Chaotic Particle Trajectories in High-Intensity Finite-Length Charge Bunches

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Abstract

A Vlasov-Maxwell equilibrium for a charged particle bunch is given in the beam frame by the distribution function that is a function of the single-particle Hamiltonian $f = f(H)$, where in an axisymmetric cylinder $H = \mathbf{p}^2/2m + \kappa_\perp r^2/2 + \kappa_z z^2/2 + q\phi(r,z)$, the kinetic energy is $\mathbf{p}^2/2m$, $\kappa_\perp$ and κ_z are the external focusing coefficients in the transverse and longitudinal directions, and ϕ is the electrostatic potential determined self-consistently from Poisson's equation $\nabla^2\phi = -4\pi q \int d^3p f(H)$. The self-field potential ϕ introduces a coupling between the otherwise independent r and z motions. Under quite general conditions, this leads to chaotic particle motion. Poisson's equation is solved using a spectral method in z and a finite-difference method in r, and a Picard iteration method is used to determine ϕ self-consistently. For the thermal equilibrium distribution $f = A\exp(-H/T)$, the single-particle trajectories display chaotic behavior. The properties of the chaotic trajectories are characterized.

Thermal Equilibrium

- External focusing potential $V_{ext} = \kappa_\perp r^2/2 + \kappa_z z^2/2$,
  $\kappa_\perp$ and κ_z are the external focusing coefficients.

- Hamiltonian

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_z^2}{2m} + \frac{\kappa_\perp r^2}{2} + \frac{\kappa_z z^2}{2} + q\phi, \tag{1}$$

  where $p_r = m\dot{r}$, $p_\theta = mr^2\dot{\theta}$ and $p_z = m\dot{z}$.

- Distribution function

$$\text{thermal equilibrium} \;\; f(H) = \frac{n_0}{(2\pi mT)^{3/2}} \exp(-\mathcal{H}/T). \tag{2}$$

- Poisson's equation

$$\nabla^2\phi = -4\pi q n, \tag{3}$$

  with number density $n = n_0 \exp(\kappa_\perp r^2/2 + \kappa_z z^2/2 + q\phi)$.

Normalization

- normalized lengths $r_b = \sqrt{T/\kappa_\perp}$, $\bar{r} = r/r_b$ and $\bar{z} = z/r_b$, normalized potential, $\bar{\phi} = q\phi/T$,

$$\overline{\nabla}^2\bar{\phi} = -2s_b \exp\left(-\bar{r}^2/2 - \eta\bar{z}^2/2 - \bar{\phi}\right), \qquad (4)$$

- dimensionless parameters

$$\boxed{\eta = \kappa_z/\kappa_\perp} \qquad \boxed{s_b = (4\pi q^2 n_0/m)/2\omega_\perp^2}$$

- Normalized Hamiltonian : $\bar{\mathcal{H}} = \mathcal{H}/T$,

$$\bar{\mathcal{H}} = \frac{\bar{p}_r^2}{2} + \frac{\bar{p}_\theta^2}{2\bar{r}^2} + \frac{\bar{p}_z^2}{2} + \frac{\bar{r}^2}{2} + \eta\frac{\bar{z}^2}{2} + \bar{\phi}, \qquad (5)$$

$\bar{p}_r = \bar{r}'$, $\bar{p}_\theta = \bar{r}^2\theta'$, $\bar{p}_z = \bar{z}'$, and $' = d/d\bar{t}$ where $\bar{t} = t\,\omega_\perp$

- Hereafter, we will use the normalized equations: the 'bars' will be dropped and the 'dot' will denote the derivative with respect to the normalized time.

Picard Solution

- A mixed finite-difference, spectral method is used

$$\phi(r,z) = \sum \phi_n(r)\,\cos(nkz), \tag{6}$$

$\phi_n(r)$ interpolates $\phi_{n,i}$ given on a radial grid.

- Laplacian operator : $\nabla^2 = \partial_r^2 + r^{-1}\partial_r + \partial_z^2$.

- The radial derivatives are approximated by the first order expressions. ∇^2 becomes a tri-diagonal matrix for each harmonic.

- This allows a Picard iterative solution for the potential

given n, solve for ϕ		given ϕ, solve for ϕ

- He we use $N_r = 100$,$N_z = 50$,$R_w = 20$ and $L_z = 20$.

Phase Space

- For each selection of the dimensionless parameters (η, s_b), a point in phase space is described by $(r, \theta, z, p_r, p_\theta, p_z)$.

