A chaotic collection of thoughts on stochastic transport

what are the issues that M3D must consider to accurately determine heat transport

which analytical and numerical methods may complement the M3D algorithm

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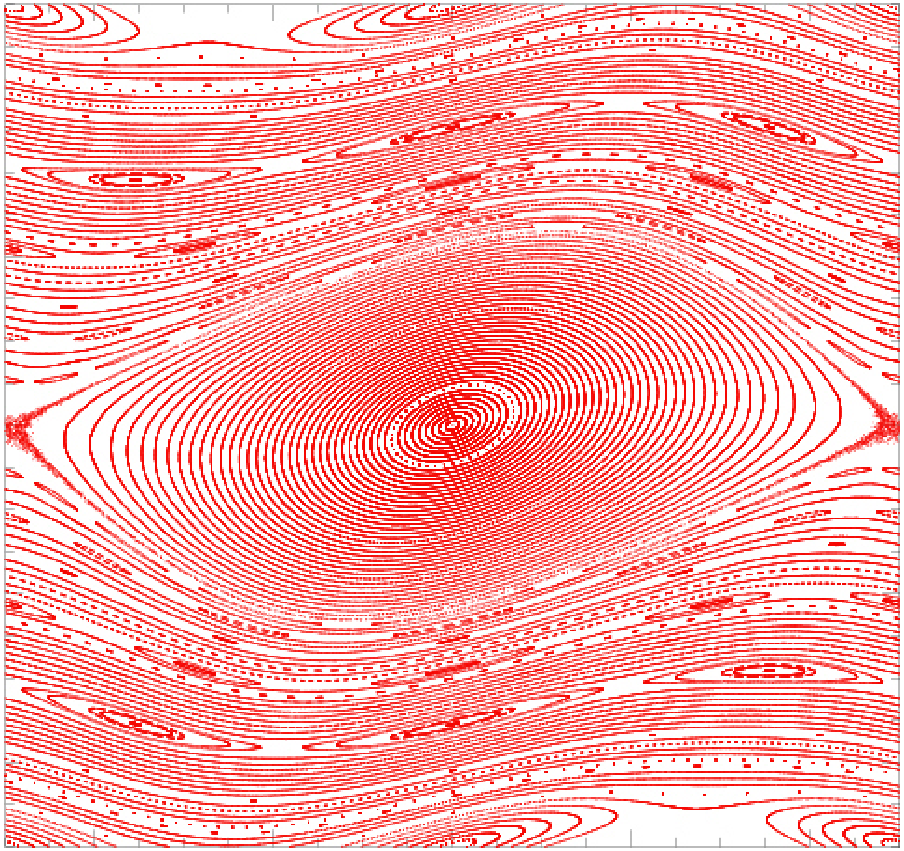
thanks for interesting discussions with Josh Breslau, Ravi Samtaney, Roscoe White, Allen Boozer

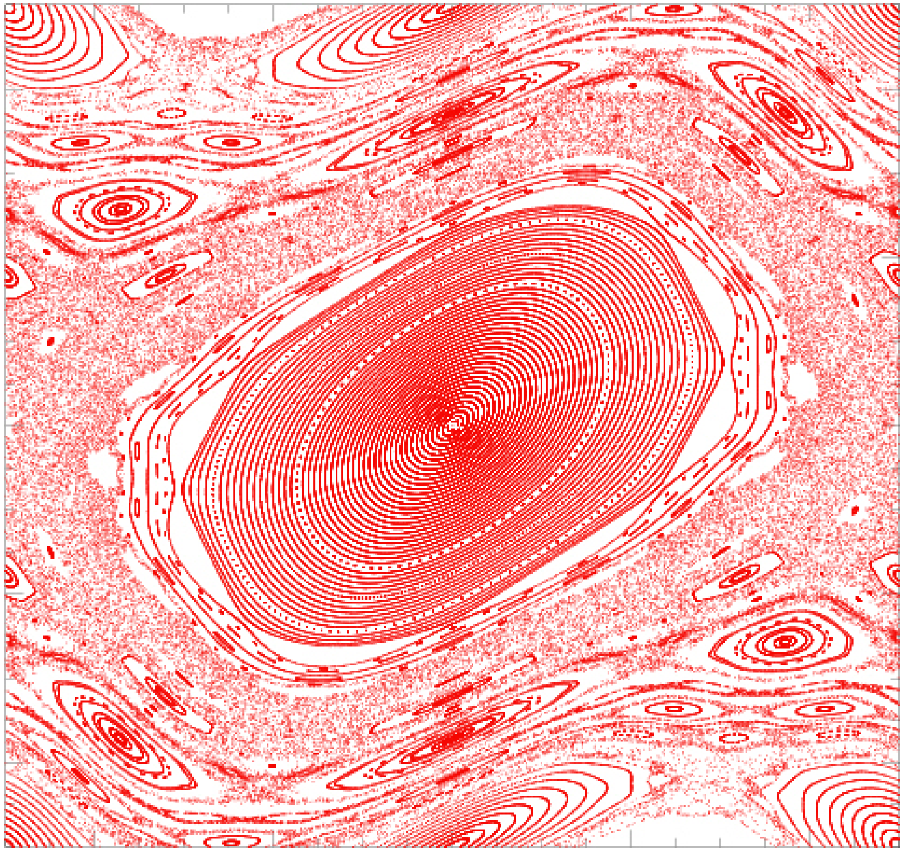
How is heat transported across islands and chaotic regions ?

$$\mathbf{q} = -\kappa_{\|}\nabla_{\|}T - \kappa_{\perp}\nabla_{\perp}T \ , \ \nabla\cdot\mathbf{q} = S \ , \ \kappa_{\|} \gg \kappa_{\perp}$$

heat flux **q**

heat source **S**





A spectrum of analytical and numerical approaches exist . .

1) <u>Analytical</u>
   1) tearing mode theory following Fitzpatrick (island but no chaos)
   2) Rechester-Rosenbluth (strong chaos)
2) <u>Monte Carlo</u>
   1) particle pushing (statistical)
3) <u>Point Gaussian model</u>
   1) pushing local gaussians
4) <u>Magnetic Coordinate approach</u>
   1) locate KAM surfaces
5) <u>Markov model of transport through cantori</u>
   1) following MacKay, Meiss, Percival
6) <u>Full Numerical Simulation</u>
   1) finite elements, M3D, NIMROD,

I will discuss each of these approaches

Fitzpatrick [PoP 2(3), 825, 1995] applied tearing mode analysis to determine scale island width

