so that the pressure-balance condition, $p + B_z^2/2\mu_0$ = constant, is satisfied. However, for low values of β, the non-uniformity of B_z is very small and will be neglected in our analysis. The suffix '0' will be used to denote equilibrium quantities, e.g. $n_0(x)$, $p_0(x)$ and B_{z0}.

The new element in our description of a plasma that is needed to produce drift waves is the full so-called 'generalized' Ohm's law, introduced in equation (8.13), namely

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta\mathbf{j} + \frac{\mathbf{j} \times \mathbf{B} - \nabla p_e}{ne}. \tag{21.1}$$

Before embarking upon our stability analysis, we must address the question of whether the use of this generalized Ohm's law, rather than the simple version which omits the last two terms on the right-hand side of equation (21.1), has any effect on our description of the *equilibrium* configuration. Clearly, such an effect *does* arise, since satisfying the independent force-balance condition, $\mathbf{j} \times \mathbf{B} = \nabla p$, where $p = p_e + p_i$, will leave an uncanceled term in ∇p_i on the right-hand side of equation (21.1). Thus, it will not be possible to have *both* $\mathbf{u} = 0$ *and* $\mathbf{E} = 0$ in the equilibrium configuration. Physically, we are encountering here the contribution to the fluid velocity from the *ion diamagnetic drift* which we discussed previously in Chapter 7. Specifically, substituting $\mathbf{j} \times \mathbf{B} = \nabla(p_e + p_i)$ on the right-hand side of equation (21.1) and neglecting, for now, the resistivity term, we can solve equation (21.1) for $\mathbf{u}_\perp$, obtaining

$$\mathbf{u}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\mathbf{B} \times \nabla p_i}{neB^2}. \tag{21.2}$$

Equation (21.2) tells us that the fluid (mass) velocity across the magnetic field is the sum of the $\mathbf{E} \times \mathbf{B}$ drift and the ion diamagnetic velocity, as we would have expected, since the ions make the dominant contribution to the plasma mass. Clearly, in a non-uniform plasma, $\mathbf{u}$ and $\mathbf{E}$ cannot *both* be zero in equilibrium. If we have an equilibrium in which the plasma is at rest, i.e. $\mathbf{u} = 0$, there will necessarily be a non-zero electric field $\mathbf{E}$, and, conversely, if the equilibrium has $\mathbf{E} = 0$, we will need to take into account a non-zero mass-velocity $\mathbf{u}$.

For present purposes, however, we can simplify the analysis by restricting ourselves to the case where the *ion pressure vanishes*, while the electron pressure does not vanish. Physically, this corresponds to a situation where $T_i \ll T_e$, which is a legitimate (and not uncommon) case to consider. Since the equilibrium ion diamagnetic drift is essentially zero, this allows us to assume that $\mathbf{E}_0 = \mathbf{u}_0 = 0$. There would be no *fundamental* difficulty in pursuing the more general case with non-zero ion pressure, for example by keeping a non-zero equilibrium $\mathbf{E}$ field in the stability analysis of a static (i.e. $\mathbf{u} = 0$) equilibrium, but the algebraic complexity would be greater, without adding much more insight into the underlying drift-wave physics.

The plasma is uniform and of infinite extent in the y and z directions. Thus we can assume that perturbations take the form of plane waves in these two directions, so that any perturbation quantity $\psi_1(\mathbf{x}, t)$ can be written

$$\psi_1(\mathbf{x}, t) = \hat{\psi}_1(x)\exp(-\mathrm{i}\omega t + \mathrm{i}k_y y + \mathrm{i}k_z z) \tag{21.3}$$

where $\hat{\psi}_1(x)$ is the amplitude of the wave-like perturbation. Once again, since the equilibrium varies in the x direction, we cannot Fourier decompose into sinusoidal modes in the x direction, but rather must search for eigenfunctions $\hat{\psi}_1(x)$. Our method of analysis will be generally similar to that employed in the derivation of the Rayleigh–Taylor and resistive-tearing instabilities in Chapters 19 and 20, respectively, except that here we have $k_z \neq 0$, implying that the perturbations have a variation *along* the main equilibrium magnetic field. However, we will look for waves satisfying

$$k_z \ll k_y \tag{21.4}$$

and the outcome of our analysis will show that this inequality is valid for a typical drift-wave instability.

For our initial derivation of the drift waves, we will keep the *magnetic* perturbations as well as the electric-field perturbations, but we will then show that, for low-β plasmas, the magnetic perturbations are unimportant relative to the perturbed electric fields $\mathbf{E}$ and the associated $\mathbf{E} \times \mathbf{B}$ flow velocities. If the magnetic perturbations are neglected from the outset, so that the perturbed electric field can be assumed to be derivable from a scalar electric potential, i.e. $\mathbf{E} = -\nabla\phi$, the analysis of drift waves is simplified considerably. We will indeed discuss this 'electrostatic' limit after we have developed the analysis for the more general case. The value of first analyzing the more general 'finite-β' case in some detail is that it demonstrates the connection to the slow shear Alfvén waves discussed in the previous two Chapters (and in Chapter 18), and it shows explicitly how the new drift-wave branch of the spectrum arises at frequencies much lower than all Alfvén wave frequencies, i.e. $\omega \ll k_z v_A \ll k_y v_A$.

21.2 THE PERTURBED EQUATION OF MOTION IN THE INCOMPRESSIBLE CASE

We begin with the perturbed equation of motion

$$\begin{aligned}\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} &= -\nabla p_1 + (\mathbf{j} \times \mathbf{B})_1 \\ &= -\nabla \left(p_1 + \frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{\mu_0} \right) + \frac{1}{\mu_0} [(\mathbf{B} \cdot \nabla)\mathbf{B}]_1 \end{aligned} \tag{21.5}$$

where, as usual, we use the suffix '1' to denote perturbed quantities. Noting that the equilibrium magnetic field is entirely in the z direction, the two components of equation (21.5) perpendicular to this equilibrium field are

$$-\mathrm{i}\omega\rho_0 u_x = -\frac{\partial}{\partial x}\left(p_1 + \frac{B_{z0}B_{z1}}{\mu_0} \right) + \frac{\mathrm{i}k_z}{\mu_0} B_{z0} B_x \tag{21.6}$$

$$-\mathrm{i}\omega\rho_0 u_y = -\mathrm{i}k_y \left(p_1 + \frac{B_{z0}B_{z1}}{\mu_0} \right) + \frac{\mathrm{i}k_z}{\mu_0} B_{z0} B_y. \tag{21.7}$$

Here, and henceforth in this Chapter, we omit the suffix '1' from perturbed quantities whose equilibrium values are zero, e.g. u_x, u_y, B_x and B_y. In deriving equations (21.6) and (21.7), we have noted that $\mathbf{B}_0$ has only a component in the z direction, so $(\mathbf{B}_1 \cdot \nabla)\mathbf{B}_0$ does not contribute anything to the x and y components of equation (21.5).

