Influence of pressure-gradient and average-shear on ballooning stability

semi-analytic expression for ballooning growth rate

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Motivation

- Recent work on 2^{nd} stability in stellarators has shown that
  - some stellarators do [WARE et al., PRL 2002],
  - some stellarators do not [HEGNA & HUDSON, PRL 2001],

  possess 2^{nd} stable regions.
- What property of the configuration determines 2^{nd} stability ?

Outline

- The method of profile variations [GREENE & CHANCE, NF 1983] is applied to stellarator configurations [HEGNA & NAKAJIMA, PoP 1996].
- The profile variations (and the self-consistent coordinate response) produce variations in the ballooning coefficients.
- Eigenvalue perturbation theory is used to obtain an analytic expression for the *ballooning growth rate,* γ, as a function of the pressure-gradient and average-shear: $\gamma = \gamma(\iota', p')$ *(for constant geometry).*
- The analytic expression determines if pressure-gradient is stabilizing or destabilizing, and suggests if a 2^{nd} stable region will exist.

Three approaches will be compared

- Equilibrium reconstruction
  - *multiple equilibrium calculations*

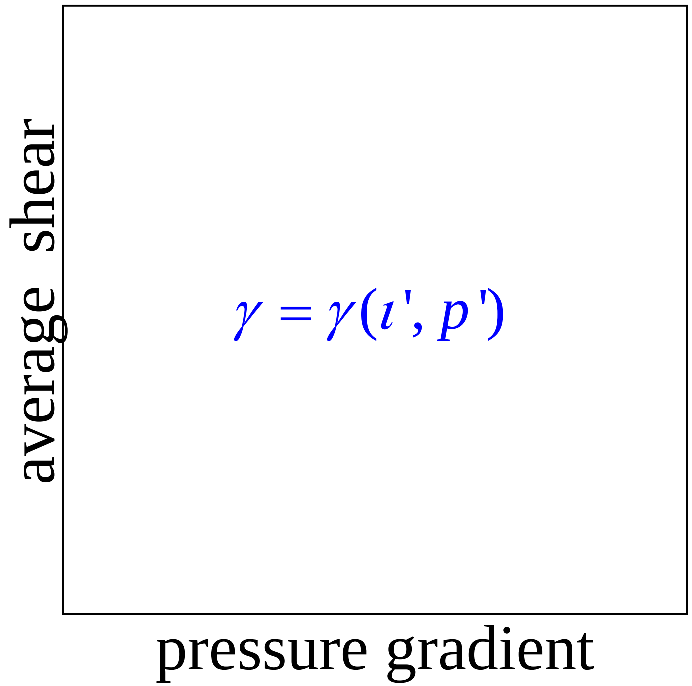
    (vary $p'(\psi)$ & $\iota'(\psi)$; VMEC)
  - *multiple ballooning calculations*
- Profile-variations

  [*GREENE&CHANCE,1983*]
  - *single equilibrium calculation*

    (construct semi-analytic equilibria)
  - *multiple ballooning calculations*
- Analytic expression

  (extension of profile-variations)
  - *single equilibrium calculation*
  - *single ballooning calculation*

to be described



The pressure-gradient and average-shear profiles are varied *(method of profile variations)*

- We begin with a full solution to an MHD equilibrium (VMEC)

- An analytic variation in p & ι is imposed

$$p(\psi) = p^{(0)}(\psi) + \mu \; \delta p(y)$$

$$\iota(\psi) = \iota^{(0)}(\psi) + \mu \; \delta\iota(y)$$

- μ is formally small parameter
- ψ_b is surface of interest

where $y = \dfrac{\psi - \psi_b}{\mu}$

- variation in the gradients is zero-order in μ
- two free parameters ($\delta\iota'$, $\delta p'$)

$$p' = p^{(0)}{}' + \cancel{\mu} \;\; \delta p' \, \cancel{\mu}^{-1}$$

$$\iota' = \iota^{(0)}{}' + \cancel{\mu} \;\; \delta\iota' \, \cancel{\mu}^{-1}$$

