126 : Constructing integrable full-pressure full-current free-boundary stellarator magnetohydrodynamic equilibria.

S.R.Hudson*, D.A.Monticello*, A.H.Reiman*, D.J.Strickler† and S.P.Hirshman†

*Princeton Plasma Physics Laboratory, P.O. Box 451, Princeton NJ 08543, USA.
†Oak Ridge National Laboratory, P.O. Box 2009, Oak Ridge TN 37831, USA.

Abstract. For the (non-axisymmetric) stellarator class of plasma confinement devices to be feasible candidates for fusion power stations it is essential that, to a good approximation, the magnetic field lines lie on nested flux surfaces; however, the inherent lack of a continuous symmetry implies that magnetic islands are guaranteed to exist. Magnetic islands break the smooth topology of nested flux surfaces and chaotic field lines result when magnetic islands overlap. An analogous case occurs with $1\frac{1}{2}$-dimension Hamiltonian systems where resonant perturbations cause singularities in the transformation to action-angle coordinates and destroy integrability. The suppression of magnetic islands is a critical issue for stellarator design, particularly for small aspect ratio devices.

Techniques for 'healing' vacuum fields and fixed-boundary plasma equilibria have been developed, but what is ultimately required is a procedure for designing stellarators such that the self-consistent plasma equilibrium currents and the coil currents combine to produce an integrable magnetic field, and such a procedure is presented here for the first time.

Magnetic islands in free-boundary full-pressure full-current stellarator magnetohydrodynamic equilibria are suppressed using a procedure based on the Princeton Iterative Equilibrium Solver [A.H.Reiman & H.S.Greenside, *Comp. Phys. Comm.*, 43:157, 1986.] which iterates the equilibrium equations to obtain the plasma equilibrium. At each iteration, changes to a Fourier representation of the coil geometry are made to cancel resonant fields produced by the plasma. As the iterations continue, the coil geometry and the plasma simultaneously converge to an equilibrium in which the island content is negligible. The method is applied to a candidate plasma and coil design for the National Compact Stellarator eXperiment [G.H.Neilson *et.al.*, *Phys. Plas.*, 7:1911, 2000.].

The magnetic field lines of toroidal plasma confinement devices, such as stellarators, are $1\frac{1}{2}$ dimensional Hamiltonian systems and magnetic flux-surfaces are the analog of constant action surfaces [1]. This may be seen by noting that in arbitrary toroidal coordinates (r,θ,ζ) any vector, in particular the magnetic vector potential, may be written $\mathbf{A} = \psi\nabla\theta - \chi\nabla\zeta + \nabla g$, where ψ, χ and g are functions of (r,θ,ζ): from which $\mathbf{B} = \nabla\psi\times\nabla\theta + \nabla\zeta\times\nabla\chi$. Using the toroidal angle ζ as the independent (time) coordinate, and considering $\chi = \chi(\psi,\theta,\zeta)$, the magnetic field line flow equations may be recast in a form identical to Hamilton's equations: $d_\zeta\theta = \partial_\psi\chi, d_\zeta\psi = -\partial_\theta\chi$.

Integrable $1\frac{1}{2}$ dimensional Hamiltonians naturally occur only in systems with a continuous symmetry, and stellarators have no continuous symmetry. Integrability can be studied by perturbing an integrable field $\mathbf{B_0}$. Writing $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ and $p = p_0 + p_1$, the perturbed system is in equilibrium if $\mathbf{B}_0\cdot\nabla p_1 + \mathbf{B}_1\cdot\nabla p_0 = 0$. In magnetic coordinates this becomes $\iota\partial_\theta p_1 + \partial_\zeta p_1 = -p_0' B_1^\psi/B_0^\zeta$. If this can be non-trivially solved for p_1, new magnetic coordinates exist and the perturbed state preserves integrability; however, the coefficients of $p_1 = \sum p_{1mn}\cos(m\theta - n\zeta)$ are given by the coefficients of (B_1^ψ/B_0^ζ) divided by $(\iota m - n)$. At rational rotational-transform surfaces, $\iota = n/m$, a singularity exists. This is the classical problem of small denominators and magnetic islands are formed with width in ψ given by $[(B_1^\psi/B_0^\zeta)_{mn}/\iota' m]^{1/2}$. Islands and the chaotic field lines caused by island overlap result in poor plasma confinement.

Changes in coil geometry will change $(B_1^\psi/B_0^\zeta)_{mn}$ and can be used to reduce islands and their associated stochastic regions. It is perhaps impossible to completely eliminate all resonant perturbation terms in non-symmetric systems, but this is too stringent a requirement as sufficiently small islands will have little, if any, effect on plasma confinement. All that is required for practical purposes is that the magnetic islands occupy less than a tolerable percentage of the plasma volume. Such a magnetic field is said to have 'good-flux-surfaces'.