Balancing $\kappa_{\|}(\mathbf{b}\cdot\nabla)^2 \sim \kappa_{\perp}(\nabla_{\perp})^2$

gives scale island width

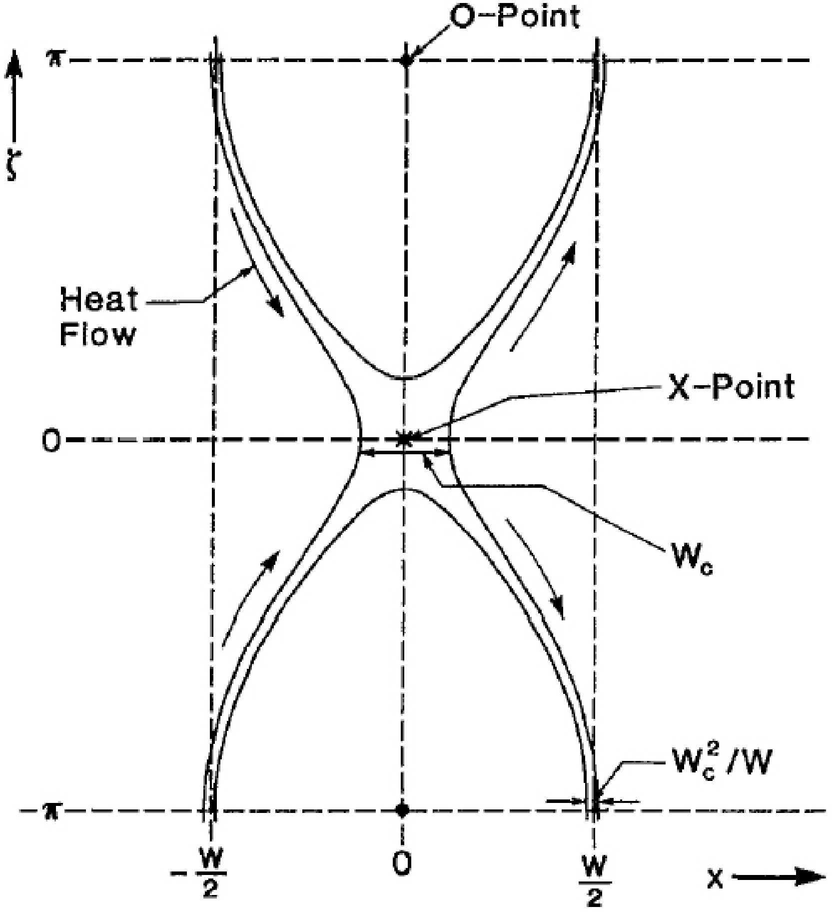
$$\frac{W}{r} \propto \left(\frac{\kappa_{\|}}{\kappa_{\perp}}\right)^{1/4}$$

for $w \ll W$, $\kappa_{\perp}$ dominates

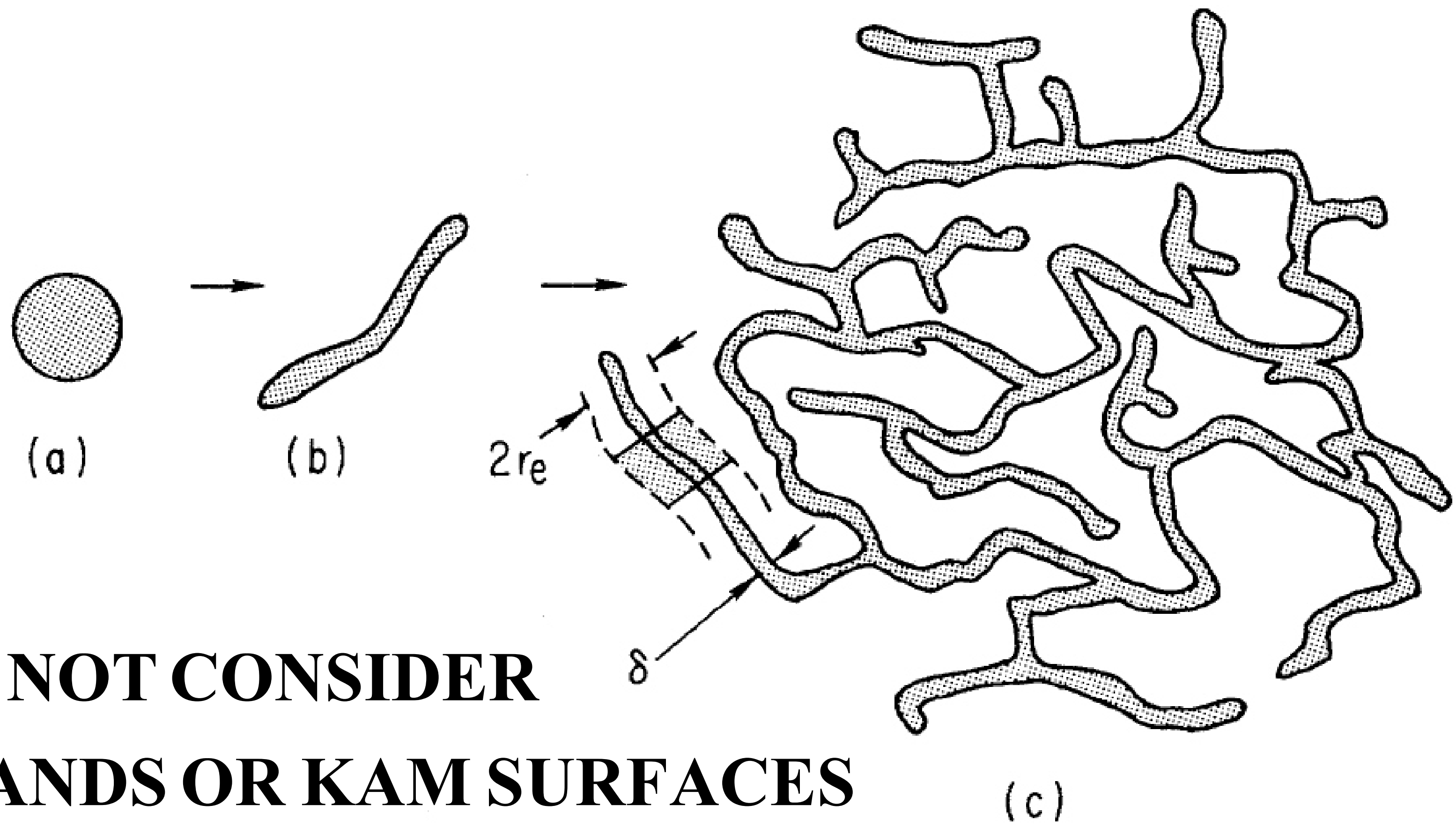
for $w \gg W$, $\kappa_{\|}$ dominates

heat transported along boundary layer

DID NOT CONSIDER CHAOS



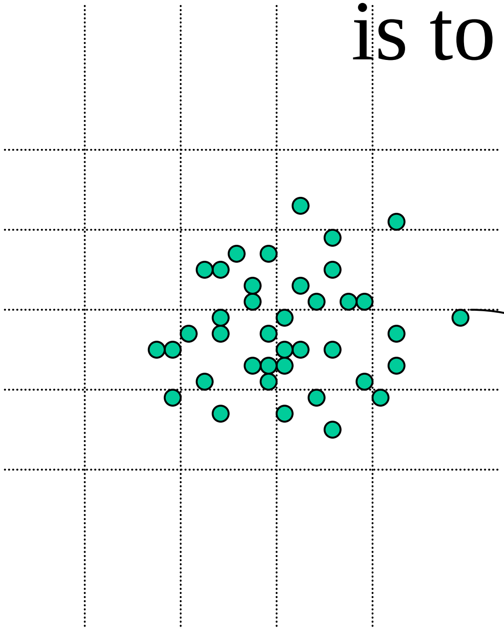
Rechester & Rosenbluth [PRL 40(1), 38, 1978] considered the enhancement to particle diffusion due to the stochastic field