We now argue that the term in B_{z1} in equations (21.6) and (21.7) contributes significantly to the right-hand side of these equations, i.e. to the force arising from the gradient of the magnetic pressure, even for B_{z1} values that are so small that they do not make a significant contribution to the divergence of the magnetic field. Using equation (21.7) for our estimates, we see that the contribution from B_{z1} to the perturbed magnetic-pressure gradient is comparable to the contribution from B_y if

$$B_{z1} \sim (k_z/k_y)B_y. \tag{21.8}$$

The condition that the perturbed magnetic field be divergence-free is

$$\frac{\partial B_x}{\partial x} + \mathrm{i}k_y B_y + \mathrm{i}k_z B_{z1} = 0 \tag{21.9}$$

and we see immediately that the contribution from B_{z1} is negligible compared with that from B_y if $k_z \ll k_y$, as we have assumed. Thus, the divergence-free magnetic field condition becomes essentially

$$\frac{\partial B_x}{\partial x} + \mathrm{i}k_y B_y = 0 \tag{21.10}$$

the same as for the Rayleigh–Taylor (flute) and resistive-tearing instabilities, both of which had $k_z = 0$.

We will also see below that B_{z1} values which contribute significantly to the magnetic-pressure gradients in equations (21.6) and (21.7) are much smaller than those which would arise if we allowed the main magnetic field to be compressed significantly. Thus, as in the case of the Rayleigh–Taylor (flute) and resistive-tearing instabilities, we want to look for solutions with the property that the flow $\mathbf{u}$ is such that the B_z field is not compressed. The consequence of these approximations is that the perturbed magnetic-field component B_{z1} will play no role in determining the plasma flows and density perturbation, and it will not, finally, appear anywhere else in our analysis, except in equations (21.6) and (21.7).

Accordingly, it is convenient to use our now-familiar technique for eliminating B_{z1} from equations (21.6) and (21.7), namely taking the x derivative of equation (21.7) and subtracting $\mathrm{i}k_y$ times equation (21.6). We obtain

$$\begin{aligned} -\mathrm{i}\omega\left(\frac{\partial}{\partial x}(\rho_0 u_y) - \mathrm{i}k_y\rho_0 u_x\right) &= \frac{\mathrm{i}k_z B_{z0}}{\mu_0}\left(\frac{\partial B_y}{\partial x} - \mathrm{i}k_y B_x\right) \\ &= -\frac{k_z B_{z0}}{\mu_0 k_y}\left(\frac{\partial^2 B_x}{\partial x^2} - k_y^2 B_x\right) \end{aligned} \tag{21.11}$$

where we have used $\partial/\partial x$ of equation (21.10) to obtain the second form of the right-hand side.

Any flow $\mathbf{u}$ that arises will be associated with an electric field $\mathbf{E}_\perp \approx \mathbf{u} \times \mathbf{B}$ and will result in compression of the magnetic field B_z, described by

$$\frac{\partial B_z}{\partial t} \approx [\boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B})]_z \tag{21.12}$$

the approximate equality indicating that some smaller terms in Ohm's law are being neglected. Equation (21.12), to first order, gives

$$-\mathrm{i}\omega B_{z1} \approx -B_{z0}\left(\frac{\partial u_x}{\partial x} + \mathrm{i}k_y u_y\right). \tag{21.13}$$

Unless the right-hand side of equation (21.13) effectively vanishes, there would arise from compression of the B_z field a perturbation of magnitude given roughly by $B_{z1} \sim B_{z0}k_y u_y/\omega$. If we were to substitute this into equation (21.7), we would find the ratio of the inertia term on the left to the term in B_{z1} on the right to be $\omega^2/k_y^2 v_A^2$, where v_A is the Alfvén speed, $B/(\rho_0\mu_0)^{1/2}$. Similarly, if we used the term u_x in equation (21.13) to eliminate B_{z1}, then we would substitute this into equation (21.6) and would find the ratio of the inertia term on the left to the term in B_{z1} on the right to be $\omega^2/k_x^2 v_A^2$, where $k_x \sim \partial/\partial x$. Since we will find that drift waves are generally characterized by $k_x \sim k_y$, these two estimates are similar. However, we want to look for frequencies *much smaller* than $k_y v_A$ (at most of order $k_z v_A$, where $k_z \ll k_y \sim k_x$), so we cannot allow the large B_{z1} values that would arise if this degree of compression of the B_z field were to occur. Thus, we can write

$$\frac{\partial u_x}{\partial x} + \mathrm{i}k_y u_y = 0. \tag{21.14}$$

We note that this is *not*, in this case, the condition for *exactly* incompressible fluid flow, which would involve an additional term $\mathrm{i}k_z u_z$ on the left-hand side of equation (21.14). Indeed, it is only the flow perpendicular to the magnetic field that is required to be incompressible; an arbitrary flow *along* the field can be added without contributing anything to the compression of the magnetic field. Nonetheless, for most cases of interest, including drift waves, both k_z and u_z are relatively small, so that a term $\mathrm{i}k_z u_z$, even if added to the left-hand side of equation (21.14), would make little difference.

Our argument for incompressibility, which has been invoked for the Rayleigh–Taylor (or flute) instability, the resistive-tearing instability and now for the drift wave, can be expressed in terms of the various types of Alfvén waves discussed in Chapter 18. Essentially, these three instabilities all arise in the *linearly polarized shear Alfvén wave branch* of the low-frequency 'spectrum', rather than in the *magnetosonic wave branch*. The physical reason for this is that these shear Alfvén waves do not require the large amount of energy that would be needed to compress the magnetic field, with the result that they are most easily driven unstable by relatively weak sources of free energy. Since perpendicular compression is not involved, the shear Alfvén waves can also have much smaller frequencies, in the case $k_z \ll k_y$. For the case of drift waves, for which we will derive a dispersion relation that displays explicitly the coupling to the shear Alfvén waves, we will find frequencies in the range

$\omega < k_z v_A$ (often $\omega \ll k_z v_A$), to be compared with the much larger frequencies, $\omega \sim k_y v_A$, characteristic of the magnetosonic waves.

Using the incompressibility condition, equation (21.14), to substitute for u_y in terms of u_z on the left-hand side, equation (21.11) becomes

$$\frac{\omega\rho_0}{k_y}\left(\frac{\partial^2 u_x}{\partial x^2} - k_y^2 u_x\right) = -\frac{k_z B_{z0}}{\mu_0 k_y}\left(\frac{\partial^2 B_x}{\partial x^2} - k_y^2 B_x\right) \qquad (21.15)$$

where, on the left-hand side, we have made the simplifying assumption that ρ_0 is not strongly varying with x on the scale of distances over which the perturbations vary significantly. Basically, we are assuming here that the effective wavelength of the perturbation in the x direction is much shorter than the scale-length of the equilibrium density variation.

