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A Computational Method in Plasma Physics

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CHAPTER 2

The Variational Principle

2.1 The Magnetostatic Equations

In magnetohydrodynamics, the equilibrium and stability of a toroidal plasma with density ρ, pressure $p = \rho^\gamma$ and internal energy $e = p/(\gamma - 1)$, confined by a strong magnetic field B, can be analyzed by means of a variational principle [4] for the potential energy

$$E = \iiint \left(\frac{B^2}{2} + \frac{p}{\gamma - 1} \right) dx_1 \, dx_2 \, dx_3,$$

subject to appropriate constraints. Stationary points correspond to equilibrium solutions; and if the energy has a local minimum, the equilibrium is considered to be stable by definition.

We present a new nonlinear formulation [7] of the standard variational principle of magnetohydrodynamics which is related to that of Kruskal and Kulsrud [28]. Our main objective is to recast the variational principle so that it can more easily be implemented as a computer code. This is to be achieved by using a simple domain for the independent variables in three dimensions, a simple way of introducing constraints, and a minimization procedure that leads to a well-posed problem for a system of partial differential equations involving an artificial time parameter.

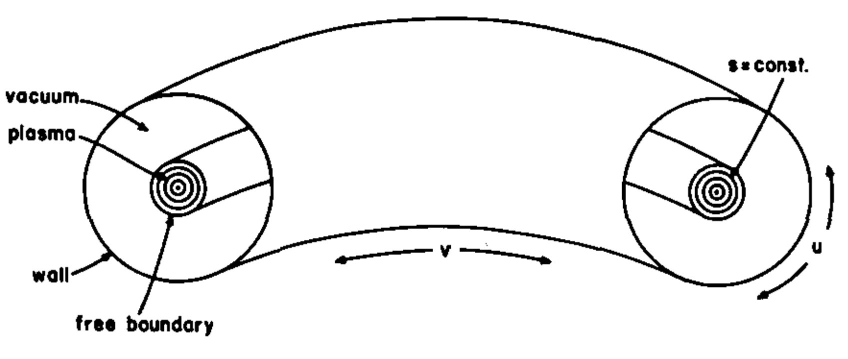

Fig. 2.1 Toroidal geometry.

Let the plasma be contained in a toroidal region Ω_1 of space that is separated by a sharp boundary Γ from an outer vacuum region Ω_2 bounded by a conducting wall C (see Fig. 2.1). We assume that a nested toroidal family of flux surfaces $s =$ const. exists in the plasma region such that $s = 0$ corresponds to the magnetic axis and $s = 1$ corresponds to the free boundary. We denote by u and v variables such as angles with unit periods in the poloidal and toroidal directions, respectively.

Let us minimize the potential energy E subject to the following five constraints:

1. It is required that $\nabla \cdot B = 0$ everywhere.
2. The toroidal and poloidal fluxes within each flux surface in the plasma region are fixed, so that

$$\iint\limits_{s \le s_0} B \cdot dS = F_T(s_0), \qquad \iint\limits_{s \le s_0} B \cdot dS = F_P(s_0),$$

   where the first integral is evaluated over a disk $v =$ const. and the second integral is evaluated over an annular surface $u =$ const.
3. The mass within each flux tube has a fixed value

$$\iiint\limits_{s \le s_0} \rho \, dV = M(s_0).$$

4. The total toroidal and poloidal fluxes in the vacuum are fixed. These two conditions can be expressed in the form

$$\iint\limits_{v = \text{const.}} B \cdot dS = F_T^V, \qquad \iint\limits_{u = \text{const.}} B \cdot dS = F_P^V.$$

5. The free surface Γ and the outer wall C are flux surfaces on which the normal component of B vanishes, i.e.,

$$B \cdot \nu = 0.$$

Note the difference between the flux constraints in the plasma and vacuum regions. In the plasma region the distributions of flux are fixed as functions of s, while in the vacuum region only the two total fluxes are preserved.

The Euler equations for this variational principle are the equations of magnetostatics. In the plasma region we have

$$\nabla p = J \times B, \qquad J = \nabla \times B,$$

where J is the current. On the other hand, in the vacuum region

$$\nabla \times B = 0.$$

The sum $\frac{1}{2}B^2 + p$ of the magnetic and fluid pressures remains continuous across the free boundary Γ. The plasma and vacuum regions must be treated by different methods because the constraints are different in each of them.

The assumption of a nested toroidal family of flux surfaces is justified by the fact that in the time evolution of a magnetohydrodynamic system, the magnetic lines are carried by the fluid and, therefore, the topology is preserved. This leads to a sufficiently simple model, so that the fully three-dimensional problem can be analyzed numerically by solution on a large-scale computer. A more general case allowing for the creation of so-called "islands" has been treated for two-dimensional or axially symmetric geometry by Grad et al. [21].

2.2 Flux Constraints in the Plasma

The first practical difficulty in dealing with the three-dimensional problem is how to prescribe the constraints in a way appropriate for numerical computation. The equation $\nabla \cdot B = 0$ can be integrated by representing the magnetic field B as the cross product [22],

$$B = \nabla s \times \nabla \psi,$$

of the gradients of two scalar flux functions s and ψ. Assuming that the loci $s = \text{const.}$ are a nested family of toroidal flux surfaces, we can use s as a Lagrangian coordinate. That is, we switch the role of dependent and independent variables, and s becomes one of our coordinates. Then we prescribe the flux constraints by prescribing the periods of the multiple-valued flux function ψ.