- The azimuthal angle θ is ignorable, thus the angular momentum p_θ is a constant of the motion. Each particle's trajectory will lie on a constant energy surface.

- A phase space subset in then specified by (p_θ, H), and a point in this space is given by (r, z, p_r).

- Note that given (p_θ, H) and (r, z, p_r), p_z is then constrained by Eq.(5).

- Poincaré section : $z = 0$ with $p_z > 0$.

Chaos and Periodic Orbits

- Regular and chaotic trajectories are interspersed in phase space.
- Regular motion lies on invariant surfaces where the frequency ratio is irrational.
- Resonance zones, or islands, will emerge where the frequency ratio between the r and z motions is rational.
- Associated with each island chain, are the stable and unstable orbits, which appear as O and X points on the Poincaré plot.
- Chaotic trajectories arise near the unstable X point.
- If the islands are so large that they overlap with nearby islands, then regions of extended chaos will be produced.

Zero self-field : Action-angle coordinates

- For the case that the self-field potential is zero, the r and z motions are independent and the dynamics is integrable.

- The 'action' coordinates are

$$j_r = (\alpha - p_\theta)/2 \qquad \theta_r = \cos^{-1}[(r^2-\alpha)/\beta] \tag{7}$$

$$j_z = (z^2\eta^{\frac{1}{2}} + p_z^2/\eta^{\frac{1}{2}})/2 \qquad \theta_z = \tan^{-1}(\eta^{\frac{1}{2}} z/p_z), \tag{8}$$

  where $\alpha = p_r^2/2 + p_\theta^2/2r^2 + r^2/2$, $\beta = \sqrt{\alpha^2 - p_\theta^2}$.

- $H = 2j_r + p_\theta + \eta^{\frac{1}{2}} j_z$.

- $\omega_r/\omega_z = 2/\eta^{\frac{1}{2}}$

- resonance will exist when $2/\eta^{\frac{1}{2}} = p/q$, where p, q are integers.