In cases such as this, where the wavelength of a perturbation is much shorter than the scale-length over which an equilibrium varies, we can use the 'WKB approximation', introduced in Chapter 15. The perturbation will adopt an approximately wave-like form, although the local wave-number k_x will adjust itself gradually to local conditions. For any general perturbed quantity $\psi_1(x)$, the WKB approximation is adopted by writing

$$\psi_1(x) = \hat{\psi}_1 \exp\left(\mathrm{i}\int^x k_x \mathrm{d}x\right) \qquad (21.16)$$

where the amplitude $\hat{\psi}_1$ and the effective wave-number k_x are both *slowly varying* functions of x, i.e. they vary on the scale of the equilibrium variation. A full application of the WKB approximation allows actual eigenfunctions to be obtained, i.e. forms for $\hat{\psi}_1(x)$ as well as for $k_x(x)$, but for present purposes it is sufficient simply to introduce a wave-number k_x, as in equation (21.16), implying that the perturbation is wave-like in x. (Effectively, the WKB approximation generates eigenfunctions by approximating to successive orders in an expansion in $(k_x L_n)^{-1}$, where L_n is the typical scale-length of the density non-uniformity; equation (21.16) represents the lowest-order eigenfunction.) When x derivatives are taken, we may simply use the rule $\partial/\partial x \to \mathrm{i}k_x$, just as if the perturbation were *exactly* of plane-wave form.

Applying this technique to equation (21.15), we obtain

$$-\frac{\omega\rho_0}{k_y} k_\perp^2 u_x = \frac{k_z B_{z0}}{\mu_0 k_y} k_\perp^2 B_x \qquad (21.17)$$

where $k_\perp^2 = k_x^2 + k_y^2$. Equation (21.17) may be rewritten

$$\omega u_x = -k_z v_A^2 B_x / B_{z0}. \qquad (21.18)$$

Equation (21.18) is as much information as we can obtain from the perpendicular components of the perturbed equation of motion, because we have now reduced the independent variables to two, namely u_x and B_x, which will be related to each other also through Ohm's law.

21.3 THE PERTURBED GENERALIZED OHM'S LAW

We turn next to the generalized Ohm's law for the first-order perturbed quantities, namely

$$\mathbf{E}_1 + \mathbf{u}_1 \times \mathbf{B}_0 = \eta \mathbf{j}_1 + \frac{1}{ne}(\mathbf{j} \times \mathbf{B} - \nabla p_e)_1 \tag{21.19}$$

which, when coupled with Faraday's law, i.e.

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla \times \mathbf{E}_1 \tag{21.20}$$

must yield another relation between B_x and u_x to combine with equation (21.18). Substituting equation (21.19) into equation (21.20) and employing our usual expansion of $\nabla \times (\mathbf{u}_1 \times \mathbf{B}_0)$ (see, for example, equation (19.6)), we obtain

$$\frac{\partial \mathbf{B}_1}{\partial t} = (\mathbf{B}_0 \cdot \nabla)\mathbf{u}_1 - (\mathbf{u}_1 \cdot \nabla)\mathbf{B}_0 - \mathbf{B}_0(\nabla \cdot \mathbf{u}_1) - \nabla \times \left(\eta \mathbf{j}_1 + \frac{1}{ne}(\mathbf{j} \times \mathbf{B} - \nabla p_e)_1\right). \tag{21.21}$$

Examination of the size of the various terms in the generalized Ohm's law shows that the additional terms on the right-hand side of equation (21.19), i.e. the last two terms, are of much more importance in the component *parallel* to the magnetic field than they are in the components *perpendicular* to the field. To see this, we simply note that the equation of motion tells us that

$$(\mathbf{j} \times \mathbf{B} - \nabla p_e)_1 \approx \rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\mathrm{i}\omega \rho_0 \mathbf{u}_1 \tag{21.22}$$

and so the ratio of the magnitude of the last two terms on the right-hand side in the perpendicular components of equation (21.19) to the magnitude of the second term on the left-hand side is of order $\omega\rho_0|\mathbf{u}_1|/ne|\mathbf{u}_1|B \approx \omega M/eB \approx \omega/\omega_{ci}$, where ω_{ci} is the Larmor frequency of the ions. For waves, with $\omega \ll \omega_{ci}$, these additional terms on the right-hand side in the perpendicular components of equation (21.19) are unimportant and may be neglected. However, the new terms must be retained in the *parallel* component of the generalized Ohm's law, which becomes

$$E_\parallel = \eta j_\parallel - \frac{1}{ne}\nabla_\parallel p_e. \tag{21.23}$$

Noting that the equilibrium magnetic field is entirely in the z direction, equation (21.23) to first order in the perturbations can be written

$$E_z = \eta j_z - \frac{1}{ne}\left(\mathrm{i}k_z p_{e1} + \frac{B_x}{B_{z0}}\frac{\mathrm{d}p_{e0}}{\mathrm{d}x}\right) \tag{21.24}$$

where, in accordance with our usual convention, we have dropped the suffix '1' from the perturbed quantities E_z and j_z, whose equilibrium values are zero. Note the appearance of the third term on the right-hand side of equation (21.24), which arises from observing that the operator $\nabla_\parallel$ means $(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})$, where $\hat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{B}$, so that

$$\begin{aligned}(\nabla_\parallel p_e)_1 &= [(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla}) p_e]_1 = \hat{\mathbf{b}}_0 \cdot \boldsymbol{\nabla} p_{e1} + \hat{\mathbf{b}}_1 \cdot \boldsymbol{\nabla} p_{e0} \\ &= \mathrm{i}k_z p_{e1} + \frac{B_x}{B_{z0}} \frac{\mathrm{d}p_{e0}}{\mathrm{d}x}. \end{aligned} \tag{21.25}$$

(Strictly, we should note that j_y is non-zero in the equilibrium, and hence will require a small but non-zero $-u_{x0}B_{z0} = \eta j_{y0}$. As we saw in Chapter 12, this u_{x0} is the fluid velocity due to collisional diffusion. In the perturbed form of equation (21.23), there will be an additional term $\eta B_y j_{y0}/B_{z0}$ on the right-hand side. This extra term is very small, since the resistivity η is generally very small; comparing it with the last term on the right-hand side of equation (21.24), we find it to be of relative order ν_{ei}/ω_{ce}, where we have written η in term of ν_{ei} and assumed $B_x \sim B_y$.)