Since B must be single-valued, the most general expression for ψ is

$$\psi = f_1(s)u + f_2(s)v + \lambda(s, u, v),$$

where λ is periodic in u and v. We have for the toroidal flux

$$F_T(s_0) = \iint_{s \leq s_0} B \cdot dS = - \iint_{s \leq s_0} \nabla \times (\psi \nabla s) \cdot dS = \oint \psi \, ds,$$

where the line integral is taken along a curve which bounds the slit disk $v = \text{const.}$, $s \leq s_0$. A cut must be introduced along the ray $u = 0$ to make ψ single valued, and the sign is chosen so that the flux is positive in the direction of increasing v. Evaluation of the line integral yields

$$F_T(s_0) = - \int_0^{s_0} f_1(s) ds,$$

where the only nontrivial contribution comes from the cut. Finally, differentiation gives $F_T'(s) = - f_1(s)$.

A similar calculation for the poloidal flux yields $F_P'(s) = f_2(s)$ and, therefore,

$$\psi = -F_T'(s)u + F_P'(s)v + \lambda(s, u, v).$$

The ratio dF_P/dF_T, which we denote by $\mu(s)$, is called the rotational transform.

Clearly, only one function of s need be prescribed to define the fluxes. In the simplest case, where we take s to be the toroidal flux itself so that $F_T(s) = s$, ψ assumes the special form

$$\psi = -u + \mu(s)v + \lambda(s, u, v).$$

However, for practical purposes it is convenient to retain the more general form because it allows us more freedom in the choice of a computational mesh and helps in treating cases where there is a reversal of sign for the main toroidal field.

2.3 The Ergodic Constraint

The relation

$$B \cdot \nabla p = 0$$

implies that the magnetic lines are real characteristics of the magnetostatic equations. In order to formulate a well-posed problem, we want to eliminate these real characteristics from our system of equations. If we assume that the magnetic lines on each flux surface are ergodic, then p must be a function of s alone. It is, therefore, natural to introduce this ergodic constraint on every toroidal flux surface. However, if we wish to arrive at a valid stability analysis we must show that the corresponding relation $\rho = \rho(s)$ yields a minimum of the total internal energy for all choices of $\rho(s, u, v)$ satisfying the basic mass constraint 3 of Section 2.1.

Let

$$D = \frac{\partial(x_1, x_2, x_3)}{\partial(s, u, v)}$$

be the Jacobian of the transformation to the coordinate system s, u, and v. An application of Hölder's inequality asserts that for any fixed s and $\gamma > 1$,

$$M'(s) = m(s) = \iint \rho D\, du\, dv \leq \left(\iint \rho^\gamma D\, du\, dv\right)^{1/\gamma} \left(\iint D\, du\, dv\right)^{(\gamma-1)/\gamma}$$

and equality holds if and only if $\rho = \text{const}$. Hence for fixed $m(s)$ and D, the internal energy

$$E_i = \frac{1}{\gamma - 1} \iiint \rho^\gamma D\, ds\, du\, dv$$

becomes a minimum when

$$\rho = \rho(s) = \frac{m(s)}{\iint D\, du\, dv}.$$

A similar proof of the admissibility of the ergodic constraint can be carried out for $0 < \gamma < 1$, too.

In summary, through the substitutions that have been described we have integrated analytically two of the four magnetostatic equations, while preserving our stability criteria. Moreover, we have eliminated the real characteristics of the system, and that will lead to the formulation of a well-posed problem. The flux and mass constraints have been incorporated explicitly in the formulation, so that an unconstrained minimization problem for the plasma region results.

Observe that the relationship between the magnetic field B and the flux functions s and ψ can be written in the invariant form

$$B_j = \frac{\partial(s, \psi, x_j)}{\partial(x_1, x_2, x_3)}.$$

Thus the expression for the energy E_1 in the plasma region reduces to

$$E_1 = \iiint \frac{D_1^2 + D_2^2 + D_3^2}{2D}\, ds\, du\, dv + \frac{1}{\gamma - 1}\int \frac{m(s)^\gamma\, ds}{(\iint D\, du\, dv)^{\gamma-1}},$$

where $D_j = \partial(s, \psi, x_j)/\partial(s, u, v)$.

2.4 Coordinate System in the Plasma

We introduce modified cylindrical coordinates r, θ, and z defined by the formulas

$$x_1 = (l + r)\cos\theta,$$
$$x_2 = (l + r)\sin\theta,$$
$$x_3 = z,$$

where l may be interpreted as the large radius of a torus which becomes a cylinder in the limit as $l \to \infty$. Because we have already integrated the equation $B \cdot \nabla p = 0$, we are free to impose on the transformation from the coordinates s, u, and v to r, θ, and z the important restriction

$$\theta = 2\pi v$$

specifying the toroidal angle. Under this hypothesis, the energy E_1 takes the same form as it does in rectangular coordinates, except that we now have

$$D_1 = -\frac{\partial(\psi, r)}{\partial(u, v)}, \qquad D_2 = -L(1 + \varepsilon r)\psi_u, \qquad D_3 = -\frac{\partial(\psi, z)}{\partial(u, v)},$$

$$D = L(1 + \varepsilon r)\frac{\partial(r, z)}{\partial(s, u)},$$

where $L = 2\pi l$ and $\varepsilon = 1/l$. The ratios D_j/D represent the components of the magnetic field B in the r, θ, and z directions, respectively. We can set $\varepsilon = 0$, keeping L fixed, to obtain a cylinder of length L with periodic boundary conditions. Note also that with the special prescription for v only two-dimensional Jacobians are needed.

