Equilibria and Stability in Partially Relaxed Plasma-Vacuum Systems

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ABSTRACT

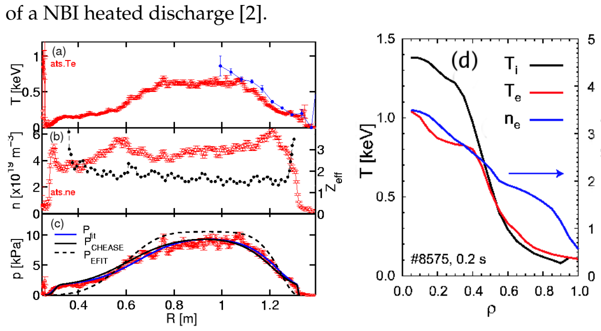
- The variational principle for relaxed toroidal plasma-vacuum systems with pressure is applied to axially periodic cylinders.
- Plasma comprises multiple Taylor-relaxed regions, with each region separated by an ideal MHD barrier of zero width.
- Extends the plasma-vacuum analysis of Kaiser and Uecker, Quartly Jnl of Mech. Appl. Math. 57(1-17), 2004.
- First stage of an attempt to describe a stepped-pressure profile in full 3D.
- May describe formation of internal transport barriers in magnetic confinement fusion experiments.

1 Introduction

1.1 Transport Barriers in Tokamaks

- At sufficiently high heating power, fusion plasmas self-organise to produce internal transport barriers.

Example: MAST discharges showing ITB formation. (a)-(c) show 7085, a high performance D-D discharge [1], (d) shows TRANSP reconstructions of a NBI heated discharge [2].



- While descriptive theories for these barriers exist : *e.g.*
  - shear flow suppression of turbulence [3],
  - chaotic magnetic field line dynamics [4],

  they don't explain *why* the plasma self-organises into this state.
- A possible explanation is that these are constrained minimum energy states.

1.2 Taylor Relaxation

- In a turbulent, resistive plasma, flux tubes do not have independant existence [5]. Infinity of constraints replaced by single constraint

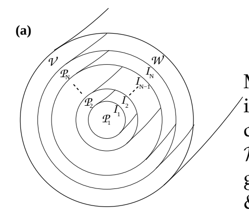
$$K_0 = \int_V \mathbf{A}\cdot\mathbf{B} d\tau \qquad (1)$$

- Minimum magnetic energy solutions, which are constrained by the total helcity are Beltrami fields

$$\nabla\times\mathbf{B} = \mu\mathbf{B} \qquad (2)$$

2 Multiple Interface Plasma Vacuum Model

- Model built upon Kaiser and Uecker [6], Spies [7] and Spies [8]. System comprises:
  - N plasma regions $\mathcal{P}_i$ in relaxed states.
  - Regions separated by ideal MHD barrier $\mathcal{I}_i$.
  - Enclosed by a vacuum $\mathcal{V}$,
  - Encased in a perfectly conducting wall $\mathcal{W}$.



Magnetic geometry, showing ideal MHD barriers $\mathcal{I}_i$, the conducting wall $\mathcal{W}$, plasma regions $\mathcal{P}_i$ and the vacuum $\mathcal{V}$. In region $\mathcal{P}_i$ plasma has pressure P_i, & equilibrium field $\mathbf{B}$.

- Energy functional can be written:

$$W = U - \sum_{i=1}^{N}\mu_i H_i/2 - \sum_{i=1}^{N}\nu_i M_i \qquad (3)$$

with

$$U_i = \int_{\mathcal{R}_i} d\tau^3\left(\frac{P_i}{\gamma-1}+\frac{B_i^2}{2}\right), M_i = \int_{\mathcal{R}_i} d\tau^3 P_i^{1/\gamma}, \qquad (4)$$

$$H_i = \int_{\mathcal{R}_i} d\tau^3 \mathbf{A}\cdot\nabla\times\mathbf{A} + \qquad (5)$$

$$\oint_{C_{pi}^<}\mathbf{dl}\cdot\mathbf{A}\oint_{C_{ti}^<}\mathbf{dl}\cdot\mathbf{A} - \oint_{C_{pi}^>}\mathbf{dl}\cdot\mathbf{A}\oint_{C_{ti}^>}\mathbf{dl}\cdot\mathbf{A} \qquad (6)$$