The construction of vacuum magnetic fields with good-flux-surfaces is not trivial [2], but is simpler than the case when a plasma is present. The complexity arises as the plasma currents modify $(B_1^\psi/B_0^\zeta)_{mn}$, and the self-consistent solution requires that the plasma equilibrium field and the coil field combine to give zero resonant component at the rational rotational-transform surfaces. Previous studies of stellarator MHD equilibria have computed finite pressure equilibria with islands and even showed that 'self-healing' [3] can occur.

The article gives a procedure for adjusting the coil geometry to remove islands at the operating plasma configuration, without degrading the previous optimization of either the coils or the self-consistent plasma equilibrium. Plasma and coil design optimization relies on equilibrium codes. The fastest equilibrium codes presuppose the existence of perfect magnetic surfaces — the existence or size of magnetic islands cannot be addressed. To do this a more computationally intensive code is used to remove the islands by iterating between the plasma equilibrium and the coil geometry.

A recent article [4] presented a method by which high-pressure full-current *fixed*-boundary solutions may be constructed with good-flux-surfaces. Although stellarator coils must balance the normal field B_n produced by the plasma currents on the plasma surface, balancing B_n at each point on a arbitrary surface generically leads to singular coil currents. Nevertheless, a number of constraints on the normal field distribution may be satisfied.

The fixed-boundary healing work [4] showed that a given (m,n) island is controlled by a single spatial distribution of normal field B_n. One constraint that must be achieved is these distributions be nulled. This is achievable since, in most cases, it is the few low-order islands which are most problematic. Additional constraints in the present context are properties that must be preserved — in particular the optimized plasma properties (ideal stability, quasi-axisymmetry, ...) and the optimized engineering properties of the coil design.

Compared to traditional stellarator designs, the problem of resonances is enhanced for the National Compact Stellarator eXperiment (NCSX) [5], which is the present motivation for this work. NCSX is both compact, thus the lack of symmetry is pronounced, and has large transform per period and large shear, which results in multiple low order resonances. NCSX will have significant plasma current, thus the rotational transform profile will be quite different to that of the vacuum state, and will operate at high plasma pressure, thus the magnetic surfaces will be different to those of the vacuum. The introduction of a technique for 'healing' a given plasma-coil configuration to ensure good-flux-surfaces at the operating plasma configuration is critical for the design. The operating configuration of NCSX has been designed using an optimization algorithm to maximize quasi-axisymmetry (for control of particle orbits), subject to the constraint that the plasma is stable at an averaged plasma energy above 4% of the averaged magnetic energy.

Our method is based on the free-boundary Princeton Iterative Equilibrium Solver (PIES) code [6] which iterates the MHD equilibrium equations to solve for plasma equilibria in stellarator geometry and allows for the existence of islands. Island suppression is achieved by adding to the standard PIES algorithm a procedure that alters the coil geometry at each iteration so that the coil magnetic field cancels the resonant components of the plasma magnetic field. By continuously adjusting the coils as required, the inherent non-linearity of the plasma response is effectively controlled. The changes in coil geometry are constrained to preserve engineering constraints on minimum bend radius and minimum coil-coil separation, as well as the plasma constraint of ideal kink stability. As the iterations continue, the coil geometry and the plasma simultaneously converge to an island-free, stable-plasma with buildable coils.

The total magnetic field is the sum of the plasma field, $\mathbf{B}_P$, and the magnetic field produced by the confining coils, $\mathbf{B}_C$, which is a function of a set of Fourier harmonics, ξ, that describe the coil geometry, at the nth iteration

$$\mathbf{B}^n = \mathbf{B}_P^n + \mathbf{B}_C(\xi^n). \tag{1}$$

The initial plasma state is provided by the VMEC code [7], which imposes the artificial constraint that the plasma has nested flux-surfaces, and the initial coil geometry is provided by the COILOPT code [8]. The method presented in this article can be considered as removing the constraint of nested surfaces and allowing the VMEC initialization to relax into an equilibrium, potentially with broken flux-surfaces (islands), while making adjustments to the coils to remove islands as they develop. The PIES iterations solve for the plasma current $\mathbf{J}$ given $\mathbf{B}$ and given pressure profile p

$$\nabla p = \mathbf{J}^{n+1} \times \mathbf{B}^n. \tag{2}$$

A magnetic-differential equation $\mathbf{B} \cdot \nabla(J_\parallel / B) = \nabla \cdot J_\perp$ gives the parallel current which is solved using magnetic coordinates [9]. The PIES code allows the field topology to break up into islands and chaos. In such regions the magnetic-differential equation need not be solved because the current and pressure profiles are flattened, which eliminates the singular parallel currents. The plasma magnetic field is then solved given $\mathbf{J}$, and blended to provide numerical stability :

$$\mathbf{J}^{n+1} = \nabla \times \mathbf{B}_P \quad ; \quad \mathbf{B}_P^{n+1} = \alpha \mathbf{B}_P^n + (1-\alpha)\mathbf{B}_P. \tag{3}$$