We could consider ψ as a Lagrangian coordinate, i.e., as a fixed function of s, u, and v, and minimize E_1 over all periodic mappings $r(s, u, v)$, $z(s, u, v)$ of the cube

$$\Omega: 0 \le s \le 1; \qquad 0 \le u \le 1; \qquad 0 \le v \le 1$$

onto a specified plasma region. Variation of the independent variables r and z and the fact that p is a function of s alone would show that the Euler equations of the new extremal problem reduce to the magnetostatic equations. However, there are several difficulties with such a formulation. The solution for ψ is not unique, since we can add any function of s alone to ψ without changing the values of B. This is reflected in the fact that the solution for the mapping is not unique, and corresponding compatibility conditions due to the toroidal geometry must be satisfied. Moreover, the magnetic axis is a singular curve in our coordinate system, which makes it difficult to write equations for r and z at $s = 0$. Finally, the boundary condition for r and z at $s = 1$ is nonlinear.

It is more effective to replace the physical coordinates r and z as dependent variables by a combination of the flux function $\psi = \psi(s, u, v)$ and a dimensionless radius $R = R(s, u, v)$ related to r and z by the formulas

$$r = r_0(v) + R(s, u, v)[r_1(u, v) - r_0(v)],$$

$$z = z_0(v) + R(s, u, v)[z_1(u, v) - z_0(v)],$$

where $r = r_0(v)$, $z = z_0(v)$ are the equations of the magnetic axis and $r = r_1(u, v)$, $z = z_1(u, v)$ are the equations of the free boundary Γ. The function R serves to define the geometry of the flux surfaces $s = \text{const}$. The boundary conditions on R require that it be periodic in u and v and that $R = 0$ at $s = 0$, and $R = 1$ at $s = 1$. There are no boundary conditions on ψ other than the poloidal and toroidal periodicity requirements already indicated. The functions r_0 and z_0 must be periodic and they must be found as part of the answer to the minimum problem. In terms of R, the Jacobian D reduces to the simple expression

$$D = LH(1 + \varepsilon r)RR_s,$$

where

$$H(u, v) = (z_1 - z_0)\frac{\partial r_1}{\partial u} - (r_1 - r_0)\frac{\partial z_1}{\partial u}.$$

2.5 First Variation of the Potential Energy

Making perturbations $\delta\psi$, δR, δr_0, and δz_0 of the dependent variables ψ, R, r_0, and z_0, we obtain, after integration by parts, an expression of the form

$$\delta E_1 = -\iiint (L_1(\psi)\delta\psi + L_2(R)\delta R)ds\,du\,dv$$
$$- \int (L_3(r_0)\delta r_0 + L_4(z_0)\delta z_0)dv$$

for the first variation of the energy E_1 in the plasma region. A calculation shows that the operators $L_1(\psi)$, $L_2(R)$, $L_3(r_0)$ and $L_4(z_0)$ occurring here are defined by the relations

$$L_1(\psi) = \frac{\partial}{\partial u}\frac{[r_v^2 + L^2K^2 + z_v^2]\psi_u - [r_u r_v + z_u z_v]\psi_v}{D}$$
$$+ \frac{\partial}{\partial v}\frac{[r_u^2 + z_u^2]\psi_v - [r_u r_v + z_u z_v]\psi_u}{D},$$

$$L_2(R) = (r_1 - r_0)\left\{\frac{\partial}{\partial u}\frac{\psi_v[\psi_v r_u - \psi_u r_v]}{D} + \frac{\partial}{\partial v}\frac{\psi_u[\psi_u r_v - \psi_v r_u]}{D}\right.$$
$$\left. + L\varepsilon\left(\frac{DP}{LK} - \frac{LK\psi_u^2}{D}\right)\right\}$$
$$+ (z_1 - z_0)\left\{\frac{\partial}{\partial u}\frac{\psi_v[\psi_v z_u - \psi_u z_v]}{D} + \frac{\partial}{\partial v}\frac{\psi_u[\psi_u z_v - \psi_v z_u]}{D}\right\}$$
$$- LHR\frac{\partial}{\partial s}(PK),$$

$$L_3(r_0) = \iint\left[(1 - R)\left\{\frac{\partial}{\partial u}\frac{\psi_v[\psi_v r_u - \psi_u r_v]}{D} + \frac{\partial}{\partial v}\frac{\psi_u[\psi_u r_v - \psi_v r_u]}{D}\right.\right.$$
$$\left.\left. + L\varepsilon\left(\frac{DP}{LK} - \frac{LK\psi_u^2}{D}\right)\right\} - PLKRR_s\frac{\partial z_1}{\partial u}\right]ds\,du,$$

$$L_4(z_0) = \iint\left[(1 - R)\left\{\frac{\partial}{\partial u}\frac{\psi_v[\psi_v z_u - \psi_u z_v]}{D} + \frac{\partial}{\partial v}\frac{\psi_u[\psi_u z_v - \psi_v z_u]}{D}\right\}\right.$$
$$\left. + PLKRR_s\frac{\partial r_1}{\partial u}\right]ds\,du,$$

where $P = \frac{1}{2}B^2 + p$ and $K = 1 + \varepsilon r$.

The Euler equations $L_1(\psi) = 0$ and $L_2(R) = 0$, asserting that E_1 is a stationary functional of ψ and R, imply that for magnetostatics,

$$\nabla s \cdot J = 0, \qquad \nabla\psi \cdot J = p'(s).$$

The first of these can be viewed as an elliptic equation for ψ in its dependence on the two variables u and v within the flux surfaces $s = \text{const}$. Similarly, the second is an elliptic equation for s embedded in the two-dimensional flux surfaces $\psi = \text{const}$. These two equations, together with the ergodic constraint $p = p(s)$, imply $\nabla p = J \times B$. They are of nonstandard type in three-dimensional space.