The self-consistent coordinate response is determined

- the coordinates are similarly adjusted to preserve $\nabla p = \boldsymbol{J}\times\boldsymbol{B}$:

$$\mathbf{x}(\psi,\theta,\zeta) = \mathbf{x}^{(0)}(\psi,\theta,\zeta) + \mu\ \mathbf{x}^{(1)}(y,\theta,\zeta).$$

- To zero order in μ, the local shear is changed

$$s = s^{(0)} + \left(1 + \partial_\eta D_{\delta\iota'}\right)\delta\iota' + \partial_\eta D_{\delta p'}\delta p'.$$

- This equation is :
  - exact at the surface of interest;
  - valid for arbitrarily large $(\delta\iota', \delta p')$;
  - the coefficients are determined by the original equilibrium:

later will keep $(\delta\iota')^2$, $(\delta\iota'\ \delta p')$, $(\delta p')^2$, .

(simply solved using Fourier representation).

$$\frac{\partial}{\partial\eta} = \frac{\partial}{\partial\zeta} + \iota\frac{\partial}{\partial\theta}$$

Pfirsch-Schluter : $\lambda = -\left(J_{\|} - \oint J_{\|}\right)/\,p'V'$

$$\frac{\partial}{\partial\eta}D = \frac{\delta\iota'}{\oint 1/g^{\psi\psi}}\left(\frac{1}{g^{\psi\psi}} - \oint\frac{1}{g^{\psi\psi}}\right) - \frac{\delta p' V'(G+\iota I)}{\oint 1/g^{\psi\psi}}\left(\frac{\lambda}{g^{\psi\psi}}\oint\frac{1}{g^{\psi\psi}} - \frac{1}{g^{\psi\psi}}\oint\frac{\lambda}{g^{\psi\psi}}\right).$$

The coefficients of the ballooning equation are changed

- The ballooning equation can be written $\left[\frac{\partial}{\partial\eta} P \frac{\partial}{\partial\eta} + Q\right]\xi = \gamma \sqrt{g}^{2} P\xi,$

where $\gamma = -\omega^2$, and the ballooning coefficients are

$$P = \frac{B^2}{g^{\psi\psi}} + g^{\psi\psi} L^2 \ \ , \ \ Q = 2p'\sqrt{g}\,(G + \iota I)\left(\kappa_n + L\kappa_g\right) \ \ , \ \ L = \int_{\eta_k}^{\eta} s(\eta')d\eta'.$$

- After the profile variation, and induced local shear variation

$$L = L^{(0)} + \left(\eta + D_{\delta\iota'}\right)\delta\iota' + D_{\delta p'}\delta p',$$

may be resolved numerically;
but further analytic progress possible

the perturbed ballooning equation may be written

$$\left[\frac{\partial}{\partial\eta}\left(P + \delta P\right)\frac{\partial}{\partial\eta} + \left(Q + \delta Q\right)\right]\left(\xi + \delta\xi\right) = \left(\gamma + \delta\gamma\right)\sqrt{g}^{2}\left(P + \delta P\right)\left(\xi + \delta\xi\right).$$

- The perturbed coefficients are

$$\delta P = P_{p'}\ \delta p' + P_{\iota'}\ \delta\iota' + P_{p'p'}\ (\delta p')^2 + P_{p'\iota'}\ \delta p'\delta\iota' + P_{\iota'\iota'}\ (\delta\iota')^2,$$

$$\delta Q = Q_{p'}\ \delta p' + Q_{\iota'}\ \delta\iota' + Q_{p'p'}\ (\delta p')^2 + Q_{p'\iota'}\ \delta p'\delta\iota' + Q_{\iota'\iota'}\ (\delta\iota')^2.$$

Eigenvalue perturbation theory gives analytic expression for change in ballooning growth rate