DID NOT CONSIDER

ISLANDS OR KAM SURFACES

Monte Carlo : A potentially exact approach for determining particle transport, is to directly follow particles . .



$$\Delta L_{||} = \sqrt{\kappa_{||}\Delta t}$$

follow particles under parallel flow

$$\Delta L_{\perp} = \sqrt{\kappa_{\perp}\Delta t}$$

add random diffusion

1) can handle islands, chaos, KAM surfaces . . .
2) need lots of particles to give accurate statistics

The following few slides will introduce the Gaussian model

The gaussian plays a special role in diffusion

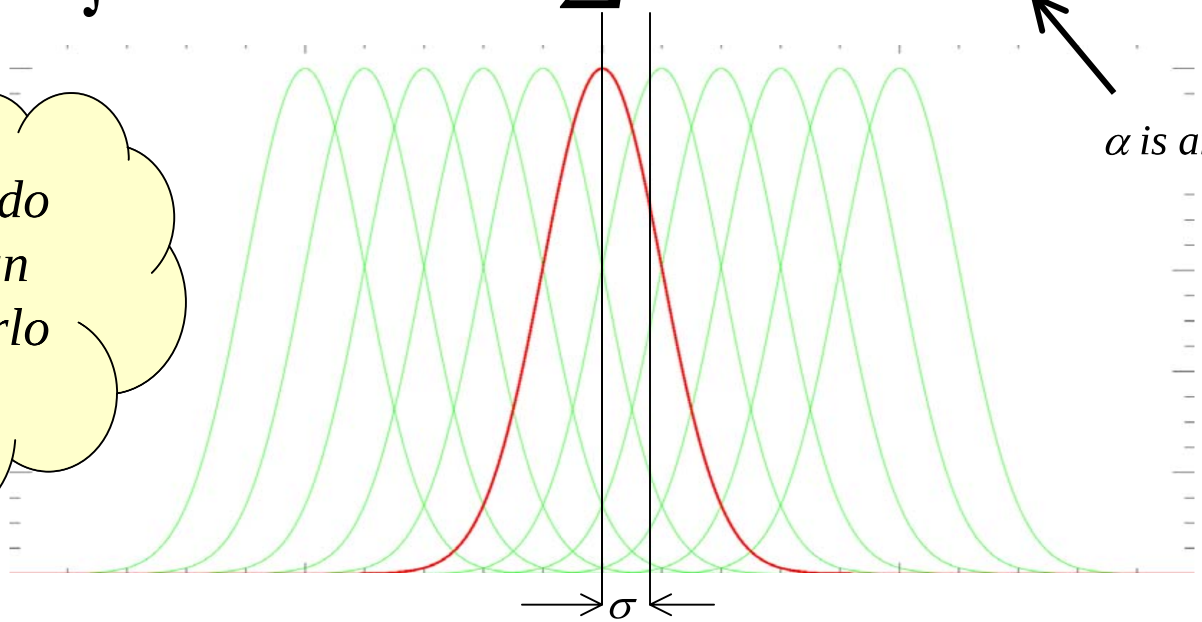
$g_\sigma(x) = \dfrac{1}{\sigma}\exp\left(\dfrac{-x^2}{2\sigma^2}\right)$ solves $\partial_t T = D\partial_{xx}^2 T$ with $\sigma = \sqrt{2Dt + \sigma_0^2}$.

Model : approximate T by summation of gaussians . . .

Consider $f(x) = \int dx'\ f(x')\ \delta(x - x') \rightarrow \sum \Delta x\ f_i\ g_\sigma(x - x_i)\ \alpha$

α is an overlap constant

hope is to do better than Monte Carlo method



By representing T as sum of gaussians, effect of diffusion is approximated.

$$\mathbf{x} = (x, y),\ \mathbf{\Gamma} = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{pmatrix}$$

In 2D, $\mathrm{T}(\mathbf{x}) = \sum_{i,j} T_{i,j} \exp\left[-\frac{1}{2} (\mathbf{x} - \mathbf{x}_{i,j})^T \cdot \mathbf{\Gamma}^2 \cdot (\mathbf{x} - \mathbf{x}_{i,j}) \right] \alpha.$

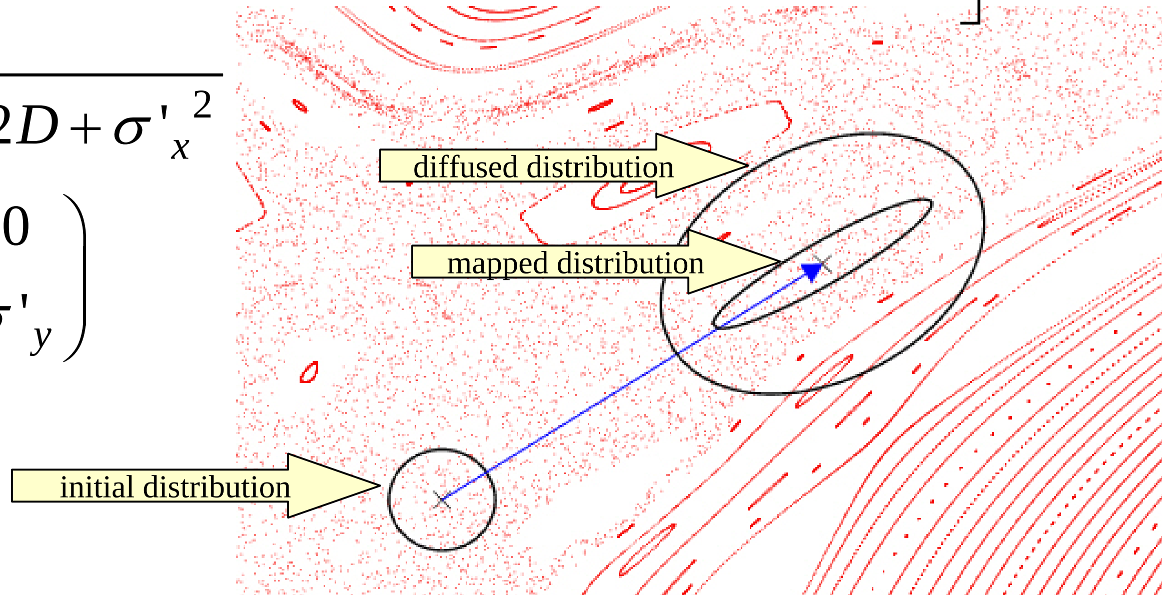
U *is rotation matrix*
W *is elongation matrix*

Under map, $\mathbf{x}' = \mathbf{F}\mathbf{x}$, gaussian is rotated & stretched

$$\mathrm{T}(\mathbf{x}) = \sum_{i,j} f_{i,j} \exp\left[-\frac{1}{2} (\mathbf{x} - \mathbf{x}'_{i,j})^T \cdot \mathbf{U} \cdot \mathbf{W}^{-1} \cdot \mathbf{\Gamma}^2 \cdot \mathbf{W}^{-1} \cdot \mathbf{U}^T \cdot (\mathbf{x} - \mathbf{x}'_{i,j}) \right] \alpha,$$

and then diffuses $\sigma'_x \to \sqrt{2D + \sigma'^{\,2}_x}$

where $\mathbf{\Gamma}' = \mathbf{\Gamma} \cdot \mathbf{W} = \begin{pmatrix} \sigma'_x & 0 \\ 0 & \sigma'_y \end{pmatrix}$