The corresponding Euler equations $L_3(r_0) = 0$ and $L_4(z_0) = 0$ for r_0 and z_0 can be written as

$$L_3(r_0) = \iint (1 - R)(\nabla p - J \times B) \cdot \hat{e}_r D \, ds \, du = 0$$

and

$$L_4(z_0) = \iint (1 - R)(\nabla p - J \times B) \cdot \hat{e}_z D \, ds \, du = 0,$$

where $\hat{e}_r$ and $\hat{e}_z$ are unit vectors in the r and z directions. They represent a weighted average of the magnetostatic forces on the cross section $v = \text{const}$.

For the case in which there is no vacuum region, the functions r_1 and z_1 represent the equations of the outer conducting wall C, and the problem reduces to minimizing the expression for the potential energy E_1 in the plasma region alone as a functional of ψ, R, r_0, and z_0. When a vacuum region is present, the location of the free boundary Γ must be found by considering the contribution of the vacuum region to the potential energy. This is the subject of the next section.

2.6 Vacuum Region and Force-Free Fields

Consider the total potential energy, which we write as

$$E = E_1 + E_2 = \iiint_{\Omega_1} \left(\frac{1}{2} B^2 + \frac{p}{\gamma - 1}\right) dV + \iiint_{\Omega_2} \frac{1}{2} B^2 \, dV,$$

where dV is the volume element and Ω_1 and Ω_2 are the plasma and vacuum regions, respectively (see Fig. 2.1). The variation of E with respect to the magnetic field B and pressure p in the plasma region has already been discussed. Now we consider the problem of minimizing E with respect to variations of the vacuum magnetic field B and the free boundary Γ subject to the vacuum flux constraints 4 and 5 of Section 2.1. Again we emphasize that in this case it is not the flux distribution $\mu(s)$ but just the total poloidal and toroidal fluxes in the vacuum region that are fixed.

At first glance, one might think that we could proceed in the same manner as before and express the vacuum magnetic field as $B = \nabla s \times \nabla \psi$ to satisfy $\nabla \cdot B = 0$. However, there are several reasons why that is not the right way to proceed.

First of all, the flux surfaces in the vacuum region will not, in general, be nested tori for genuinely three-dimensional geometry [32]. While in the plasma region the topology of the magnetic lines has an invariant structure, no such requirement holds for the vacuum magnetic field, and topological assumptions are not natural.

Second, even if we were to accept the constraint of nested toroidal surfaces, we would have to minimize the energy E_2 with respect to the function

$$F(s) = \int_{s0}^{s} \mu(\sigma)d\sigma$$

over the vacuum region. Making a variation $\delta\mu$ shows that the corresponding Euler equation has the form

$$I(s) = \oint B \cdot dx = \text{const.}$$

where I is the net current through the toroidal flux tube $s = \text{const}$. It is only when sufficient regularity is assumed that this equation, together with the force-free field condition $J \times B = 0$, implies $J = 0$. In more general cases, F might be continuous but not differentiable and the answer to the minimum problem might be a weak solution with surface current sheets through the points where μ is discontinuous. On such surfaces, the magnetic lines need not be ergodic and there could be surface currents with alternating signs.

Because of these considerations, we introduce a reciprocal variational problem [5,6]. It is well known that if we minimize E_2 with respect to B subject to $\nabla \cdot B = 0$, the corresponding Euler equation is $\nabla \times B = 0$. We now consider the reciprocal problem, which is to find the stationary point for E_2 subject to $\nabla \times B = 0$ and subject to a suitable formulation of the flux constraints 4. The corresponding Euler equation is $\nabla \cdot B = 0$, and both problems have the same solution. However, in the reciprocal case the vacuum energy is a maximum with respect to variations of B, as we shall prove in the next section. The advantage of the constraint $\nabla \times B = 0$ is that it is easily imposed by means of a scalar potential, but its disadvantage is that we have to deal with a minimax problem rather than a straight minimum problem for E. In principle, one should determine the vacuum field for each position of the free boundary Γ and then minimize with respect to the other dependent variables. In practice, though, it suffices to iterate the vacuum field equation more often than the rest of the dependent variables.

2.7 Variation of the Vacuum Field

We integrate the constraint $\nabla \times B = 0$ by setting

$$B = \nabla\phi = c_1\nabla\phi_1 + c_2\nabla\phi_2,$$

where the potentials ϕ_1 and ϕ_2 are associated with unit currents $\int d\phi_j$ in the poloidal and toroidal directions, respectively, but have zero periods in the conjugate directions. The Euler equation stating that ϕ is a stationary point of E_2 implies that $\Delta\phi_i = 0$ in Ω_2, and that $\partial\phi_i/\partial\nu = 0$ on Γ and C. Note that we have here a *natural boundary condition in the sense of the calculus of variations*, which means that the boundary condition is a consequence of the extremal property.