Using equation (21.24) for the parallel component of the generalized Ohm's law, but assuming that $\mathbf{E}_\perp = -\mathbf{u} \times \mathbf{B}$ is a satisfactory approximation for the perpendicular components, so that the vector inside the curl operator in the last term in equation (21.21) retains only its component parallel to $\mathbf{B}$, i.e. $[\eta j_\parallel - (\nabla_\parallel p_e)/ne]_1 \hat{\mathbf{b}}$, the x and y components of equation (21.21) can be written

$$\begin{aligned} -\mathrm{i}\omega B_x &= \mathrm{i}k_z B_{z0} u_x - \mathrm{i}k_y \left[\eta j_z - \frac{1}{ne}\left(\mathrm{i}k_z p_{e1} + \frac{B_x}{B_{z0}}\frac{\mathrm{d}p_{e0}}{\mathrm{d}x}\right)\right] \\ -\mathrm{i}\omega B_y &= \mathrm{i}k_z B_{z0} u_y + \frac{\partial}{\partial x}\left[\eta j_z - \frac{1}{ne}\left(\mathrm{i}k_z p_{e1} + \frac{B_x}{B_{z0}}\frac{\mathrm{d}p_{e0}}{\mathrm{d}x}\right)\right] \end{aligned} \tag{21.26}$$

although the second of these equations is redundant once equations (21.10) and (21.14) have been established, and so it is not used further in our analysis. We now use Ampere's law, $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$, together with equation (21.10) to express B_y in terms of B_x, to obtain an expression for j_z in terms of B_x:

$$\begin{aligned} j_z &= \frac{1}{\mu_0}\left(\frac{\partial B_y}{\partial x} - \mathrm{i}k_y B_x\right) \\ &= \frac{\mathrm{i}}{\mu_0 k_y}\left(\frac{\partial^2 B_x}{\partial x^2} - k_y^2 B_x\right) \\ &\approx -\frac{\mathrm{i}k_\perp^2}{\mu_0 k_y} B_x \end{aligned} \tag{21.27}$$

with $k_\perp^2 = k_x^2 + k_y^2$ and where the WKB approximation has been invoked in the final step. From equation (21.26), we now obtain

$$\omega B_x + k_z B_{z0} u_x = -\frac{i\eta}{\mu_0} k_\perp^2 B_x - \frac{k_y}{ne}\left(ik_z p_{e1} + \frac{B_x}{B_{z0}}\frac{dp_{e0}}{dx}\right). \tag{21.28}$$

The electron pressure perturbation p_{e1} still needs to be eliminated in favor of B_x and u_x. It will be determined by an equation of state, which relates p_{e1} to the density perturbation n_{e1}, which in turn will be determined from the perturbed continuity equation. Physically, the most appropriate assumption will be that the electrons are isothermal, which is equivalent to assuming that the electron thermal conductivity is sufficiently large to maintain a uniform temperature T_e *along* the magnetic field, i.e.

$$\mathbf{B} \cdot \nabla T_e = 0. \tag{21.29}$$

Allowing for the possibility of a temperature gradient *across* the field in the equilibrium, i.e. $T_{e0} = T_{e0}(x)$, the perturbed form of equation (21.29) is

$$ik_z T_{e1} + \frac{B_x}{B_{z0}}\frac{dT_{e0}}{dx} = 0. \tag{21.30}$$

Using $p_{e1} = T_{e0} n_{e1} + n_{e0} T_{e1}$, it follows that the term in parenthesis on the right-hand side of equation (21.28) is given by

$$\begin{aligned} ik_z p_{e1} + \frac{B_x}{B_{z0}}\frac{dp_{e0}}{dx} &= ik_z T_{e0} n_{e1} + \frac{B_x}{B_{z0}}\left(\frac{dp_{e0}}{dx} - n_{e0}\frac{dT_{e0}}{dx}\right) \\ &= T_{e0}\left(ik_z n_{e1} + \frac{B_x}{B_{z0}}\frac{dn_{e0}}{dx}\right). \end{aligned} \tag{21.31}$$

Equation (21.31) may be substituted into equation (21.28), which has the effect of eliminating the pressure perturbation p_{e1} in favor of the density perturbation n_{e1}.

The continuity equation to first order in the perturbations, i.e.

$$\frac{\partial n_{e1}}{\partial t} + \mathbf{u}_\perp \cdot \nabla n_{e0} + \nabla_\parallel (n_{e0} u_\parallel) = 0 \tag{21.32}$$

where we have used $\nabla \cdot \mathbf{u}_\perp = 0$, can be written

$$-i\omega n_{e1} + u_x \frac{dn_{e0}}{dx} + ik_z n_{e0} u_z = 0. \tag{21.33}$$

The perturbed velocity parallel to the equilibrium magnetic field, u_z, must be obtained from the parallel component of the equation of motion. Although we have already made use of the perpendicular components of the equation of motion, we have not yet used the parallel component, which is

$$\rho_0 \frac{\partial u_\parallel}{\partial t} = -\nabla_\parallel p_e. \tag{21.34}$$

To first order, the perturbed form of equation (21.34) becomes

$$\begin{aligned}-\mathrm{i}\omega\rho_0 u_z &= -\mathrm{i}k_z p_{e1} - \frac{B_x}{B_{z0}}\frac{\mathrm{d}p_{e0}}{\mathrm{d}x} \\ &= -T_{e0}\left(\mathrm{i}k_z n_{e1} + \frac{B_x}{B_{z0}}\frac{\mathrm{d}n_{e0}}{\mathrm{d}x}\right)\end{aligned} \tag{21.35}$$

where we have again used equation (21.31). We can substitute equation (21.35) into equation (21.33), to obtain

$$-\mathrm{i}\left(\omega - \frac{k_z^2 T_{e0}}{\omega M}\right) n_{e1} + u_x \frac{\mathrm{d}n_{e0}}{\mathrm{d}x} + \frac{k_z T_{e0}}{\omega M}\frac{B_x}{B_{z0}}\frac{\mathrm{d}n_{e0}}{\mathrm{d}x} = 0. \tag{21.36}$$

We have now obtained an expression for the density perturbation, and hence also the electron pressure perturbation, in terms of u_x and B_x.

We now substitute equation (21.31) into equation (21.28) and then substitute for n_{e1} from equation (21.36). This involves a significant amount of straightforward manipulation, which proceeds most easily by first noting that equation (21.36) can be rewritten

$$\mathrm{i}k_z n_{e1} + \frac{B_x}{B_{z0}}\frac{\mathrm{d}n_{e0}}{\mathrm{d}x} = \frac{\omega}{B_{z0}}\frac{\mathrm{d}n_{e0}}{\mathrm{d}x}\frac{\omega B_x + k_z B_{z0} u_x}{\omega^2 - k_z^2 C_s^2} \tag{21.37}$$

where $C_s = (T_e/M)^{1/2}$ is the plasma sound speed (i.e. the ion thermal speed evaluated with the electron temperature). Using this in equation (21.31), which is then substituted into equation (21.28), we obtain

$$(\omega B_x + k_z B_{z0} u_x)\left(1 - \frac{k_y v_{de}}{\omega - k_z^2 C_s^2/\omega}\right) = -\frac{\mathrm{i}\eta}{\mu_0} k_\perp^2 B_x. \tag{21.38}$$

Here

$$v_{de} = -\frac{T_{e0}}{n_{e0} e B_{z0}}\frac{\mathrm{d}n_{e0}}{\mathrm{d}x} \tag{21.39}$$

is very similar in form to the electron diamagnetic drift velocity (see Chapter 7), the minus sign coming from the electron's charge, $-e$. (Note that v_{de} is not *exactly* the electron diamagnetic drift velocity, as defined in Chapter 7, in which $\mathrm{d}p_{e0}/\mathrm{d}x$ would appear, rather than $T_e(\mathrm{d}n_{e0}/\mathrm{d}x)$. Thus, v_{de} differs from the diamagnetic drift velocity if there is a temperature gradient across the magnetic field. In magnitude and sign, however, the two velocities are, of course, generally similar.)