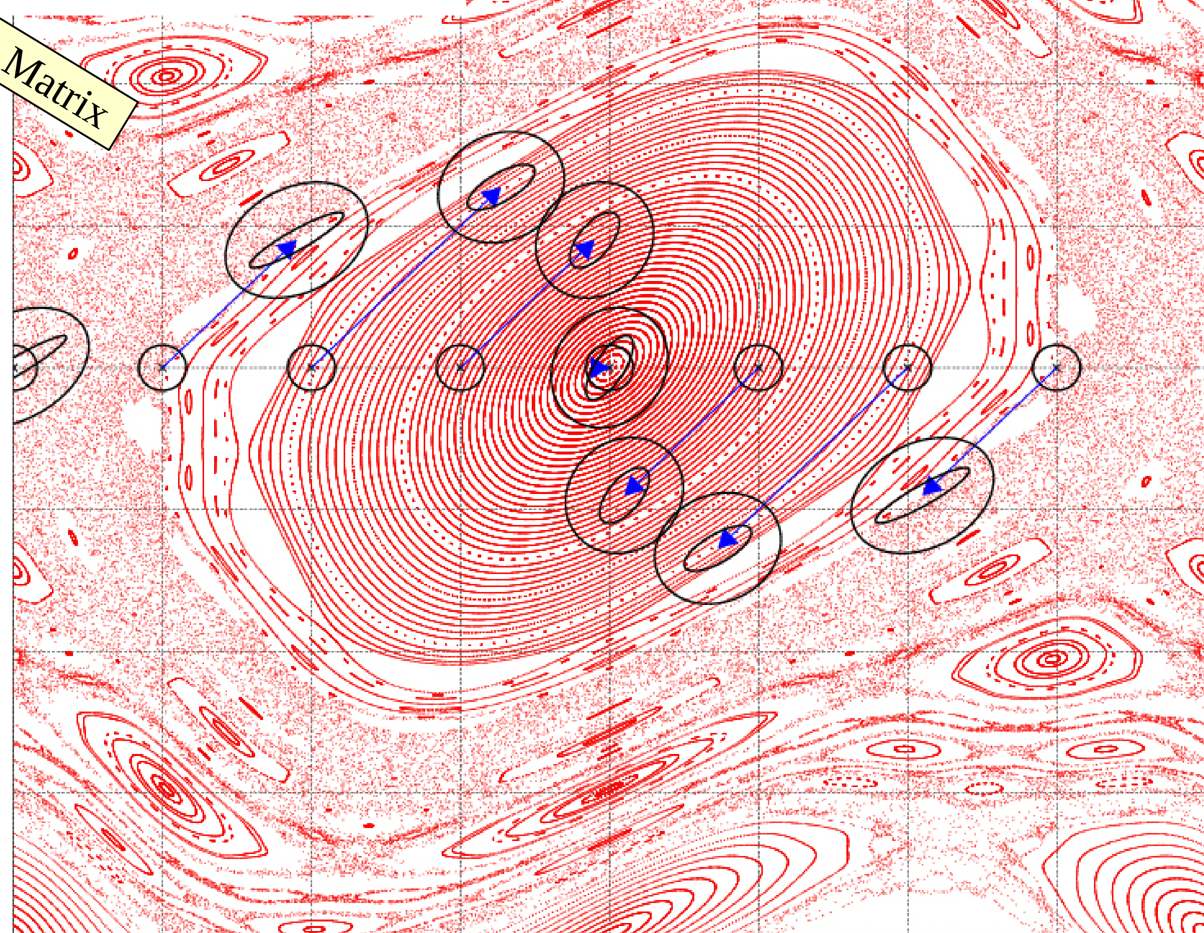


The model determines Temperature evolution

$$T_{i,j}^{n+1} = \sum_{i',j'} c_{i',j',i,j} T_{i,j}^{n} + \sum_{i',j'} c_{i',j',i,j} B_{i,j}$$

Coupling Matrix

Boundary (Source) Term



Linear System

$$\mathbf{T}^{n+1} = \mathbf{C}\ \mathbf{T}^{n} + \mathbf{B}$$

Steady state solved directly

$$\left(\mathbf{I} - \mathbf{C}\right)\mathbf{T} = \mathbf{B}$$ (use BiCGStab)