Since it is the fluxes F_P^V and F_T^V that must be fixed, we require

$$a_{11}c_1 + a_{12}c_2 = F_P^V,$$

$$a_{21}c_1 + a_{22}c_2 = F_T^V,$$

where the matrix A with elements

$$a_{ij} = \iiint\limits_{\Omega_2} \nabla\phi_i \cdot \nabla\phi_j \, dV, \qquad i, j = 1, 2,$$

is called the inductance matrix. Let c denote the vector with components c_1 and c_2, and f the vector with components F_P^V and F_T^V. We have

$$E_2 = \tfrac{1}{2}\iiint\limits_{\Omega_2} (\nabla\phi)^2 \, dV = \tfrac{1}{2}c'Ac,$$

where c' is the transpose of c. The matrix A is symmetric and positive definite, and using the flux constraints we obtain

$$E_2 = \tfrac{1}{2}f'A^{-1}f.$$

We intend to prove that E_2 is a maximum with respect to variations of ϕ for $Ac = f$ fixed. Let A_0 be the matrix corresponding to the stationary solutions with $\Delta\phi_i = 0$, $\partial\phi_i/\partial\nu = 0$. Since A and A_0 are symmetric and positive definite, we can find an orthogonal basis such that

$$c'A_0c = \lambda_1\xi_1^2 + \lambda_2\xi_2^2, \qquad c'Ac = \xi_1^2 + \xi_2^2,$$

where ξ_1 and ξ_2 are the components of c in the new basis, and λ_1 and λ_2 are the roots of the equation $|A_0 - \lambda A| = 0$. Correspondingly, we can write

$$f'A_0^{-1}f = \mu_1\eta_1^2 + \mu_2\eta_2^2, \qquad f'A^{-1}f = \eta_1^2 + \eta_2^2,$$

where μ_1 and μ_2 are the roots of $|A_0^{-1} - \mu A^{-1}| = 0$. This implies that $\mu_i = 1/\lambda_i$. Now by Dirichlet's principle we have

$$c'A_0c \le c'Ac$$

for all c and, therefore, $\lambda_i \le 1$. As a consequence, $\mu_i \ge 1$, so that

$$f'A_0^{-1}f \ge f'A^{-1}f$$

for any ϕ_i, which completes the proof.

2.8 Variation of the Free Boundary

Next we consider the variation of the energy E as a functional of the position of the free boundary Γ. Let δv be an arbitrary perturbation of Γ along its outer normal. A direct calculation shows that [6]

$$\delta E_2 = -\tfrac{1}{2}c'\delta Ac,$$

and Hadamard's variational formula for harmonic functions [17] implies that

$$\delta a_{ij} = -\iint_\Gamma \nabla\phi_i \cdot \nabla\phi_j \delta v\, dS,$$

where dS is the surface area element on Γ. A similar calculation for the plasma region shows that

$$\delta E_1 = -\iint_\Gamma (\tfrac{1}{2}B_1^2 + p)\delta v\, dS,$$

which, together with the above equations, leads to

$$\delta E = -\iint_\Gamma (\tfrac{1}{2}B_1^2 + p - \tfrac{1}{2}B_2^2)\delta v\, dS,$$

where B_1 and B_2 stand for the limiting values of the magnetic field coming from the plasma and the vacuum, respectively. This means that $\frac{1}{2}B^2 + p$ must be continuous across Γ for a solution of the variational problem.

2.9 Coordinate System in the Vacuum

The basic difficulty in developing a numerical scheme for a free boundary problem is to handle the changing shape and location of the free boundary. If the problem involves only two independent variables, conformal mapping techniques can be used to solve it in a fixed auxiliary domain [10]. Then the region of the solution is determined as the conformal image of that domain.

In the general case of three independent variables no such conformal mapping exists, but the basic idea of mapping the physical region onto a fixed auxiliary domain Ω can still be used. Since the mapping is not conformal, the resulting equations in Ω will be more complicated, which in turn means that finding the solution will require a greater amount of computation. However, the advantage of having a fixed domain in which to solve difference equations far outweighs the disadvantage of the extra computation. This also provides an ideal framework in which to use fast, direct methods for solving the resulting difference equations. We shall have more to say about that later.

We choose to formulate the vacuum energy problem as a minimum problem for Dirichlet's integral $\iiint (\nabla\phi)^2\,dV$ rather than to start from Laplace's equation. This approach has several advantages, one of them being that the boundary condition $\partial\phi/\partial\nu = 0$ is a consequence of the minimization and need not be imposed as a special requirement. Another advantage is that the resulting Euler equation will be in conservation form and, therefore, the compatibility condition for a solution of the Neumann problem will be satisfied. These properties can easily be extended to difference equations by using a discrete variational principle in a fashion suggested by the finite element method.

We start with the cylindrical coordinate system of Section 2.4 and put $\theta = 2\pi v$ again. Consider the mapping of the cube

$$\Omega: 0 \le s \le 1; \qquad 0 \le u \le 1; \qquad 0 \le v \le 1$$

onto the vacuum region Ω_2 given by

$$r = r_1(u, v) + s[r_2(u, v) - r_1(u, v)],$$

$$z = z_1(u, v) + s[z_2(u, v) - z_1(u, v)],$$

where $r = r_2(u, v)$ and $z = z_2(u, v)$ are the equations of the outer conducting wall C and $r = r_1(u, v)$, $z = z_1(u, v)$ are, as before, the equations of the free boundary Γ. The Dirichlet integral can be written as

$$\iiint_{\Omega_2} (\nabla\phi)^2\,dV = \iiint (a\phi_s^2 + b\phi_u^2 + c\phi_v^2 + 2d\phi_s\phi_u + 2e\phi_s\phi_v + 2f\phi_u\phi_v)\,ds\,du\,dv,$$

where

$$L = \frac{2\pi}{\varepsilon}, \qquad K = 1 + \varepsilon r, \qquad \Delta = \frac{\partial(r, z)}{\partial(s, u)},$$

and

$$a = \frac{LK(r_u^2 + z_u^2 + e^2)}{\Delta}, \qquad b = \frac{LK(r_s^2 + z_s^2 + f^2)}{\Delta},$$