21.4 THE DISPERSION RELATION FOR DRIFT WAVES

By combining equation (21.38) with the relation between u_x and B_x obtained from the perpendicular components of the equation of motion, i.e. equation (21.18), we obtain the dispersion relation for the waves under investigation, namely

$$\left(\omega - \frac{k_z^2 v_A^2}{\omega}\right)\left(1 - \frac{k_y v_{de}}{\omega - k_z^2 C_s^2/\omega}\right) = -\frac{i\eta}{\mu_0} k_\perp^2. \tag{21.40}$$

In the limit of zero resistivity, we see that there are two distinct branches of the dispersion relation. One branch has

$$\omega = k_z v_A \tag{21.41}$$

and clearly corresponds to the shear Alfvén wave. The second branch has a dispersion relation

$$\omega - k_y v_{de} - \frac{k_z^2 C_s^2}{\omega} = 0 \tag{21.42}$$

and corresponds to the 'drift waves'. In a uniform plasma, for which $v_{de} = 0$, this is just the ion sound wave encountered in Chapter 16, with $k\lambda_D \ll 1$. Since equation (21.42) is quadratic in ω, for given values of k_y and k_z there are two branches of the drift wave, i.e. two possible values of ω, as shown in Figure 21.1. The branch for which ω has the same sign as $k_y v_{de}$ (upper curve in Figure 21.1) is usually called the 'electron drift wave'; the other branch (lower curve in Figure 21.1) is usually called the 'ion branch' of the drift wave, although, for reasons that will soon be apparent, this branch is of less interest. In the limit in which $k_z C_s \ll k_y v_{de}$, the electron drift wave has the frequency

$$\omega \approx k_y v_{de}. \tag{21.43}$$

(The ion branch of the drift wave as shown in Figure 21.1 violates the convention introduced in Chapter 15 that real frequencies ω are taken to be positive. If we are interested in this branch, we can satisfy the convention by simply reversing the sign of k_y. Physically, the ion branch of the drift wave propagates in the direction opposite to that of the electron diamagnetic drift.)

Problem 21.1: By solving the quadratic equation, equation (21.42), for ω exactly, draw a more accurate version of Figure 21.1, plotting the dimensionless frequency $\omega/k_y v_{de}$ versus the dimensionless quantity $k_z C_s/k_y v_{de}$.

Equation (21.40) indicates that the effects of non-zero resistivity are to couple the shear Alfvén and drift-wave branches of the spectrum together and to add an imaginary part (either a growth rate or a damping decrement, depending on sign) to the frequencies of each of the branches.

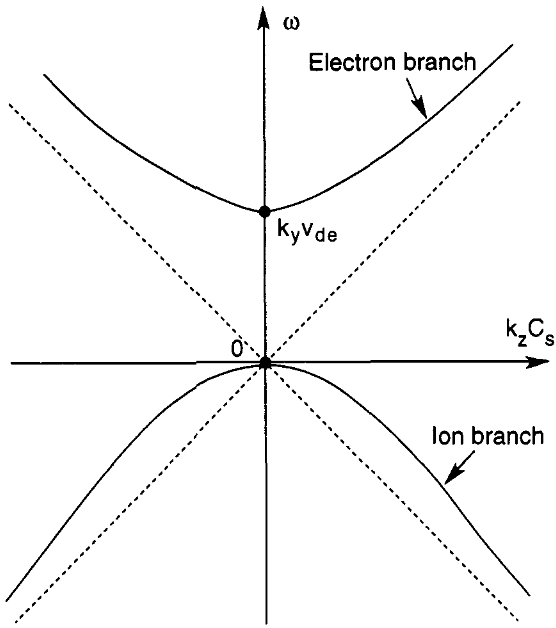

Figure 21.1. Electron and ion branches of the drift-wave dispersion relation. Both branches approach asymptotes $\omega = \pm k_z C_s$.

In order to proceed further, we must consider the typical magnitudes of the various frequencies appearing in equation (21.40). First, we note that $C_s \equiv (T_e/M)^{1/2}$ and $v_A \equiv B/(\mu_0 n M)^{1/2}$, so that

$$C_s/v_A = (\mu_0 n T_e)^{1/2}/B \approx (\beta/2)^{1/2} \tag{21.44}$$

indicating that the sound-wave frequency, $k_z C_s$, is very much smaller than the shear Alfvén wave frequency in all cases where the plasma β value is very small.

Second, we note that $v_{de} = T_e/eBL_n$, where $L_n = n/(\mathrm{d}n/\mathrm{d}x)$, the scale length of the density non-uniformity, so that

$$v_{de}/C_s = (MT_e)^{1/2}/eBL_n \approx r_{Ls}/L_n \tag{21.45}$$

where $r_{Ls} = (MT_e)^{1/2}/eB = C_s/\omega_{ci}$, the average Larmor radius of the ions evaluated as if the ions had the *electron* temperature. The ion Larmor radius

r_{Li} in a magnetized plasma is usually very much smaller than any macroscopic scale-length. Furthermore, although our treatment of drift waves has assumed, for simplicity of analysis, that $T_i \ll T_e$, the disparity in the two temperatures is not usually sufficient to make r_{Ls} more than a few times, at most, larger than r_{Li}. Thus, in many cases of interest, we can assume that $v_{de} \ll C_s$. It follows that the ratio of the two frequencies appearing in the drift-wave dispersion relation, equation (21.42), namely $k_y v_{de}/k_z C_s$, is very small, unless

$$k_z \ll k_y \tag{21.46}$$

or, more specifically, $k_z/k_y \sim r_{Ls}/L_n$ for the two frequencies $k_y v_{de}$ and $k_z C_s$ to be comparable.

Since a finite number of wavelengths $\lambda_y = 2\pi/k_y$ and $\lambda_z = 2\pi/k_z$ must 'fit' into the plasma in the y and z directions, respectively, it follows that our 'plane plasma slab' must be much more extended in the z direction than in the y direction, by roughly the ratio L_n/r_{Ls}, for the drift wave to be clearly distinguishable from the ion sound wave. If the plasma slab is infinite in both y and z directions, as it strictly is within our model, then all k_y and k_z values are allowed but, as we will see, the most unstable perturbations will be much more extended in the z direction. The infinite plasma slab will be a good representation of a finite-size plasma, provided the wavelengths in both y and z directions are much shorter than the y and z dimensions of the finite plasma, respectively.