this is really advection-diffusion

$$\partial_t T = \mathbf{v} \cdot \nabla \mathbf{T} + \mathbf{D} \nabla^2 T$$

& operator splitting

An example Temperature profile is shown

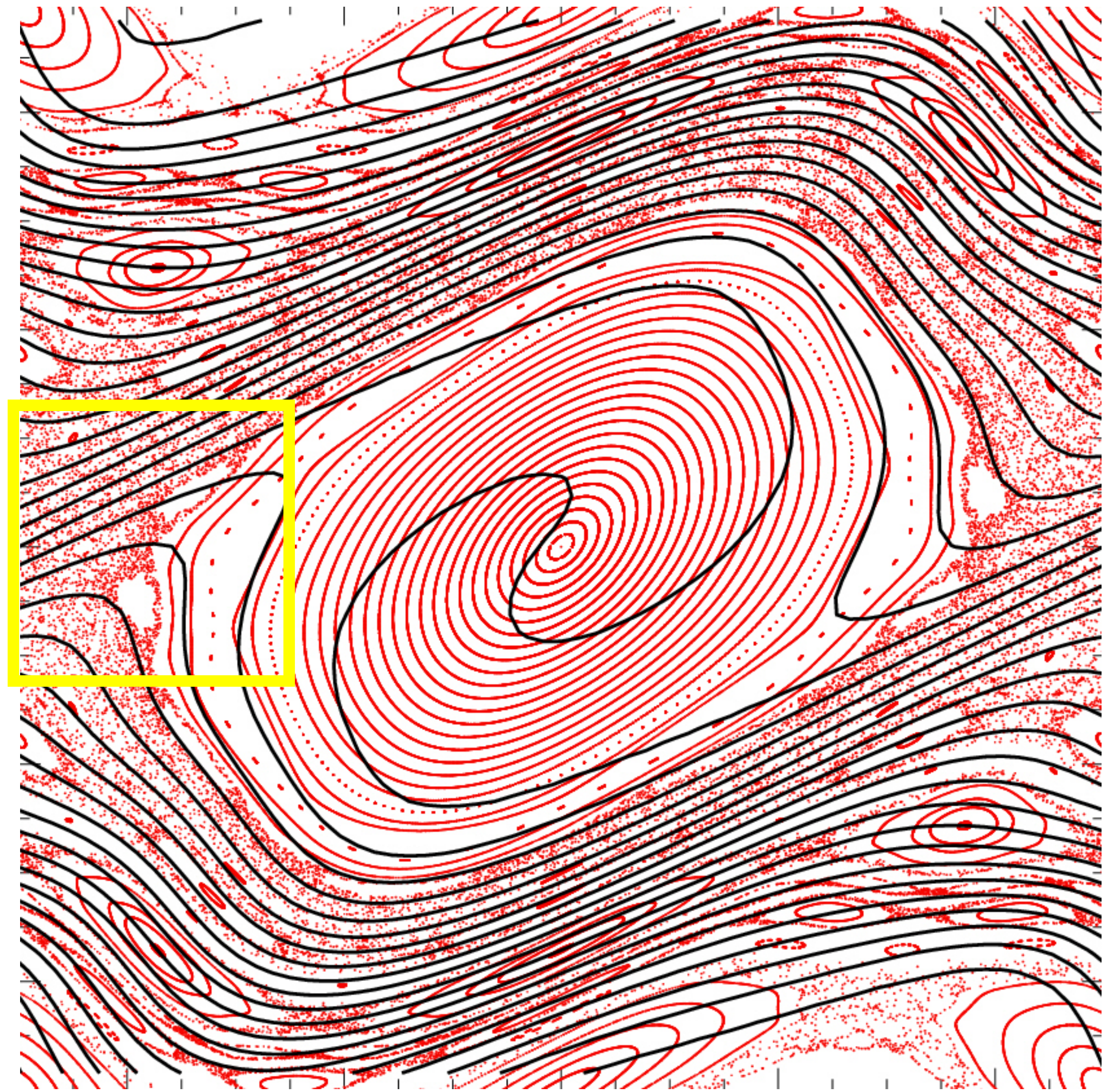
1) Steady state T is shown

boundary condition $T(r \le 1) = 1$

2) island width suff. large

so $\kappa_{||}(\mathbf{b}\cdot\nabla)^{2} \sim \kappa_{\perp}(\nabla_{\perp})^{2}$

unstable manifold

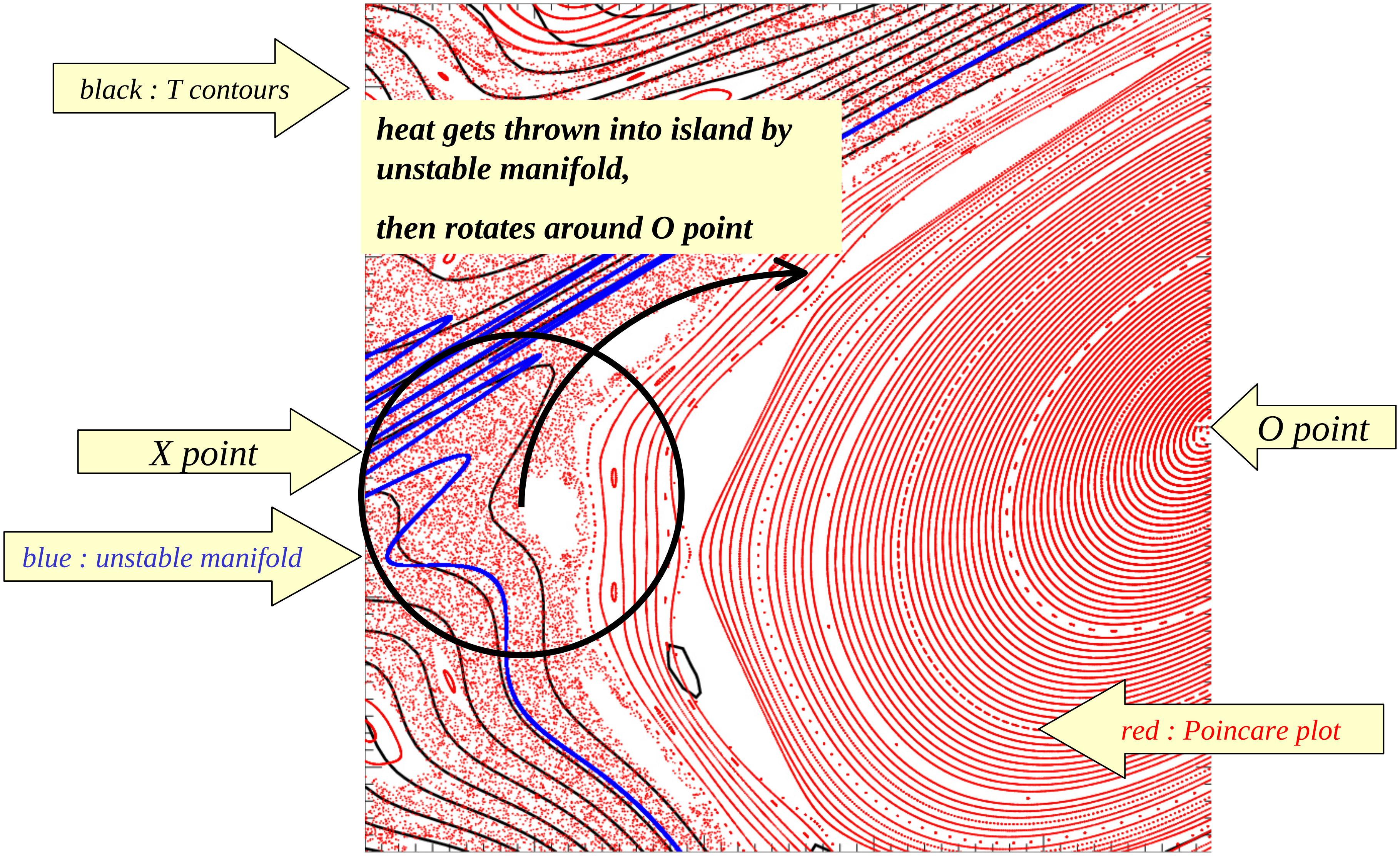


3) What is influencing T ?

here standard map is used

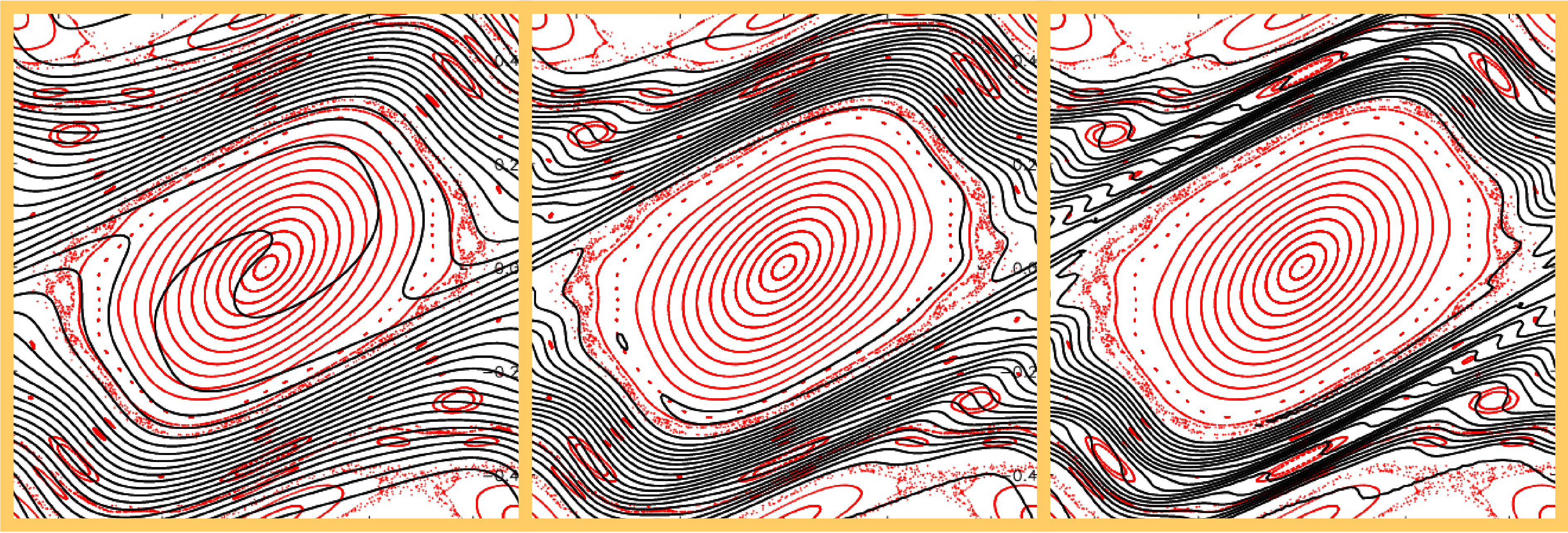
The unstable manifold W^u determines Temp profile