$$c = \frac{\Delta}{LK}, \qquad d = \frac{LK(ef - r_u r_s - z_u z_s)}{\Delta}$$

$$e = \frac{r_u z_v - r_v z_u}{LK}, \qquad f = \frac{r_v z_s - r_s z_v}{LK}.$$

The periodicity conditions on ϕ_1 and ϕ_2 now become

$$\phi_i(s, u + 1, v) = \phi_i(s, u, v) + \delta_{i1},$$

$$\phi_i(s, u, v + 1) = \phi_i(s, u, v) + \delta_{i2}$$

for $i = 1, 2$, where δ_{ij} is the Kronecker delta. In the new coordinates s, u, v over Ω, the Euler equation, equivalent to Laplace's equation, appears in the conservation form

$$\frac{\partial}{\partial s}(a\phi_s + d\phi_u + e\phi_v) + \frac{\partial}{\partial u}(b\phi_u + d\phi_s + f\phi_v) + \frac{\partial}{\partial v}(c\phi_v + e\phi_s + f\phi_u) = 0,$$

with boundary conditions at $s = 0, 1$ given by

$$a\phi_s + d\phi_u + e\phi_v = 0.$$

To specify the free boundary variation, we write the equations of Γ in terms of a dimensionless radius $g = g(u, v)$ as

$$r_1(u, v) = r_3(v) + g(u, v)[r_2(u, v) - r_3(v)],$$

$$z_1(u, v) = z_3(v) + g(u, v)[z_2(u, v) - z_3(v)],$$

where $r_3(v)$, $z_3(v)$ are the equations of a curve defining a new origin of coordinates in each meridian plane $v = \text{const}$. This closed curve can be chosen to follow the shape of the outer wall, so that g becomes a slowly varying function of u and v.

Making a perturbation δg, we obtain after integration by parts the variation of the energy due to a shift of the free boundary in the form

$$\delta E = -\iint M(g)\delta g\, du\, dv.$$

Here $M(g)$ is defined, following Section 2.8, by the formula

$$M(g) = LK[(r_2 - r_3)z_u - (z_2 - z_3)r_u][\tfrac{1}{2}B_1^2 + p - \tfrac{1}{2}B_2^2].$$

From Section 2.4 we obtain the expression for the plasma magnetic pressure

$$B_1^2 = \left(\frac{[r_u^2 + z_u^2]\psi_v^2 + [(LK)^2 + r_v^2 + z_v^2]\psi_u^2 - 2[r_u r_v + z_u z_v]\psi_u\psi_v}{(LK)^2(HRR_s)^2}\right).$$

On the other hand, using the boundary conditions for ϕ_i on Γ, we have for the vacuum magnetic pressure there

$$B_2^2 = \left(\frac{[r_u^2 + z_u^2]\phi_v^2 + [(LK)^2 + r_v^2 + z_v^2]\phi_u^2 - 2[r_u r_v + z_u z_v]\phi_u\phi_v}{([LK]^2[r_u^2 + z_u^2] + [r_v z_u - r_u z_v]^2)}\right).$$

The free boundary condition is, of course, just the first-order partial differential equation $M(g) = 0$ for g.

This completes our formulation of the variational principle. The formulas that we have derived will be the main tool to set up a minimization procedure to be described in the next section.

2.10 Accelerated Paths of Steepest Descent

We propose to solve the magnetostatic boundary value problem for ψ, R, r_0, z_0, and g by considering paths of steepest descent associated with the minimum energy principle for E. We assume that for any g we have solved the vacuum *equations for the potentials ϕ_l exactly, so that the vacuum energy is a functional* of g alone.

Letting the unknown functions depend on an artificial time parameter t, we define an accelerated path of steepest descent by means of the system of partial differential equations

$$
\begin{aligned}
a_1\psi_{tt} + e_1\psi_t &= L_1(\psi), \\
a_2 R_{tt} + e_2 R_t &= L_2(R), \\
a_3(r_0)_{tt} + e_3(r_0)_t &= L_3(r_0), \\
a_3(z_0)_{tt} + e_3(z_0)_t &= L_4(z_0), \\
e_4 g_t &= M(g),
\end{aligned}
$$

where the operators on the right come from the Euler equations found in Sections 2.5 and 2.9. The form of the equations is motivated by the method of steepest descent, the conjugate gradient method, and the second-order Richardson method. The coefficients a_j are to be determined so that the artificially time-dependent system becomes hyperbolic, while the e_j are supposed to be large enough to maintain descent.

The first thing to be noticed is that the system is chosen so that the energy E becomes a decreasing function of t. If $a_j = 0$, we have the method of steepest descent and E_t is negative, as can be seen from our formulas for the first variation. Furthermore, the path is chosen in the direction of maximum descent. However, for $a_j = 0$ the system is not adequate because the type of the differential operators on the right, which, with the exception of $M(g)$, are second order in the space variables, is nonstandard. Thus we have added second-order time derivatives so as to obtain a hyperbolic system. The convergence to a steady-state solution would be prohibitively slow without such acceleration terms. This explains our use of the second-order Richardson method, which is more or less equivalent to the conjugate gradient method in the present case.

The idea is to compute solutions of the artificially time-dependent system in the limit as $t \to \infty$. If the associated plasma equilibrium is stable, the energy has a relative minimum and the answer will converge to a steady-state solution of the magnetostatic equations. If, on the other hand, the equilibrium is unstable, the energy has a saddle point, and the artificially time-dependent solution will diverge from equilibrium following essentially the most unstable eigenfunction. If only quadratic terms are kept in an expansion of E about equilibrium, this procedure reduces to the standard variational principle of magnetohydrodynamics [4].