Small self-field : Perturbation Analysis

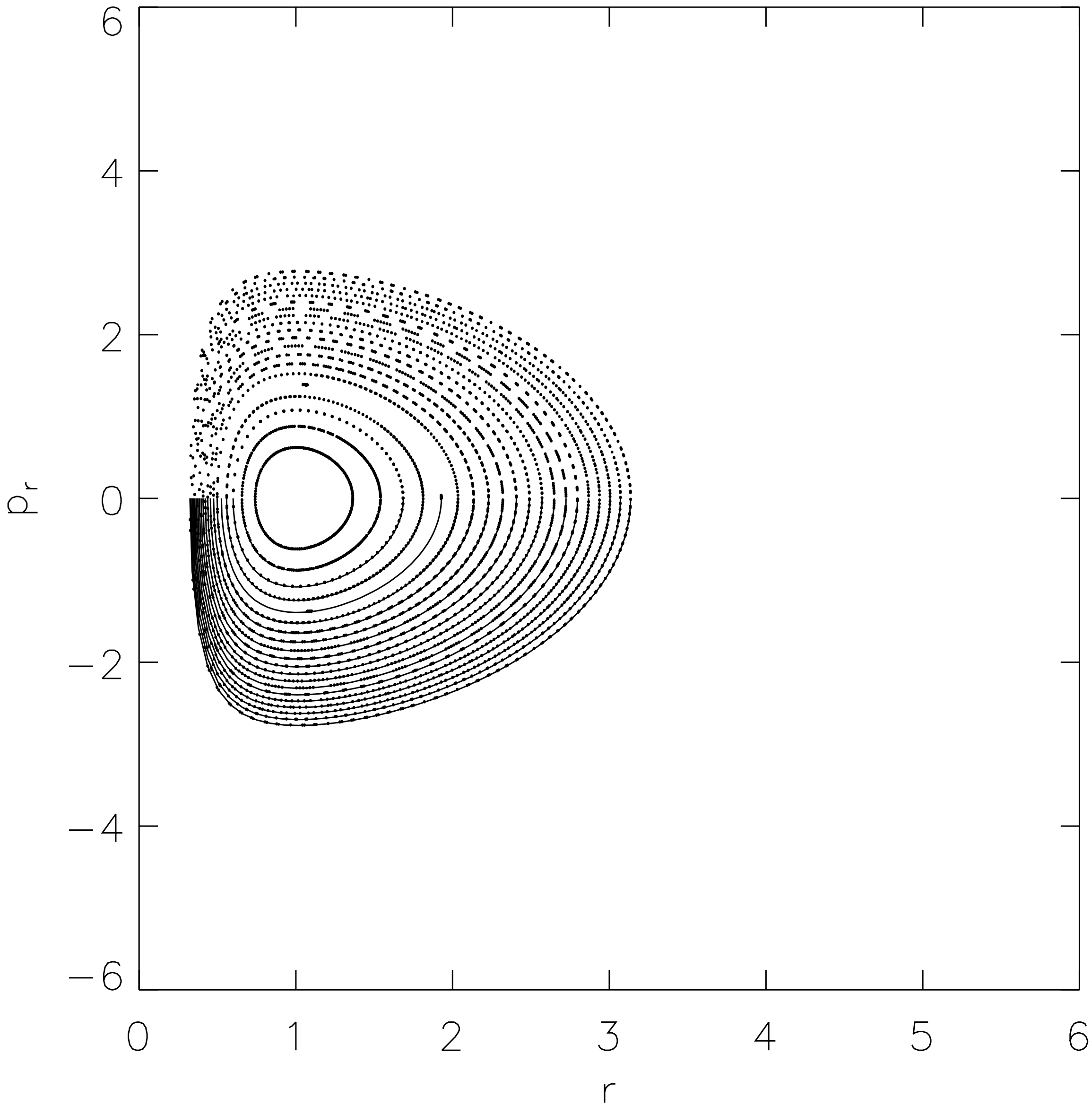
- Hamiltonian

$$H = 2j_r + p_\theta + \eta^{\frac{1}{2}} j_z + \varphi, \tag{9}$$

where $\varphi(\theta_r, \theta_z, j_r, j_z) = \phi(r(\theta_r, j_r), z(\theta_z, j_z))$.

- Writing $H = h_0 + \epsilon h_1$ where $h_0 = 2j_r + p_\theta + \eta^{\frac{1}{2}} j_z$ and $h_1 = \varphi$, action-angle coordinates (equivalently, invariants of the motion) for the perturbed motion through second order in ϵ can be constructed and compared to the exact trajectories.

- For sufficiently small self-field potential, the agreement is generally good.



Poincaré section with invariant surfaces constructed from second-order perturbation theory $(\eta, s_b) = (0.3, 0.1)$, $(p_\theta, H) = (1.0, H = 5.0)$.

Equations of Motion

- With the selection of the plane $z = 0$ as the Poincaré section, it is convenient to consider the independent integration parameter to be θ_z, rather than the time.

- The equations to be integrated then become

$$\theta_r' = \dot{\theta}_r/\dot{\theta}_z, \tag{10}$$
$$j_r' = d_t j_r/\dot{\theta}_z, \tag{11}$$
$$j_z' = d_t j_z/\dot{\theta}_z. \tag{12}$$

  where the $'$ denotes the derivative with respect to θ_z, $\dot{\theta}_r = 2 + \partial\varphi/\partial j_r$, $\dot{\theta}_z = \eta^{\frac{1}{2}} + \partial\varphi/\partial j_z$, $d_t j_r = -\partial\varphi/\partial\theta_r$, and $d_t j_z = -\partial\varphi/\partial\theta_z$.

- The mapping from the Poincaré section to itself, the Poincaré map, is now obtained by integrating these equations from $\theta_z = 0$ to $\theta_z = 2\pi$.

Lyapunov exponent

- A defining feature of chaos is that particle trajectories have an extreme sensitivity to the initial conditions.

- Particle trajectory : initial conditions $\boldsymbol{x}(0) = (\theta_r(0), j_r(0), j_z(0))$ (where, given H, j_z is constrained)

- Nearby trajectory $\boldsymbol{x}(0) + \boldsymbol{\delta x}(0)$, where δj_z is constrained to lie in the constant-energy tangent space

$$\delta j_z = -(\partial_{\theta_r}\varphi \delta\theta_r + (2 + \partial_{j_r}\varphi)\delta j_r)/(\eta^{\frac{1}{2}} + \partial_{j_z}\varphi). \tag{13}$$