To retain both branches of the drift waves shown in Figure 21.1, we take $k_y v_{de} \sim k_z C_s$, in which case the typical ordering of the frequencies in equation (21.40) is

$$k_y v_{de} \sim k_z C_s \ll k_z v_A \tag{21.47}$$

the inequality following from $\beta \ll 1$.

In this case, even with resistivity included, equation (21.40) divides into a higher-frequency branch, the shear Alfvén wave, with

$$\omega - \frac{k_z^2 v_A^2}{\omega} = -\frac{\mathrm{i}\eta}{\mu_0} k_\perp^2 \tag{21.48}$$

and a lower-frequency branch, the drift wave, with

$$\omega - k_y v_{de} - \frac{k_z^2 C_s^2}{\omega} = \frac{\mathrm{i}\eta k_\perp^2}{\mu_0} \frac{\omega^2 - k_z^2 C_s^2}{k_z^2 v_A^2}. \tag{21.49}$$

This separation into two branches of the dispersion relation (21.40) can be derived by first looking for high-frequency solutions, $\omega \sim k_z v_A$, for which the inequality given in equation (21.47) implies that the second of the two factors in parentheses on the left in equation (21.40) is approximately unity, thereby yielding equation (21.48). Next, looking for low-frequency solutions,

$\omega \sim k_y v_{\rm de} \sim k_z C_{\rm s}$, the same inequality, i.e. equation (21.47), implies that the first of the two factors in parentheses on the left-hand side of equation (21.40) is simply $-k_z^2 v_{\rm A}^2/\omega$, thereby yielding equation (21.49). The fundamental assumption that permits this division into two distinct branches of the dispersion relation is that $\beta \ll 1$, which produces a wide separation between the lower frequency drift waves with $\omega \sim k_y v_{\rm de} \sim k_z C_{\rm s}$, and the higher frequency shear Alfvén waves with $\omega \sim k_z v_{\rm A}$.

We examine the effect of resistivity first on the shear Alfvén waves. Neglecting the imaginary term from resistivity in equation (21.48), we have at lowest-order the familiar solution $\omega \sim \pm k_z v_{\rm A}$. Treating the imaginary term on the right-hand side of equation (21.48) as a small correction and allowing ω to acquire a correspondingly small imaginary part, $\omega \to \omega + {\rm i}\gamma$ (where ω and γ are now both assumed real, with $\gamma/\omega \ll 1$), the imaginary part of the left-hand side of equation (21.48) is simply $\gamma + (k_z^2 v_{\rm A}^2/\omega^2)\gamma \approx 2\gamma$, which yields $\gamma \approx -\eta k_\perp^2/2\mu_0$, indicating that the shear Alfvén waves are damped by resistivity (negative γ). The damping decrement is essentially the rate of resistive diffusion of magnetic field over a distance of order a perpendicular wavelength—a physically intuitive result unrelated to the present topic of drift-wave physics.

Carrying out a similar analysis of equation (21.49), we find a lowest-order dispersion relation for ω that is the same as equation (21.42) whose solutions are shown in Figure 21.1. Then, letting $\omega \to \omega + {\rm i}\gamma$ and equating the imaginary part of order γ on the left-hand side of equation (21.49), which is $\gamma + (k_z^2 C_{\rm s}^2/\omega^2)\gamma$, to the imaginary expression on the right-hand side, in which only the real part of the frequency ω need be used, we obtain

$$\gamma = \frac{\eta k_\perp^2}{\mu_0} \frac{\omega^2(\omega^2 - k_z^2 C_{\rm s}^2)}{k_z^2 v_{\rm A}^2(\omega^2 + k_z^2 C_{\rm s}^2)}. \tag{21.50}$$

Equation (21.50) shows that the drift wave is unstable whenever $|\omega| > |k_z C_{\rm s}|$. Referring to Figure 21.1, we see that the electron drift wave (the upper curve in Figure 21.1) is always unstable (positive γ), although the growth rate will diminish rapidly as ω approaches the asymptote $k_z C_{\rm s}$, whereas the ion branch of the drift wave (the lower curve in Figure 21.1) is always damped. The electron drift wave destabilized by resistivity is usually called the 'resistive drift instability'.

In the simple case where $k_y v_{\rm de} \gg k_z C_{\rm s}$, the frequency and growth rate of the resistive drift wave instability are given by

$$\omega = k_y v_{\rm de} \qquad \gamma = \eta k_\perp^2 k_y^2 v_{\rm de}^2/\mu_0 k_z^2 v_{\rm A}^2 \approx \nu_{\rm ei} k_\perp^2 r_{\rm Ls}^2 k_y^2 v_{\rm de}^2/k_z^2 v_{\rm t,e}^2 \tag{21.51}$$

where, in the second form of the expression for the growth rate γ, we have substituted $\eta \approx \nu_{\rm ei} m/ne^2$, where $\nu_{\rm ei}$ is the electron–ion collision frequency, and

$v_{t,e} = (T_e/m)^{1/2}$ is the electron thermal velocity. The growth rates of resistive drift instabilities tend to be quite small. Specifically, since $k_y v_{de} \ll k_z v_A$, the first expression for γ in equation (21.51) shows that the growth rate must be very small compared with the rate of resistive diffusion of magnetic field over a distance of order a perpendicular wavelength, i.e. $\eta k_\perp^2/\mu_0$. For perpendicular wavelengths much longer than the ion Larmor radius (evaluated with the electron temperature), i.e. $k_\perp r_{Ls} \lesssim 1$, and for $k_y v_{de} \lesssim k_z C_s \ll k_y v_{t,e}$, the second expression for γ in equation (21.51) shows that the growth rate must also be very much less than the electron–ion collision frequency ν_{ei}. On the other hand, since $\gamma \propto k_\perp^2 k_y^2/k_z^2$, the growth rate increases rapidly as the perpendicular wavelength decreases or as the parallel wavelength increases. Thus, for very short perpendicular wavelengths (down to some limit of order the ion Larmor radius, below which our analysis would not be valid) and for very long parallel wavelengths, the growth rates of resistive drift instabilities can be appreciable. Since the parallel wavelength is limited only by the length of the plasma slab in the z direction, drift-wave instabilities tend to be most serious for plasmas that are very extended along a straight, unidirectional magnetic field. Not surprisingly, drift waves are quite strongly affected by the introduction of magnetic shear, i.e. an equilibrium component $B_{y0}(x)$, as was discussed in the context of resistive tearing instabilities in Chapter 20.