Typically the blending parameter $\alpha = 0.99$ for NCSX style equilibria. The standard PIES algorithm makes no changes to the coil geometry and iterates through equations (2,3) to calculate the free-boundary equilibrium for a given pressure profile and coil set. The additional steps in the implementation of the coil-healing are as follows. The total magnetic field $\bar{\mathbf{B}}$ is written

$$\bar{\mathbf{B}} \quad = \quad \mathbf{B}_P^{n+1} + \mathbf{B}_C(\xi^n). \tag{4}$$

We may consider $\bar{\mathbf{B}}$ as a *nearly* integrable field and that magnetic islands are caused by fields normal to and resonant with rational rotational-transform flux-surfaces of a *nearby* integrable field.

A set of resonances that are to be suppressed is selected. The selection is determined by the rotational-transform profile. Islands associated with low-order rationals are typically the largest, but where the shear is small higher-order islands can easily overlap and result in chaotic field lines. A set of toroidal surfaces matching the selected resonances is constructed. Each such surface (in fact a quadratic-flux-minimizing surface [10]) may be considered as a rational rotational-transform flux-surface of an underlying integrable field [11], with each surface passing directly through its associated island chain and containing the stable and unstable periodic orbits. The construction of the quadratic-flux-minimizing surfaces provides an optimal magnetic coordinate system, or equivalently an optimal nearby integrable magnetic field, and in these coordinates resonant perturbation harmonics are clearly identified. The method is computationally efficient as the quadratic-flux-minimizing surfaces may be constructed exactly and only where required — at the rational rotational-transform surfaces where islands develop.

The amplitude of each of the N selected resonant field harmonics, denoted $\{\bar{B}_i : i = 1, N\}$, is calculated by Fourier decomposing the magnetic field normal to the quadratic-flux-minimizing surface. The Fourier decomposition is performed using an angle coordinate which corresponds to magnetic coordinate of the underlying integrable field on that surface. The COILOPT [8] code provides a convenient Fourier representation of the coil geometry and a set of M coil harmonics $\{\xi_j : j = 1, M\}$ is systematically varied to set $\bar{B}_i = 0$ using a Newton method. The coupling matrix, ∇B^n_{Cij}, is defined as the partial derivatives of the selected resonant harmonics of the coil magnetic field normal to the quadratic-flux-minimizing surface, which is updated every PIES iteration, with respect to the chosen coil harmonics and is calculated using finite-differences. A multi-dimensional Newton method is applied to find the coil changes $\delta\xi_j$ that set $\bar{B}_i = 0$

$$-\bar{B}_i \quad = \quad \nabla B^n_{Cij} \cdot \delta\xi^n_j. \tag{5}$$

This equation is solved for the $\delta\xi_j$ in a few iterations by inverting the $N \times M$ matrix ∇B^n_{Cij} using singular-value decomposition [12] and the coil set is adjusted $\xi^{n+1}_j = \xi^n_j + \delta\xi^n_j$, *at every PIES iteration*, such that resonant components of the combined plasma-coil field are eliminated. The algorithm returns to Eqn(1). As the iterations proceed, the coil geometry and the plasma simultaneously converge to coil geometry-plasma solution with good-flux-surfaces.