- The perturbed eigenvalue and eigenfunction have the form :

$$\delta\gamma = \gamma_{p'}\,\delta p' + \gamma_{\iota'}\,\delta\iota' + \gamma_{p'p'}\,(\delta p')^2 + \gamma_{p'\iota'}\,\delta p'\delta\iota' + \gamma_{\iota'\iota'}\,(\delta\iota')^2 + \text{ h.o. } + \ldots$$

$$\delta\xi = \xi_{p'}\,\delta p' + \xi_{\iota'}\,\delta\iota' + \xi_{p'p'}\,(\delta p')^2 + \xi_{p'\iota'}\,\delta p'\delta\iota' + \xi_{\iota'\iota'}\,(\delta\iota')^2 + \text{ h.o. } + \ldots$$

- The 1st order variations in the growth rate are :

$$\gamma_{p'} = \frac{\int \xi\,[\partial_\eta P_{p'}\partial_\eta + Q_{p'} - \gamma R_{p'}]\,\xi\,d\eta}{\int \xi\ \mathrm{R}\ \xi\ d\eta}\ ,\ \gamma_{\iota'} = \frac{\int \xi\,[\partial_\eta P_{\iota'}\partial_\eta + Q_{\iota'} - \gamma R_{\iota'}]\,\xi\,d\eta}{\int \xi\ R\ \xi\ d\eta}.$$

inner product

- The 1st order variations in the eigenfunction are :

$$[\partial_\eta P\partial_\eta + Q - \gamma R]\xi_{p'} = \gamma_{p'}R\xi - [\partial_\eta P_{p'}\partial_\eta + Q_{p'} - \gamma R_{p'}]\xi,$$

$$[\partial_\eta P\partial_\eta + Q - \gamma R]\xi_{\iota'} = \gamma_{\iota'}R\xi - [\partial_\eta P_{\iota'}\partial_\eta + Q_{\iota'} - \gamma R_{\iota'}]\xi.$$

operator inversion

- Higher order variations are similarly calculated.
- All variations are determined by a single eigenvalue-eigenvector calculation.

The theory determines . . .

- if increased p' is stabilizing or destabilizing,
- if a 2nd stable region will exist.

- Considering only to second order in $\delta p'$ variations

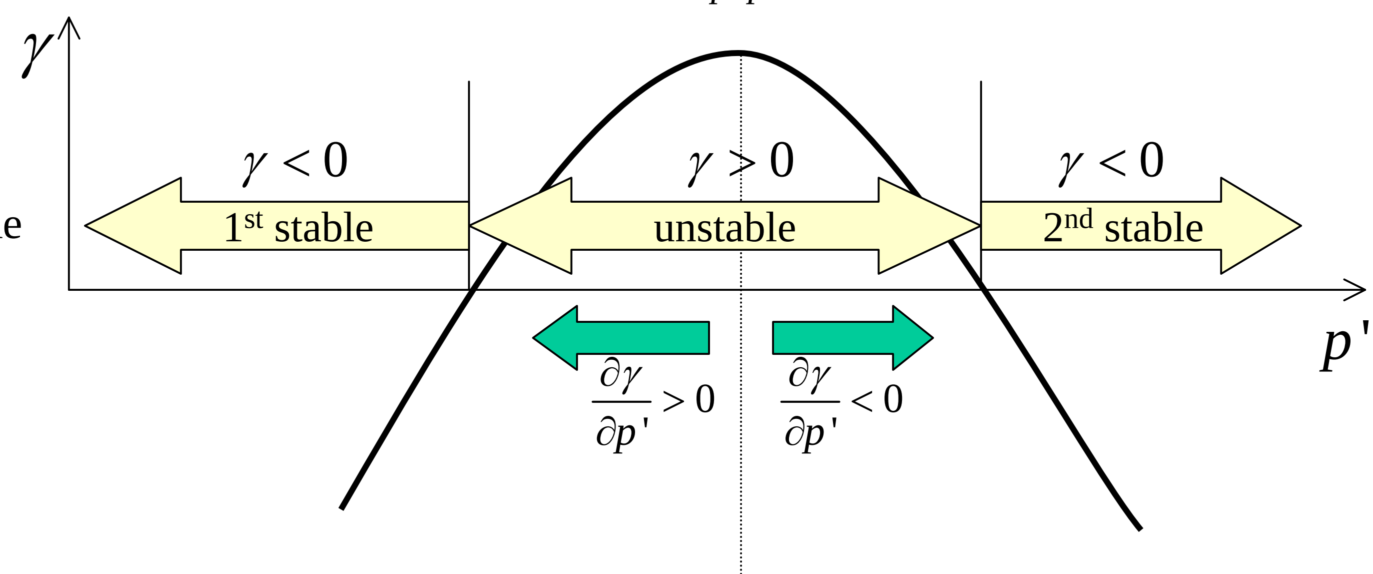
$$\delta\gamma = \gamma_0 + \gamma_{p'}\,\delta p' + \gamma_{p'p'}\,(\delta p')^2$$