$z \in W^u \Rightarrow F^j z \to z_0$ as $j \to -\infty$, where z_0 is X point



Decreasing diffusion $\kappa_\perp$ leads to increased correlation with field

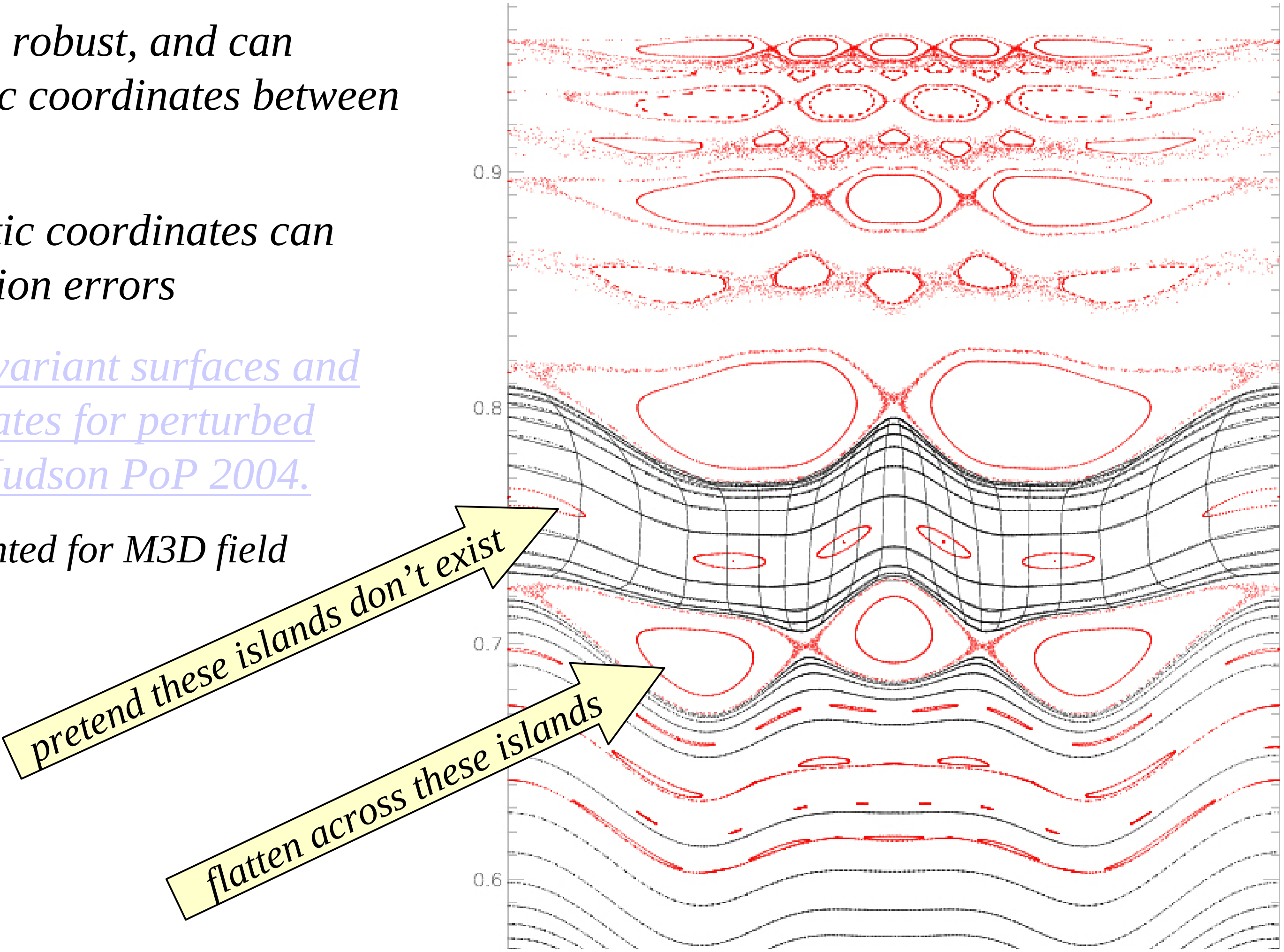
strong diffusion **weak diffusion**



1) *I am still working through the details of this method . .*
2) *The method should be directly extendable to 3D chaotic fields with arbitrary diffusion . .*
3) *The parallel motion is solved exactly, so this method may reduce numerical pollution of perpendicular transport*
4) *The model is similar to both the Monte Carlo method and to direct matrix methods . .*

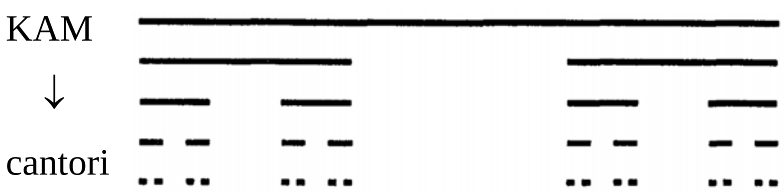
Now moving on to other approaches . .
Can use discrete magnetic coordinates

1) *KAM surfaces are robust, and can construct magnetic coordinates between islands*
2) *The use of magnetic coordinates can reduce discretization errors*
3) *[Destruction of invariant surfaces and magnetic coordinates for perturbed magnetic fields, Hudson PoP 2004.]*
4) *Procedure implemented for M3D field*



Can also use a Markov model for transport developed by MacKay, Meiss & Percival [PHYSICA D 13 (1-2): 55-81 1984]