By choosing the coefficients a_j and e_j appropriately, we are able to study questions of both equilibrium and stability with far less computational effort than would be necessary if we examined dependence on the physical time instead (cf. [11]). In some sense, our artificially time-dependent system of partial differential equations may be interpreted as a primitive model of magnetohydrodynamics. A similar approach has been proposed by Chodura and Schlüter [12].

Since our formulation is nonlinear, we can study problems that are beyond the scope of the usual linearized stability analysis. For example, we can consider the case of a solution which is linearly stable but becomes unstable under large perturbations. Conversely, we can investigate the problem of bifurcation by perturbing an unstable equilibrium and seeing if the result converges to a different stable solution. This is sometimes referred to as saturation.

The same method could be used to do linearized stability analysis for problems with axial or helical symmetry. If we impose the symmetry condition on the formula for the energy, then the variational principle leads to a two-dimensional problem for the equilibrium solution. One equation can be integrated explicitly, and we are led to a single equation for a scalar potential. If we linearize about this solution, a Fourier analysis can be done with respect to the ignorable coordinate. Here our method has the advantage that the coordinate system used for the equilibrium problem is also convenient for solving the stability problem (cf. [23]). We hope to work on implementing this approach in a future publication.

A major contribution is the great generality allowed by our coordinate system. Since this follows the motion of the plasma, we can compute solutions with large deviations from axial symmetry but still use unknown functions that are slowly varying. Thus it is relatively easy to study the effect of wall shape or compression ratio on the stability properties of the solution.

2.11 Determination of the Acceleration Coefficients

The role of the coefficients a_j is to bring the system of partial differential equations of Section 2.10 into the hyperbolic type and provide it with appropriate characteristics. The a_j will later be selected to meet the Courant–

Friedrichs–Lewy criterion for stability of analogous difference equations on a given mesh (cf. [17]). Assuming them to be fixed, the rate of convergence in t is governed by the first-order coefficients e_j. To look for the best device to accelerate the convergence, let us consider the example of a simple scalar equation. It will be obvious how our conclusions are to be generalized to handle the plasma physics problem.

Suppose that $L(\psi)$ is a linear second-order differential operator in the space variables and that we are minimizing a functional E whose first variation is

$$\delta E = -\iiint L(\psi)\delta\psi \, dV.$$

The associated paths of steepest descent are defined by

$$a\psi_{tt} + e\psi_t = L(\psi).$$

Let Ψ be an eigenfunction corresponding to a negative eigenvalue $-\omega^2$ of L and set $\psi = e^{\lambda t}\Psi$, where λ satisfies the dispersion relation

$$a\lambda^2 + e\lambda = -\omega^2.$$

In order to maintain descent, we need

$$E_t = -\iiint (a\lambda + e)\psi_t^2 \, dV < 0,$$

which requires $e/a > |\lambda|$. If we choose e independent of t, we are forced to have e/a greater than the largest value of $|\lambda|$ that occurs in a given distribution of initial data. For $e \gg a$, we have the asymptotic relation

$$\lambda \approx -\frac{\omega^2}{e}.$$

If ω is small, which is the case we are primarily interested in, the resulting convergence rate is much too slow. This can be interpreted to mean that the artificial time t scales like the square root of real time. Similar considerations apply to any positive eigenvalue of L.

To accelerate the method, we choose e to be proportional to the dominant growth rate λ, which may be either positive or negative and may vary with time. Then, according to the dispersion relation, we obtain

$$\lambda \approx \omega \text{ const.},$$

so the artificial time scales like real time rather than like its square root.

To implement this idea, consider the least-squares error

$$F(t) = \iiint L(\psi)^2 \, dV.$$

In terms of the eigenfunctions Ψ_n of L, this can be expanded as

$$F(t) = \sum A_n \exp\{2\lambda_n t\} \iiint L(\Psi_n)^2 \, dV,$$

where *for the sake of simplicity we have assumed orthogonality*. The values of $|F_t/F|$ averaged over a number of time cycles provide a good measure of the dominant growth rate λ. By setting

$$e(t) = \tau a \left| \frac{F_t}{F} \right|$$

for a suitable value of the constant $\tau \geq 1$, the best rate of convergence is achieved and simultaneously an estimate is obtained of the growth rate ω of the least favorable mode for an unstable equilibrium. To ascribe a physical meaning to this growth rate in practice, however, comparison must be made with some example in which an Alfvén transit time is known from other considerations.

The procedure we have described for acceleration by means of a variable convergence factor $e = e(t)$ significantly enhances the method of steepest descent, which is prohibitively slow in its usual formulation. The same procedure is applicable to the problem of estimating the relaxation factor for the method of successive over-relaxation in a more general context. This will be described in the next chapter in connection with the solution of Laplace's equation for the vacuum region. In practice no extra computational work is required, since the operator $L(\psi)$ must be computed in any case. Rates of convergence can be improved by as much as a factor of ten in typical cases.

For the partial differential equation of the free surface, an exception has to be made because it is only of the first order. However, no acceleration is called for in that case anyway, so the coefficient of the time derivative can be assigned in a more obvious fashion. It then turns out that the previous assertions about growth rates remain valid even with a free surface included in the model. However, for the free surface model the convergence of the solution is markedly improved if we allow the origin of the coordinate system for the free boundary to move with the plasma. This is accomplished by writing differential equations defining paths of steepest descent for $r_3(v)$ and $z_3(v)$ which only involve averages with respect to u of the free boundary equation. These are given by

$$e_5(r_3)_t = \int (1 - g)K[\tfrac{1}{2}B_1^2 + p - \tfrac{1}{2}B_2^2]z_u \, du,$$

$$e_5(z_3)_t = -\int (1 - g)K[\tfrac{1}{2}B_1^2 + p - \tfrac{1}{2}B_2^2]r_u \, du.$$

Such a rezoning minimizes mesh distortion because the origin of the coordinate system follows the plasma shape. For example, helical excursion or translation of the plasma column is described primarily by the moving origin itself rather than by large distortions of the function g. Thus truncation errors are minimized, too.