valid if $\partial^3\gamma/\partial p'^3\ \delta p' \ll \partial^2\gamma/\partial p'^2$;
in general, may need $(\delta p')^3, (\delta p')^4 \ldots$

- Second stable region will exist if $\gamma_{p'p'} < 0$

$\gamma > 0$: unstable

$\gamma < 0$: stable



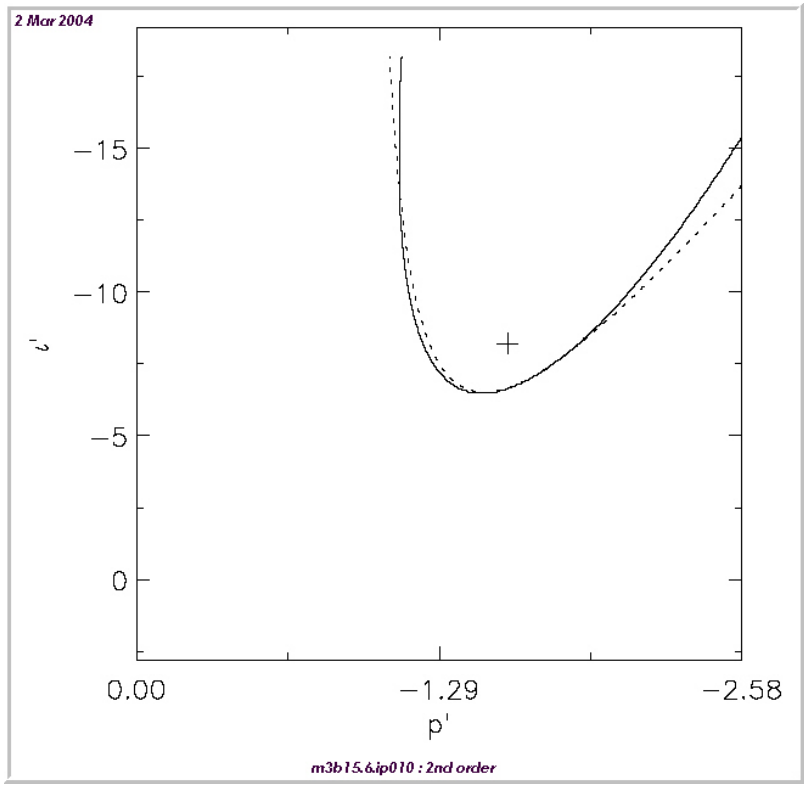
Marginal stability boundary : quasi-poloidal configuration

- quasi-poloidal configuration [WARE et al. PRL, 2002] has 2nd stable region

- solid curve is stability boundary determined by exactly re-solving ballooning equation on grid 200x200