- **First variation :** Set $\delta W = 0$, yields *partially* Taylor relaxed equilibria:

$$\begin{aligned}
&\mathcal{P}_i; \quad \nabla\times\mathbf{B} = \mu_i\mathbf{B}, \quad P_i = \text{const.}, && (7)\\
&\mathcal{I}_i; \quad \mathbf{n}\cdot\mathbf{B} = 0, \quad < P_i + 1/2B^2 >= 0, && (8)\\
&\mathcal{V}; \quad \nabla\times\mathbf{B} = 0, \quad \nabla\cdot\mathbf{B} = 0 && (9)\\
&\mathcal{W}; \quad \mathbf{n}\cdot\mathbf{B} = 0 && (10)
\end{aligned}$$

  - μ_i, ν_i are Lagrange multipliers,
  - $\mathbf{n}$, a unit vector normal to $\mathcal{I}_i$,
  - $< x >= x_{i+1} - x_i$ is the change in x across $\mathcal{I}_i$.
  - poloidal, toroidal flux constant during relaxation
- **Second variation :** Examine stability to interface displacements ξ_i by minimize $\delta^2 W$ wrt constant N_B,

$$N_B = \sum_{i=1}^{N}\int_{\mathcal{I}_i} d\sigma^2|\xi_i|^2 \qquad (11)$$

To solve, vary functional $L = \delta^2 W - \lambda N_B$. For $\mathcal{P}_i, \mathcal{I}_i, \mathcal{V}$, solutions to $\delta L = 0$ are:

$$\begin{aligned}
&\mathcal{P}_i \;;\; \nabla\times\mathbf{b} = \mu_i\mathbf{b}, && (12)\\
&\mathcal{I}_i \;;\; \xi_i^*\langle\mathbf{B}\cdot\mathbf{b}\rangle + \xi_i^*\xi_i\langle B(\mathbf{n}\cdot\nabla)B\rangle - \lambda\xi_i^*\xi_i = 0, && (13)\\
&\mathbf{n}\cdot\mathbf{b}_{i,i+1} = \mathbf{B}_{i,i+1}\cdot\nabla\xi_i + \xi_i\mathbf{n}\cdot\nabla\times(\mathbf{n}\times\mathbf{B}_{i,i+1}), && (14)\\
&\mathcal{V} \;;\; \nabla\times\mathbf{b} = 0, \quad \nabla\cdot\mathbf{b} = 0, && (15)\\
&\mathcal{W} \;;\; \mathbf{n}\cdot\mathbf{b} = 0. && (16)
\end{aligned}$$

where $\mathbf{b} = \delta B$ is the perturbed field. Solutions of $\delta L = 0$ with $L = 0$ are stable providing $\lambda > 0$.

3 Cylindrical Stepped Pressure Equilibria

- Assume plasma is cylindrically symmetric, with axial periodicity L, vacuum boundary at $r = 1$, wall at $r = r_w$.
- Solutions of B in each region read Eq.

$$\begin{aligned}
&\mathcal{P}_1 \;: \{0, \quad k_1 J_1(\mu_1 r), \quad k_1 J_0(\mu_1 r)\},\\
&\mathcal{P}_i \;: \{0, \quad k_i J_1(\mu_i r) + d_i Y_1(\mu_i r), \quad k_i J_1(\mu_i r) + d_i Y_1(\mu_i r)\},\\
&\mathcal{V} \;: \{0, \quad B_\theta^V/r, \quad B_z^V\},
\end{aligned} \qquad (17)$$

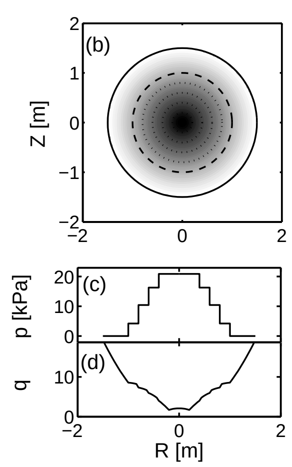
where :
  - ideal MHD barriers located at radii r_i.
  - $B_\theta^V, B_z^V, k_i, d_i \in \Re$,
  - J_0, J_1 and Y_0, Y_1 are Bessel functions.
- equilibrium prescribed: EITHER by B and r_i

$$\{k_1, ..., k_N, d_2, ..., d_N, \mu_1, ..., \mu_N, r_1, ...r_{N-1}, r_w, B_\theta^V, B_z^V\} \qquad (18)$$

OR by safety factors and fluxes

$$\{\Psi_1^t, ..., \Psi_N^t, \Psi_1^p, ..., \Psi_N^p, \Psi_V^t, \Psi_V^p, q_1^i, ..., q_N^i, q_1^o, ..., q_N^o\} \qquad (19)$$

  - q_i^i and q_i^o are safety factors on inside/outside of each interface.
  - $4N + 2$ parameters in total.



Example of a stepped-pressure plasma profile, with five ideal MHD barriers, showing : (a) a contour plot of the polodial flux, (b) the pressure profile, and (c) the safety factor, given by $q_i = \frac{2\pi}{L}\frac{B_z(r)}{B_\theta(r)}$

NOTE: Only the core necessarily has reverse shear.