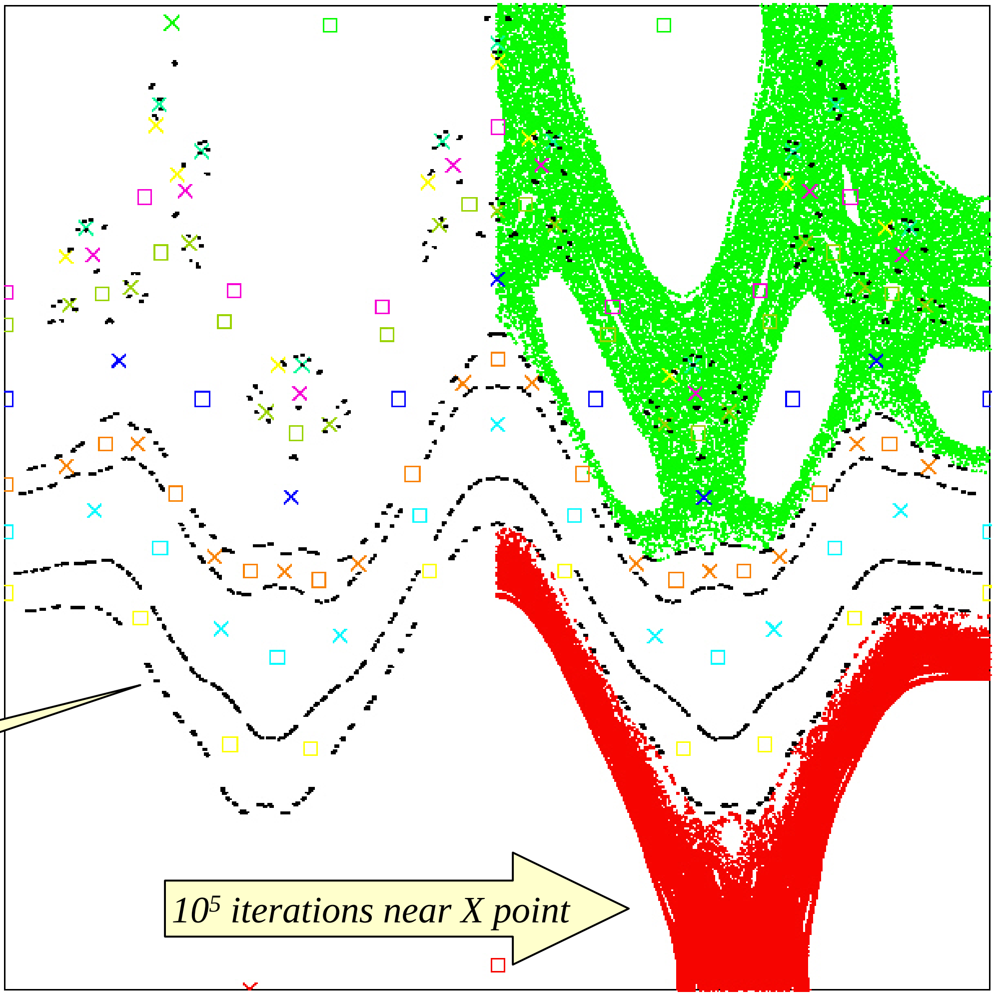
1) *KAM surfaces disintegrate into leaky Cantor-set tori=cantori*



2) *Can develop a transport model based on probability of jumping from region to region* $N_i' = \Sigma_j P_{i,j} N_j$

3) *Cantori are approximated by high order periodic orbits eg. Calculation of cantori for Hamiltonian flows, Hudson, Physical Review E, 2006.*

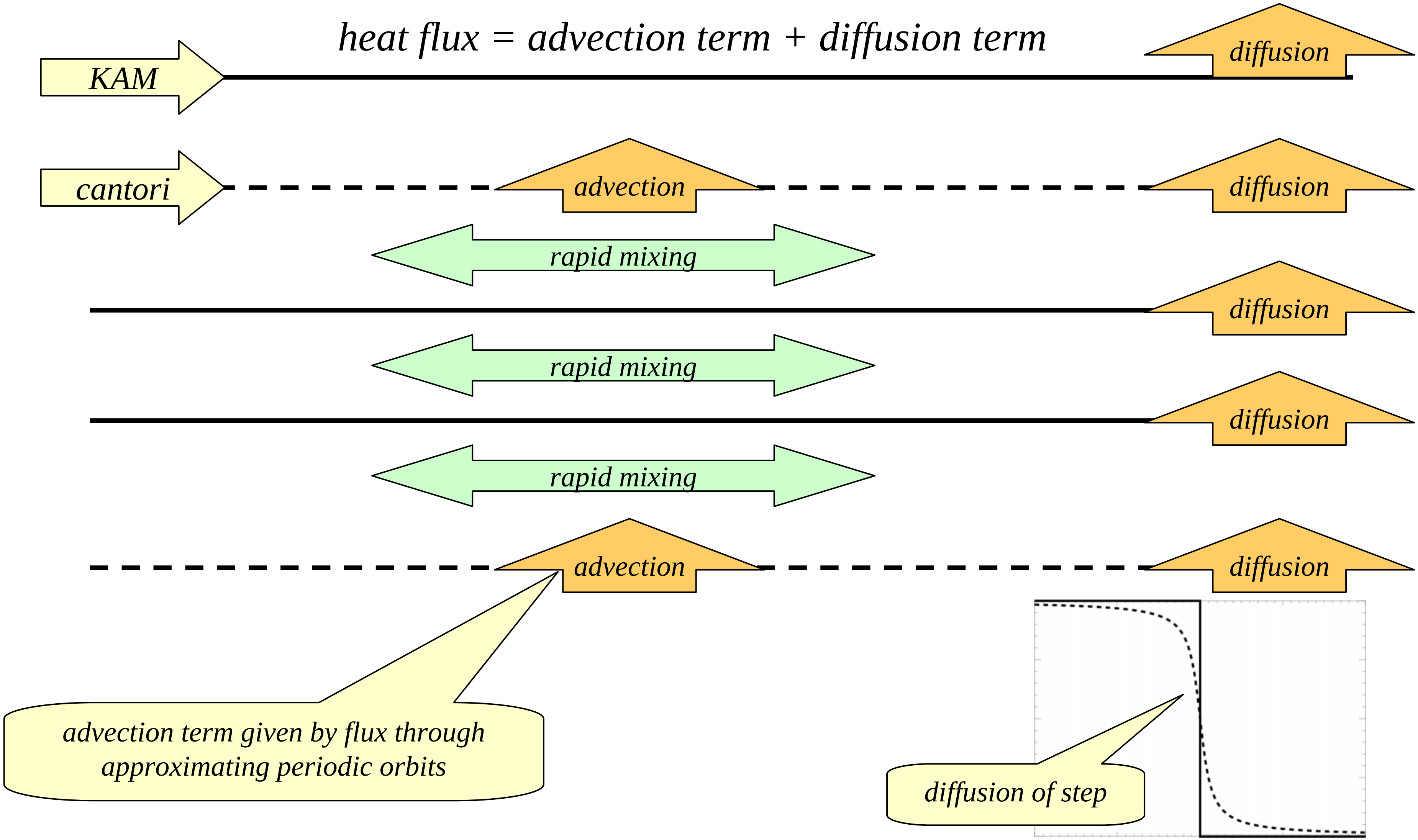
cantori can be very strong, partial barriers



A Markov model of advection-diffusion might look like this . .