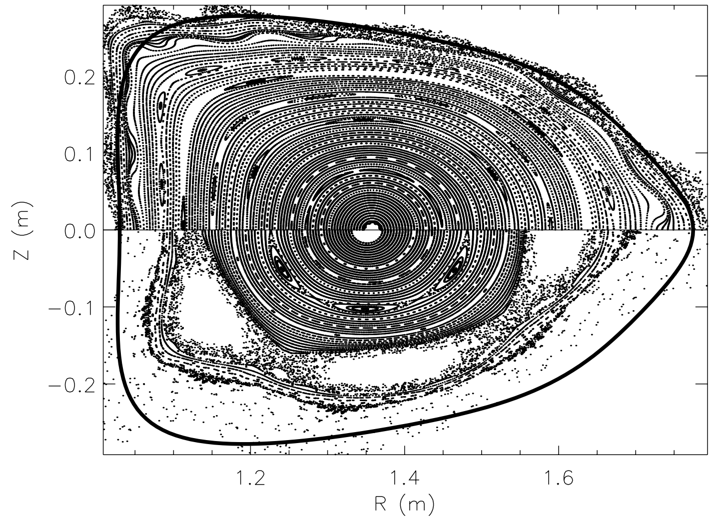
To be 'build-able', the minimum coil-curvature and coil-coil separation, for example, of the coils must exceed certain limits. Such constraints are calculated by the COILOPT code and the initial coil set, described by ξ^0, is satisfactory from an engineering perspective. The healing algorithm is modified to preserve the minimum curvature and coil-coil separation by adding to the set of resonant fields to be eliminated the (appropriately weighted) differences in minimum curvature and coil-separation of the nth coil set, described by ξ^n, from the initial coil set. This constrains the island-eliminating coil variations to lie in the nullspace of these measures of engineering acceptability. In a similar manner, the algorithm preserves kink stability. The VMEC initialization is kink-stable, and by calculating kink stability using VMEC and the TERPSICHORE code [13], the coil changes are constrained to preserve kink stability.

The method is routinely applied to NCSX [5] candidate coil and plasma designs. NCSX is a proposed proof-of-principle device with three field periods, aspect ratio A=4.4, major radius R=1.4m and magnetic field B=1.7T. The stellarator symmetric coil design consists of 18 modular coils (3 distinct coil types), 18 toroidal field coils, and six pairs of poloidal field coils and some additional trim coils. The plasma is designed to be quasi-axisymmetric to give good transport, and is stable to kink modes at $\beta \sim 4\%$, but is marginally unstable to infinite-n ballooning modes. The rotational-transform profile has $\iota \sim 0.4$ on axis, maximum $\iota \sim 0.66$ near the edge and $\iota \sim 0.65$ at the edge: including the low order resonances $\iota = 3/7, 3/6$ and $3/5$. Note that the shear vanishes near the $\iota = 6/9$ resonance.

Considering a candidate coil set (named M45) and selecting the $(n,m) = (3,6), (3,5)$ islands to be suppressed, subject to the constraint that the minimum coil curvatures, the coil-coil separation and the kink stability be preserved (9 constraints), and allowing some $m = 3,4,5,6,7,8$ modular coil harmonics to vary (36 independent variables), a healed coil-plasma state is achieved. The engineering measures are preserved and the plasma is stable with respect to kink modes. Also, the plasma retains quasi-axisymmetry and is stable to ballooning modes $n < 45$.

Several hundred iterations are required to approach convergence in both the plasma field and the coil geometry. To confirm convergence several hundred additional standard PIES iterations are performed with the coil set unchanged. A

FIGURE 1. Poincaré plot of the converged healed coil-plasma field (upper) and for the original, unhealed coils after 180 standard PIES iterations (lower) for the NCSX candidate coil set M45. The VMEC initialization boundary is shown as the thick solid line. The island content in the healed configuration is negligible, though there is some resonant $m = 18$ deformation near the zero shear location and some high order ($m = 10, 11, 12,$ and 14) island chains. For the unhealed case there is a large $m = 5$ island and the configuration deteriorates into large regions of chaos.



Poincaré plot of the final field is shown on an up-down symmetric toroidal cross-section in the upper half of fig(1). For comparison, a Poincaré plot of the unhealed configuration is shown after 180 standard PIES iterations in the lower half of the figure. The maximum coil alteration is about 2cm, which comfortably exceeds manufacturing tolerances, but is not so large that 'healing' significantly impacts other design concerns, such as diagnostic access. The coil harmonics varied actually describe the toroidal variation of the modular coils on a toroidal winding surface. The calculation shown used 63 radial surfaces, 12 poloidal and 6 toroidal modes. Similar results have been obtained using up to 93 radial surfaces and 20 poloidal modes.

The flux-surface quality of the 'healed' equilibrium shows remarkable improvement compared to the unhealed configuration. The coils have been described with a filamentary model, and a finite thickness model of the healed coils shows further improvement, in particular the $m = 18$ deformation and the high order islands are reduced. In principle, in the limit of suppressing additional islands, this approach can lead to non-axisymmetric coil-plasma configurations with integrable magnetic fields. The procedure amounts to a stellarator design optimization routine that for the first time provides a mechanism for suppressing magnetic islands, while providing ideal stability and satisfying engineering constraints and is an important advance for the design of stellarator experiments.

We thank the NCSX design team, in particular Long-Poe Ku and Guo-Yong Fu for stability analysis. One of us (SRH) is deeply indebted to Allen Boozer for suggesting the coupling matrix approach and advice regarding the text. This work was supported in part by US Department of Energy contract DE-AC02-76CH03073.

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