Problem 21.2: Using the same dimensionless quantities for the two axes, add the shear Alfvén wave, whose dispersion relation is given by equation (21.41), to the figure drawn in Problem 21.1. To do this, you need to choose a specific value of β in order to relate C_s to v_A using equation (21.44): take $\beta = 0.02$. Using equation (21.40), indicate which branches of the dispersion relation in the upper (electron) half of your figure become unstable when a small amount of resistivity η is added. By what factor must our 'plane plasma slab' be more extended in the z direction than in the y direction to allow waves with $\omega \sim k_y v_{de} \sim k_z v_A$: give your answer in terms of the quantities r_{Ls}/L_n and β.

Problem 21.3: Examine analytically the region where the two branches of the dispersion relation in the upper half of the figure which you have produced in Problem 21.2 appear to cross each other, i.e. the region $\omega \approx k_y v_{de} \approx k_z v_A$. For the purpose of this analytic calculation, you may assume $\beta \to 0$, i.e. $C_s/v_A \to 0$. By choosing some particular k_z value in this region, for example that given exactly by $k_z v_A = k_y v_{de}$, show from equation (21.40) that there is an instability with a growth rate that scales like $\eta^{1/2}$, rather than like η, for small values of the resistivity. (Hint: You will find it useful to note that the frequency is given approximately by $\omega \approx k_y v_{\mathrm{de}} = k_z v_{\mathrm{A}}$, so that equation (21.40) may then be used only to calculate the small complex correction to this frequency.) This more-rapidly growing instability arises from a coupling between the drift wave and the shear Alfvén wave.

21.5 'ELECTROSTATIC' DRIFT WAVES

The astute reader may suspect that the limit $\omega \ll k_z v_{\mathrm{A}}$, in which the lower-frequency drift wave separates from the shear Alfvén wave in the dispersion relation equation (21.40), corresponds to the case where the *magnetic* perturbations play essentially no role in the dynamics. In this sense, the drift wave is sometimes called 'electrostatic'.

We can see this by noting that our analysis of the perturbed generalized Ohm's law, with the added assumption that the perturbed electric field is constrained so as to produce negligible magnetic perturbations, is essentially sufficient by itself to produce the drift-wave dispersion relation: comparing equation (21.38) with equation (21.40), we see that the shear Alfvén wave branch of the dispersion relation arises from retaining the term ωB_x in the first factor on the left-hand side of equation (21.38). This, in turn, arises from retaining the $\dot{\mathbf{B}}$ term in the perturbed Ampere's law, i.e. the term on the left-hand side of equation (21.26). Neglecting these terms is equivalent to looking for modes in which the perturbed $\mathbf{E}$ fields adjust themselves so as to avoid producing significant magnetic perturbations. This will necessarily involve a non-zero perturbed $E_\parallel$ as well as $\mathbf{E}_\perp$, but the generalized Ohm's law allows this perturbed $E_\parallel$ to be balanced by the parallel perturbed electron pressure gradient. If we neglect the term ωB_x in the first factor of the left-hand side of equation (21.38), but keep all of the other terms, using equation (21.18) to provide another relation between u_x and B_x, we obtain the drift-wave branch of the dispersion relation, i.e. equation (21.49).

The derivation of the drift-wave dispersion relation is simplified considerably if we make this 'electrostatic' assumption from the outset. Specifically, the 'electrostatic" approximation amounts to assuming that the components of the perturbed electric field, $\mathbf{E}_1$, are related to each other by the requirement that $\nabla \times \mathbf{E}_1 = 0$, which implies that the perturbed electric field can be written as the gradient of a scalar potential ϕ, i.e.

$$\mathbf{E} = -\nabla\phi \tag{21.52}$$

where we have dropped the subscript '1', since both $\mathbf{E}$ and ϕ are zero in the equilibrium.

As we have seen, the generalized Ohm's law for the perturbed quantities, i.e. equation (21.19), divides into components perpendicular to the magnetic field, for which the approximation

$$\mathbf{u}_\perp \approx \mathbf{E}\times\mathbf{B}/B^2 \tag{21.53}$$

will suffice, and a component parallel to the magnetic field, in which all of the terms must be retained, i.e.

$$E_\parallel = \eta j_\parallel - \frac{1}{ne}\nabla_\parallel p_{\mathrm{e}}. \tag{21.54}$$

Noting that the equilibrium magnetic field is in the z direction and that the perturbed magnetic field is to be neglected, equation (21.54) to first order in the perturbations can be written

$$E_z = \eta j_z - \frac{\mathrm{i}k_z p_{\mathrm{e}1}}{ne}. \tag{21.55}$$

In the electrostatic approximation, equation (21.53) tells us that

$$u_x = E_y/B_{z0} = -\mathrm{i}k_y\phi/B_{z0} \tag{21.56}$$

so that

$$E_z = -\mathrm{i}k_z\phi = k_z B_{z0}u_x/k_y \tag{21.57}$$

in which case equation (21.55) becomes

$$k_z B_{z0}u_x = k_y\left(\eta j_z - \frac{\mathrm{i}k_z p_{\mathrm{e}1}}{ne}\right) = k_y\left(\eta j_z - \frac{\mathrm{i}k_z T_{\mathrm{e}0}}{ne}n_{\mathrm{e}1}\right). \tag{21.58}$$

In the second form of equation (21.58), we have again made the assumption that the electron temperature must remain uniform along the (now straight and unperturbed) magnetic field.

To obtain the density perturbation, $n_{\mathrm{e}1}$, in terms of u_z, we proceed in much the same way as before, i.e. we combine the continuity equation

$$-\mathrm{i}\omega n_{\mathrm{e}1} + u_x\frac{\mathrm{d}n_{\mathrm{e}0}}{\mathrm{d}x} + \mathrm{i}k_z n_{\mathrm{e}0}u_z = 0 \tag{21.59}$$

with the parallel component of the equation of motion

$$-\mathrm{i}\omega\rho_0 u_z = -\mathrm{i}k_z T_{\mathrm{e}0}n_{\mathrm{e}1} \tag{21.60}$$

(see equations (21.33) and (21.35)). We substitute for u_z from equation (21.60) into equation (21.59), thereby obtaining $n_{\mathrm{e}1}$ in terms of u_x, which is then substituted into equation (21.58). This gives

$$\left(1 - \frac{k_y v_{\mathrm{de}}}{\omega - k_z^2 C_{\mathrm{s}}^2/\omega}\right)u_x = \frac{k_y\eta}{k_z B_{z0}}j_z. \tag{21.61}$$