4 Stability

- Fourier decompose variations $\mathbf{b}$ and ξ :

$$\mathbf{b} = \widetilde{\mathbf{b}}e^{i(m\theta+\kappa z)} \qquad \xi_i = X_i e^{i(m\theta+\kappa z)} \qquad (20)$$

  - $\widetilde{\mathbf{b}}, X_i$ are complex Fourier amplitudes
  - $m \in \mathbb{Z}, \kappa \in \mathbb{Z}$ are poloidal, axial wave numbers
- In general, the plasma and vacuum regions, Eqs. (12) (15) can be re-arranged as a second order differential equation for $\widetilde{b}_z$. That is,

$$\begin{aligned}
&\mathcal{P}_i; \quad L_\pm(m)[\widetilde{b}_z(\digamma r)] = 0, \quad \digamma = |\kappa^2 - \mu^2| > 0 && (21)\\
&\mathcal{V}; \quad L_+(m)[\widetilde{b}_z(|\kappa|r)] = 0, \quad \kappa \neq 0, && (22)
\end{aligned}$$

where $L_+(m)$ is the modified Bessel ODE for $\kappa^2 > \mu^2$, and $L_-(m)$ the Bessel ODE for $\kappa^2 < \mu^2$.
- Equation (13) reduced to the eigenvalue equation,

$$\boldsymbol{\eta}\cdot\mathbf{X} = \lambda\mathbf{X} \qquad (23)$$

with $\boldsymbol{\eta}$ a $N\times N$ tridiagnonal matrix. The i'th row of $\boldsymbol{\eta}$ is the i'th interface calculation of

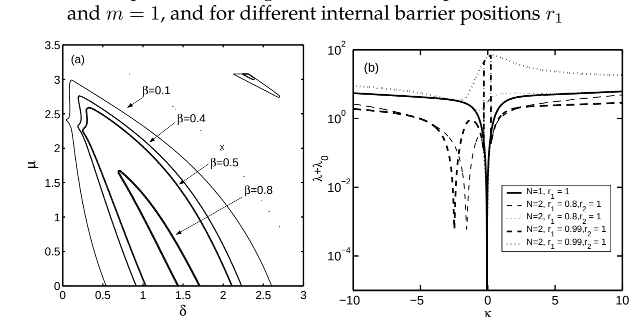
$$(\langle\mathbf{B}\cdot\mathbf{b}\rangle + \xi_i\langle B(\mathbf{n}\cdot\nabla)B\rangle)\, e^{-i(m\theta+\kappa z)} \qquad (24)$$

- Eq. (23) solved for the set of N eigenvalues $\lambda_1, ..., \lambda_N$, and eigenvectors $\mathbf{X}_1, ..., \mathbf{X}_N$.
  - η_{ij} coded into a case-selection algorithm.
  - QR algorithm used to resolve λ_i [17]
  - solutions stable providing all $\lambda_i < 0$
- **Benchmark A:** For $N = 1$, Eq. (23) reduces to eigenvalue λ, and results compared to Kaiser and Uecker [12].
  - Marginal stability parameter spans, sweeping κ over range $-K \leq \kappa \leq K$,
  - δ a measure of increase in pitch angle of $\mathbf{B}$

$$\begin{aligned}
B_{\theta,V} &= J_1(\mu_1)\cos\delta + J_0(\mu_1)\sin\delta && (25)\\
B_{z,V} &= J_0(\mu_1)\cos\delta - J_0(\mu_1)\sin\delta && (26)
\end{aligned}$$

  - Pressure described by $\beta = \frac{2||P_i||}{B^2|_{r=1+}}$
- **Benchmark B:** For $N = 2$, introduce artificial ITB with $r_1 = r_2 - \epsilon$, and no change in equilibrium. As $r_1 \to r_2$, $\lambda_2 \to 2\lambda(N = 1)$, and $\lambda_1 \to \infty$ at most unstable point.

Figure (a) shows marginal stability boundaries for $m = 1$ in $\mu_1 - \delta$ space, and for different plasma β values. The plasma has $r_w = 1.1$ and $L = 1$. The stable region is interior to each locus. The cross-hairs denote the equilibrium configuration used for the dispersion curves presented in Fig. (b), which is a dispersion curve for $N = 2$ and $m = 1$, and for different internal barrier positions r_1



6 Conclusions

- Developed multiple ideal barrier variational model.
- Shown existence of tokamak-like q profiles
- Generalized analysis for stability of multiple barriers
- Benchmarked analysis
- Begun ITB configurations scans

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