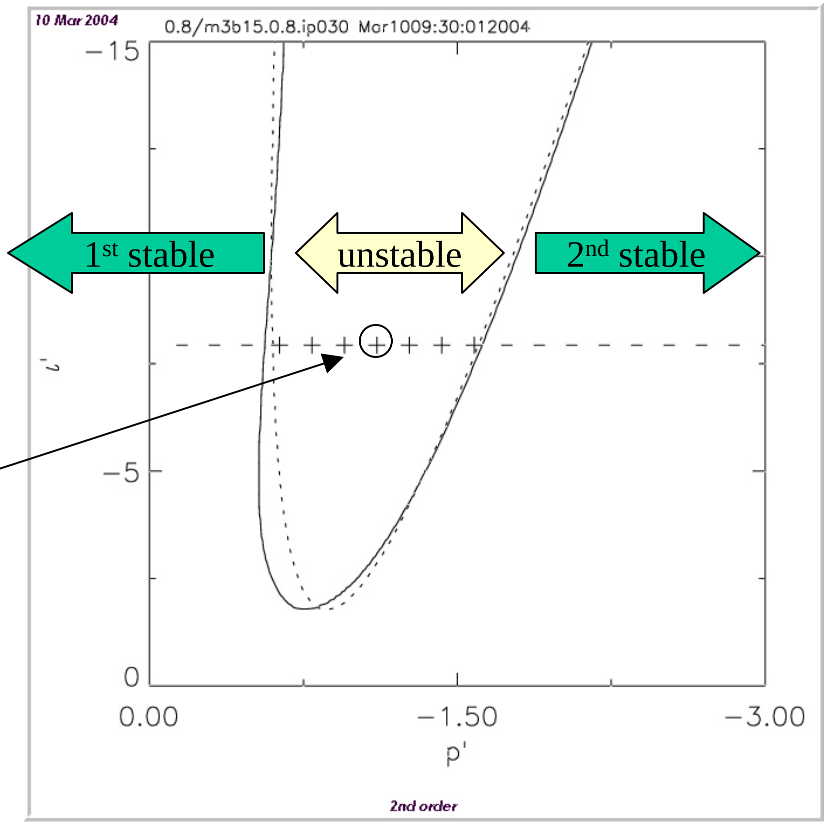
- dotted curve from analytic expression
  - including 2nd order terms
  - *single eigenfunction calculation*

- *analytic expression accurately reproduces exact stability boundary*



Stability boundary is verified by global equilibrium reconstruction

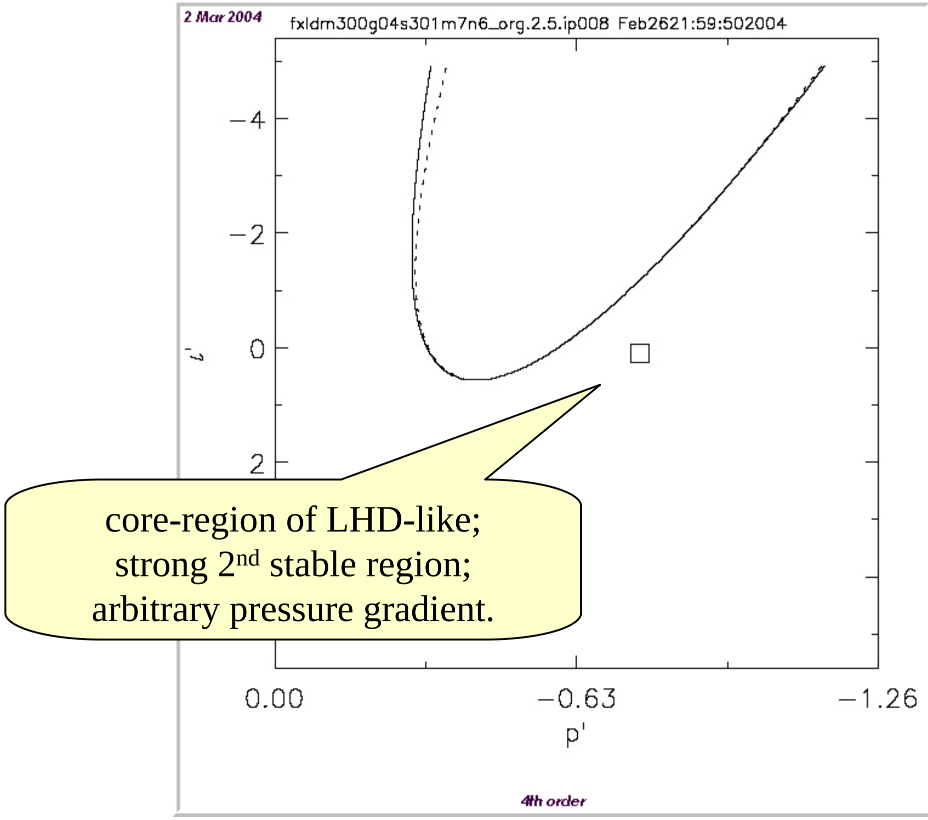
- a sequence of increasing pressure equilibria is constructed with VMEC
  - fixed boundary, fixed current profile
- For each s=0.3 surface, the ballooning stability is re-calculated
  - indicated with – or +, stable or unstable
- The marginal stability diagram is constructed
  - using original equilibrium O
    - solid : exact solution to perturbed eqn.
    - dotted : from 2nd order analytic expression