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CHAPTER 1

Introduction

1.1 Formulation of the Problem

In magnetic fusion energy research, a central role is played by toroidal devices for the confinement of a plasma. These devices are essentially of two different types, called the Tokamak and the stellarator. In a Tokamak, which is an axially symmetric configuration having a plane magnetic axis, the toroidal outward drift of the plasma is counterbalanced by the poloidal magnetic field due to a strong toroidal plasma current. In a stellarator, which is a toroidal configuration with a helically deformed magnetic axis, the toroidal drift is offset by a restoring force associated with helical windings, and the net toroidal current is negligible compared to the poloidal current producing the main theta pinch field. Extensive experimental investigations of both types of devices have been conducted. So far the Tokamak work has been more successful and at present dominates the scene.

The partial differential equations of magnetohydrodynamics define a valid isotropic continuum model for mathematical analysis of the toroidal equilibrium of a plasma. When resistivity is neglected, there is a variational principle for the combined magnetic and fluid potential energy that leads to a relatively simple theory of equilibrium and stability [4,22,28]. Even that theory is too complicated, however, to permit exact solutions of many of the problems that arise in the applications. The purpose of this book is to develop a numerical method for the solution of the magnetostatic equations and to present a computer code based on that method for the study of practical questions of equilibrium and stability in plasma physics.

Our intension is to solve problems involving genuinely three-dimensional geometry, such as those associated with the helical windings of a stellarator having no symmetry. Instead of treating the full magnetohydrodynamic equations directly, we calculate equilibria by applying the method of steepest descent to the variational principle for the plasma and vacuum potential energy in a fashion that provides significant information about stability [7,8,12,28]. Therefore, we are able to confine our attention to a reduced system of partial differential equations related to magnetostatics. This simplification does, however, raise some subtle mathematical questions about the formulation of the steady-state problem and the existence of weak solutions [7,20].

Paths of steepest descent are defined by solving an initial value problem for a system of partial differential equations that is expressed in terms of an artificial time parameter. For stable equilibria, the solution approaches a steady state as the artificial time becomes infinite. We introduce an accelerated scheme for which the partial differential equations are of the hyperbolic type. They are more primitive than the full system of magnetohydrodynamic equations, but have many similar properties. In particular, the stability properties of equilibrium solutions are the same.

A computer code has been written to implement our method of finding toroidal equilibrium. Questions of stability can be answered by examining the asymptotic behavior of solutions for large artificial time. A run with adequate resolution can be made in two hours on the CDC 6600 computer. The code is sufficiently fast and accurate to handle three space variables and time with limited computer capacity. For both equilibrium and stability calculations, it is preferable to codes requiring the solution of the full magnetohydrodynamic equations.

1.2 Discussion of Results

The computer code we have developed is most effective for the study of equilibria with medium or high values of the plasma parameter

$$\beta = 2p/(2p + B^2)$$

measuring the ratio of the fluid pressure p to the sum of the fluid pressure and the magnetic pressure $B^2/2$. It is most appropriate for examples where three-dimensional geometry and nonlinear effects play a significant role. Because the code takes into account three space variables as well as the artificial time, there is a severe limitation on how small the mesh sizes can be taken. The resulting truncation errors are not always easy to assess. In general, they take the form of artificial viscosity terms whose effect is in some sense comparable to that of a finite Larmor radius in plasma physics. Both effects tend to reduce growth rates of physically unstable modes.

We have made extensive computer studies of high β stellarators such as the Isar Tl-B at Garching and the Scyllac at the Los Alamos Scientific Laboratory. The calculations enable one to assess the effects of nonlinearity and of a diffuse pressure profile as well as of a vacuum field surrounding the plasma. Unstable equilibria can be determined by examining streak plots of the motion of the plasma corresponding to various helical distortions of the outer conducting coils. Comparable computations have been performed by Barnes and Brackbill [1] at Los Alamos using a three-dimensional code of the Harlow variety for the full magnetohydrodynamic equations. When these computations were used to redesign a set of coils for the final Scyllac experiment, they resulted in a doubling of the containment time, raising it to 50 μsec.

A principal difficulty with high β stellarators has been the instability of the gross $m = 1$, $k = 0$ mode, which shifts the whole plasma to the outer wall. Here m and k indicate the wave numbers in the poloidal and toroidal directions, respectively. Our calculations show that this mode can be stabilized by introducing coils with triangular cross sections [8]. The stabilization depends on the magnetic structure and flux constraints inside the plasma. It enhances the more usual wall stabilization that occurs for low compression ratios. A straight helically symmetric experiment to test this contention is in the construction stage at the Max Planck Institute for Plasma Physics in Garching.

The code is applicable to high β Tokamaks and to Tokamaks with superimposed helical windings. For axially symmetric geometry, it has been used to show that values of β as high as 18 percent can be achieved stable to $m = 1$ by introducing appropriate cross sections. To exhibit the nonlinear and three-dimensional features of the method, we have calculated bifurcated equilibria that are associated with nonlinear saturation of linearly unstable modes.