It remains only to relate the perturbed current density j_z to the mass velocity u_x by the equation of motion. Our procedure here is somewhat different from before, in that we do not want to express the forces arising from current-density perturbations, such as j_z, in terms of the perturbed magnetic fields, as was done in equations (21.5)–(21.11), because these perturbed magnetic fields are neglected, and so are not being otherwise calculated. Rather, we want to deal with the current-density perturbations directly. The x and y components of the perturbed equation of motion, equation (21.5), can be written

$$\begin{aligned} -\mathrm{i}\omega\rho_0 u_x &= -\frac{\partial p_1}{\partial x} + j_{y1}B_{z0} \\ -\mathrm{i}\omega\rho_0 u_y &= -\mathrm{i}k_y p_1 - j_{x1}B_{z0} \end{aligned} \tag{21.62}$$

noting that terms such as $j_z B_y$ and $j_z B_x$ will be second order in the perturbations and may therefore be omitted. Taking $\partial/\partial x$ of the second of these and subtracting $\mathrm{i}k_y$ times the first, thereby eliminating the pressure perturbation p_1 (a familiar procedure), we obtain

$$\begin{aligned} -\mathrm{i}\omega\left(\frac{\partial}{\partial x}(\rho_0 u_y) - \mathrm{i}k_y\rho_0 u_x\right) &= -B_{z0}\left(\frac{\partial j_{x1}}{\partial x} + \mathrm{i}k_y j_{y1}\right) \\ &= \mathrm{i}k_z B_{z0} j_z \end{aligned} \tag{21.63}$$

where, in the second step, we have made use of the divergence-free property of the perturbed current density. Invoking the incompressibility of $\mathbf{u}_\perp$, i.e. equation (21.14), and using the WKB approximation to express $\partial/\partial x$ as $-\mathrm{i}k_x$, equation (21.63) gives

$$j_z = \frac{\mathrm{i}\omega\rho_0}{k_y k_z B_{z0}} k_\perp^2 u_x \tag{21.64}$$

where $k_\perp^2 = k_x^2 + k_y^2$. Substituting this into equation (21.61) gives a final dispersion relation

$$\omega - k_y v_{\mathrm{de}} - \frac{k_z^2 C_\mathrm{s}^2}{\omega} = \frac{\mathrm{i}\eta k_\perp^2}{\mu_0}\frac{\omega^2 - k_z^2 C_\mathrm{s}^2}{k_z^2 v_\mathrm{A}^2} \tag{21.65}$$

exactly the same as equation (21.49). In the case where $k_y v_{\mathrm{de}} \ll k_z C_\mathrm{s}$, the frequency of the drift wave becomes simply $\omega \approx k_y v_{\mathrm{de}}$ and its growth rate is given in equation (21.51).

We conclude that magnetic-field perturbations play no essential role in the dynamics of the low-β drift wave. Rather, the drift wave is produced by a perturbed electric field, whose perpendicular components give rise to perpendicular plasma flows, and whose parallel component is force-balanced self-consistently by the perturbed electron pressure gradient along the magnetic field. Without resistivity, equation (21.54) tells us that the peaks in the electron pressure (or equivalently, the electron density) along the magnetic field coincide exactly with the peaks in the electric potential ϕ. Indeed, assuming as before that the electron temperature remains uniform along the magnetic field, equation (21.54) (without the resistivity term) has the familiar exact nonlinear solution $n_e \propto \exp(e\phi/T_{e0})$, which reflects the tendency of the electrons to adopt a Boltzmann distribution along the magnetic field. In the drift wave, without resistivity, the electron density perturbation will be exactly *in phase* with electric potential perturbations. Introduction of non-zero resistivity produces a small phase shift between the density and potential perturbations. It is this phase shift that allows the drift-wave flow pattern to extract energy from the thermal energy available in the pressure gradient of the electrons to provide for unstable growth of the wave energy.

The analysis of drift waves presented in this Chapter has made several simplifying assumptions, in particular that the equilibrium magnetic field is straight and essentially uniform and that the ions are essentially 'cold', i.e. $T_i \ll T_e$. The introduction of non-zero ion temperature, i.e. $T_i \sim T_e$, would have the predictable effect of bringing the ion diamagnetic drift into the theory, in addition to the electron diamagnetic drift. However, this would not introduce any qualitative change in the stability properties of the drift wave, at least not for the 'electron branch'. The frequency of the 'ion branch' of the drift wave would be modified and, if additional dissipative effects are included, this branch can sometimes be destabilized, but we defer this topic until we are able to treat drift waves from a 'kinetic' viewpoint (see Chapter 26). Modifications to the equilibrium geometry of greatest impact are those that eliminate very small values of the wave-vector parallel to the magnetic field, namely k_z in our case of a straight, uniform field. Finite-length limitations, or the periodic boundary conditions that would be appropriate for a toroidal plasma, rather than an infinitely long plasma slab, are examples where lower limits are imposed on k_z. If the magnetic field is slightly sheared, i.e. a component $B_y(x)$ is added to the larger B_z component (see Chapter 20), then the effective parallel component of the wave-vector becomes $k_\parallel = \mathbf{k} \cdot \hat{\mathbf{B}} \approx k_z + k_y B_y(x)/B_z$, which assumes a range of values as a function of x depending on the width of the mode in the x direction. All 'finite-length' and 'shear' effects tend to be stabilizing, but a detailed analysis of these effects is outside the scope of this book.

Of perhaps more fundamental concern is the validity of the fluid model itself, with its implied assumption that the electrons remain Maxwellian, with a temperature that remains uniform along the magnetic field. We have seen in Chapter 12 that the electron thermal diffusivity along a magnetic field is a quantity of order $v_{t,e}^2/\nu_{ei}$. For the electron temperature to remain essentially uniform along the magnetic field in the presence of a drift wave with frequency ω and wave-number k_z along the field requires that $\omega \ll k_z^2 v_{t,e}^2/\nu_{ei}$. Thus, *the* *electron collision frequency cannot become arbitrarily large* without violating our assumption of isothermal electrons and requiring a more complete fluid model including parallel temperature gradients. Moreover, inspection of the second form of the growth rate γ given in equation (21.51) shows that for $\omega \sim k_y v_{\mathrm{de}}$ the growth rate is then limited to values satisfying $\gamma/\omega \ll k_\perp^2 r_{\mathrm{Ls}}^2$. Again, we see that drift-wave growth rates are appreciable only for perpendicular wavelengths that do not exceed by much the ion Larmor radius, although it should be noted that, because of our assumption that $T_{\mathrm{i}} \ll T_{\mathrm{e}}$, our analysis has not implied an expansion in $k_\perp r_{\mathrm{Ls}}$. The validity of the fluid model also requires that *the electron collision frequency not be too small.* Specifically, for collisions to maintain a *Maxwellian* distribution along the magnetic field, the mean-free path must be shorter than the parallel wavelength, which requires $k_z v_{\mathrm{t,e}} \ll \nu_{\mathrm{ei}}$. If this latter requirement is not satisfied, a 'kinetic' version of the 'electron branch' of the drift wave must be found, which is discussed in Chapter 26.

There is a vast literature on drift waves in non-uniform plasmas. An account of the early work in the field is to be found in an article by N A Krall (1968, in *Advances in Plasma Physics 1*, edited by A Simon and W B Thompson New York: Interscience), which discusses the 'kinetic' versions of the drift wave, to be introduced in Chapter 26, as well as the fluid versions which have been described in the present Chapter.