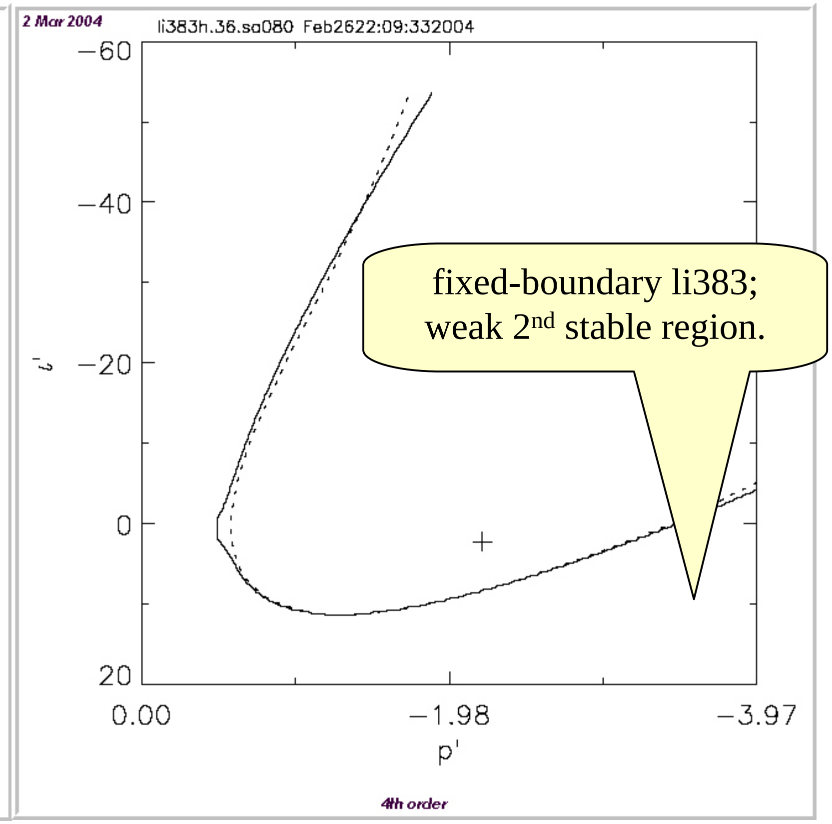


- *The stability diagram gives good prediction of global stability boundary*

Stability boundary

- LHD-like configuration
- NCSX-like configuration





solid : exact (numerical) solution to perturbed ballooning equation

dotted : from analytic expression (4th order)

Summary

- An analytic expression describing the dependence of the ballooning growth rate on pressure-gradient and shear variations is derived.
- The expression agrees well with the exact numerical solution to the perturbed ballooning equation, and agrees with stability boundaries computed with global equilibrium reconstructions.
- The expression determines :
  - if pressure-gradient is stabilizing or destabilizing
  - suggests if a 2^{nd} stable region will exist.
- Theory may be of use in stellarator optimization routines and enable deeper insight into mechanism of 2^{nd} stability.
- This approach ‘quantifies’ the strength of the second stability effect.