- The trajectories will evolve under Eq.(10-12), and the rate at which the separation $\boldsymbol{\delta x}(\theta_z)$ evolves is characterized by the Lyapunov exponent σ

$$\sigma(\boldsymbol{x}, \boldsymbol{\delta x}) = \lim_{|\boldsymbol{\delta x}(0)| \to 0} \lim_{\theta_z \to \infty} \frac{1}{\theta_z} \ln \frac{|\boldsymbol{\delta x}(\theta_z)|}{|\boldsymbol{\delta x}(0)|}. \tag{14}$$

Linearized Equations

- The limit $|\boldsymbol{\delta x}(0)| \to 0$ is most conveniently treated by linearizing Eqs.(10-12) to obtain $d\boldsymbol{\delta x}/d\theta_z = \mathbf{T}\boldsymbol{\delta x}$, where $\mathbf{T}$ is the tangent map

$$\mathbf{T} = \begin{pmatrix} \frac{\partial \theta_r'}{\partial \theta_r}, & \frac{\partial \theta_r'}{\partial j_r}, & \frac{\partial \theta_r'}{\partial j_z} \\ \frac{\partial j_r'}{\partial \theta_r}, & \frac{\partial j_r'}{\partial j_r}, & \frac{\partial j_r'}{\partial j_z} \\ \frac{\partial j_z'}{\partial \theta_r}, & \frac{\partial j_z'}{\partial j_r}, & \frac{\partial j_z'}{\partial j_z} \end{pmatrix}. \tag{15}$$

- The component of $\boldsymbol{\delta x}$ along the most unstable direction will grow most rapidly and dominate the computation. For an arbitrary initial $\boldsymbol{\delta x}$, the largest exponent will be calculated.

- After linearizing the equations, all that remains is to follow the trajectory, while evolving the tangent vector, to determine the quantity $\ln|\boldsymbol{\delta x}|/\theta_z$ as $\theta_z \to \infty$ where $|\boldsymbol{\delta x}(0)| = 1$. Typically, a trajectory must be followed hundreds of oscillations for this limit to converge.

Periodic Orbits : Lyapunov exponent

- Periodic orbit : $\theta_r(2\pi q) = \theta_r(0) + 2\pi p$ and $j_r(2\pi q) = j_r(0)$.

- Full-period tangent map, M, at the periodic orbit is obtained

$$\frac{dM}{d\theta_z} = TM, \quad \text{with initial condition } M = I. \tag{16}$$

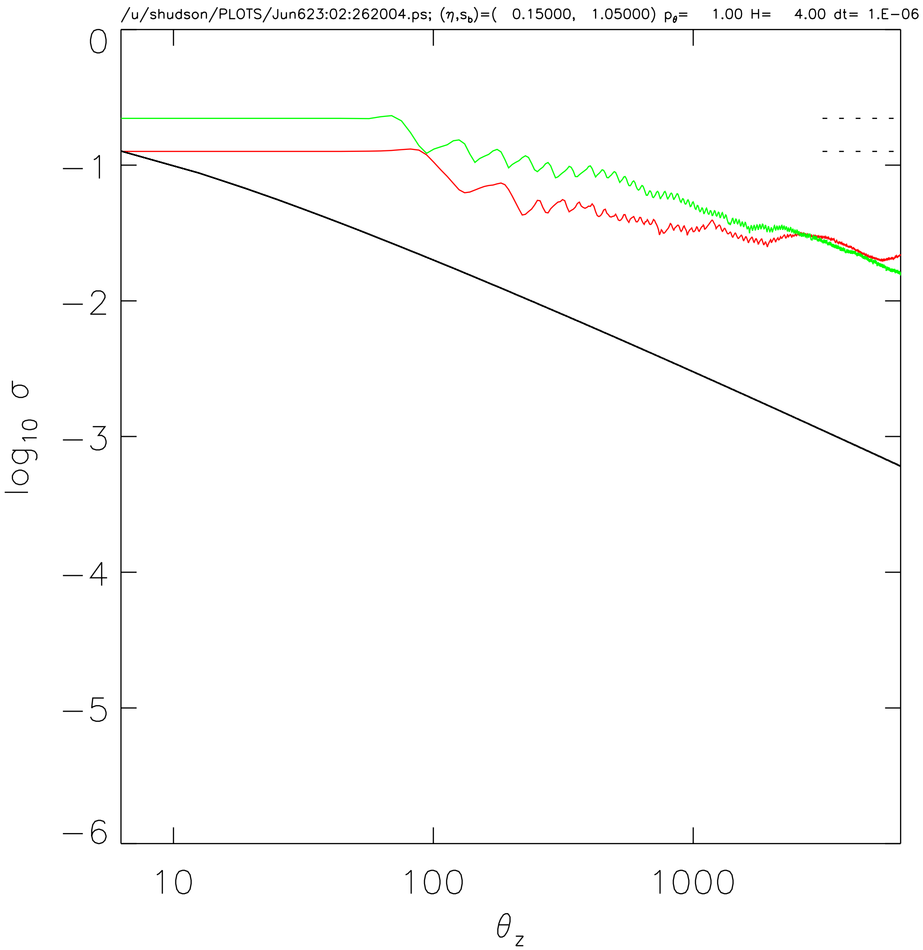
  By incorporating Eq.(13), M reduces to a 2×2 matrix.