Perhaps a stepped temperature profile . . ?

heat flux = advection term + diffusion term



And, of course, there is the full numerical simulation

1) *Such numerical simulations are probably the most reliable approach, but lack insight and need to be supported by theory .*

2) *The ideas presented on previous slides may enable a better initial guess for iterative solutions . .*

eg. Gunter et al., JCP 209, 354, 2005

$$\left.\frac{\partial}{\partial x}T\right|_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2(\Delta x)}\left((T_{i+1,j+1}+T_{i+1,j})-(T_{i,j+1}+T_{i,j})\right),$$

$$\left.\frac{\partial}{\partial y}T\right|_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2(\Delta y)}\left((T_{i+1,j+1}+T_{i,j+1})-(T_{i+1,j}+T_{i,j})\right),$$

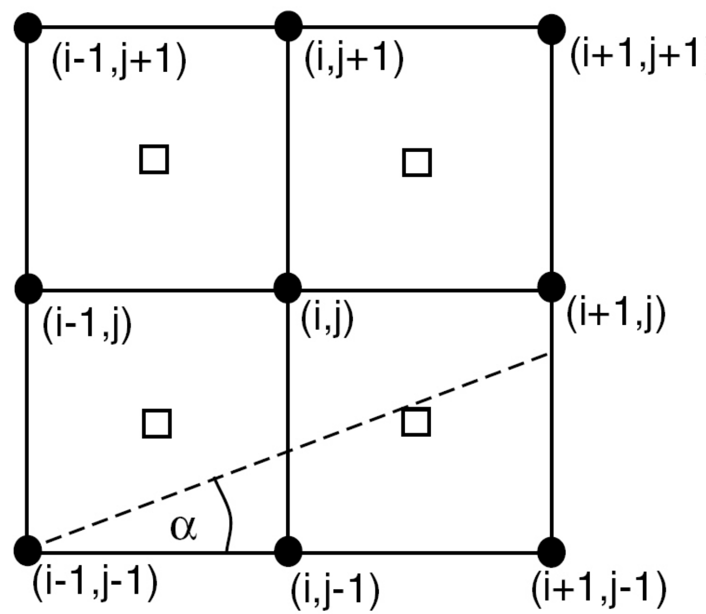
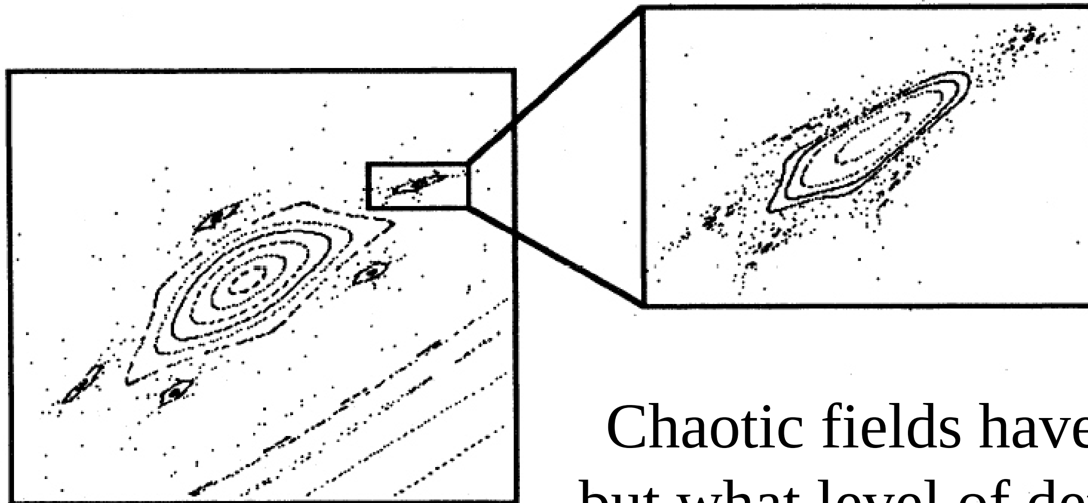


Fig. 1. Grid labelling and elementary cell.

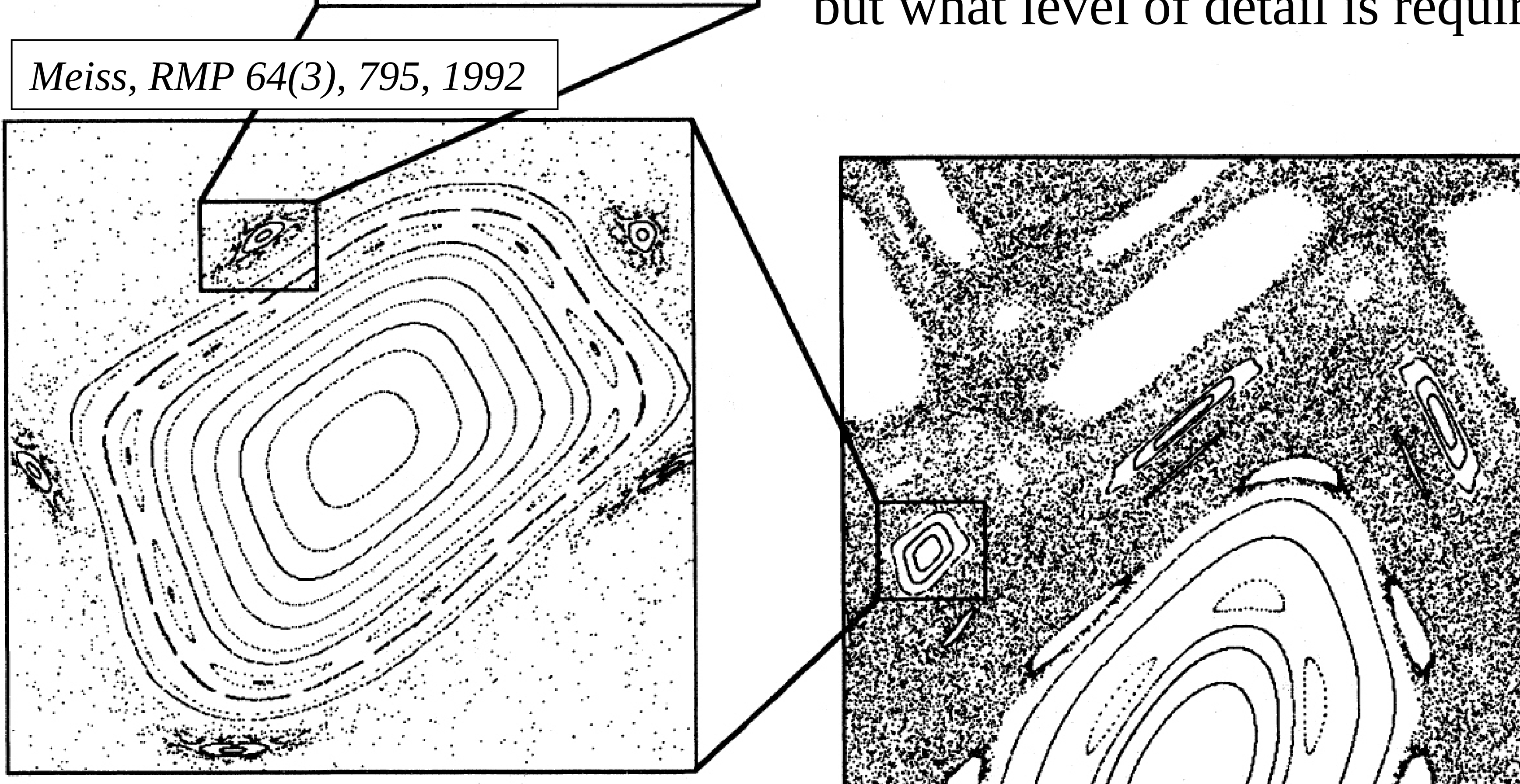
as $\frac{\kappa_{\parallel}}{\kappa_{\perp}} \to \infty,\ \mathbf{B}\cdot\nabla \mathrm{T} \to 0$

and the structure of **B**

determines T(**x**)



Chaotic fields have infinite detail,
but what level of detail is required ?



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