- If the eigenvalues are real, the orbit is unstable,

$$\text{Lyapunov exponent of periodic orbit} \quad \sigma_{pq} = \ln \lambda / 2\pi q, \tag{17}$$

  where λ is the maximum eigenvalue.

- Symmetry : periodic orbits lie on the symmetry lines $\theta_z = 0, \pi$. The search for periodic orbits becomes a one-dimensional search in j_r.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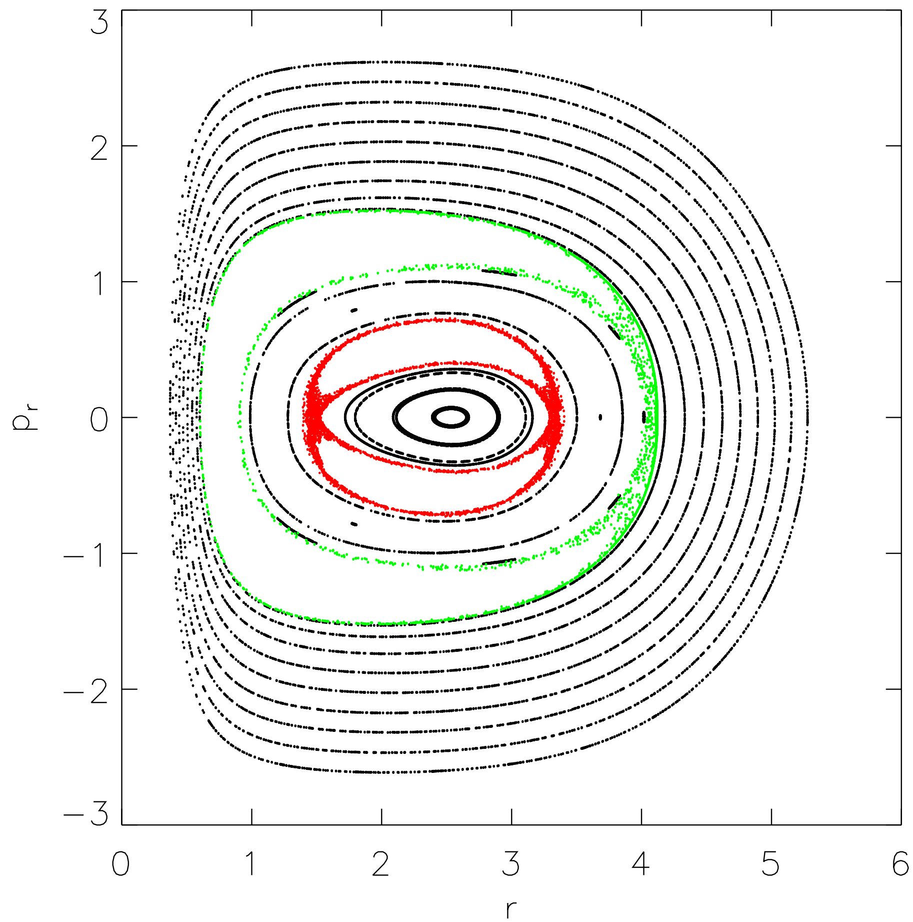
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Characterization of Chaos

- The eigenvalues of the full period tangent map are related to a quantity called the residue introduced by Greene.

- The limiting residue of an appropriate sequence of periodic orbits may be used to determine the existence, or non-existence, of an irrational (KAM) surface.

- Also, the tangent map at the periodic orbits can also be used to estimate island widths.

- This suggests that an numerically efficient method to quantify the degree of chaos would be to locate several periodic orbits (usually those with the lowest values of q are most important, and conveniently these are of the shortest length), estimate the widths of the islands associated with these periodic orbits and apply a Chirikov style island overlap criterion.