THE AUSTRALIAN NATIONAL UNIVERSITY

PRINCETON PLASMA PHYSICS LABORATORY

The MHD equilibrium problem in nonaxisymmetric toroidal plasma confinement systems

(as a novel KAM problem)

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1. Introduction

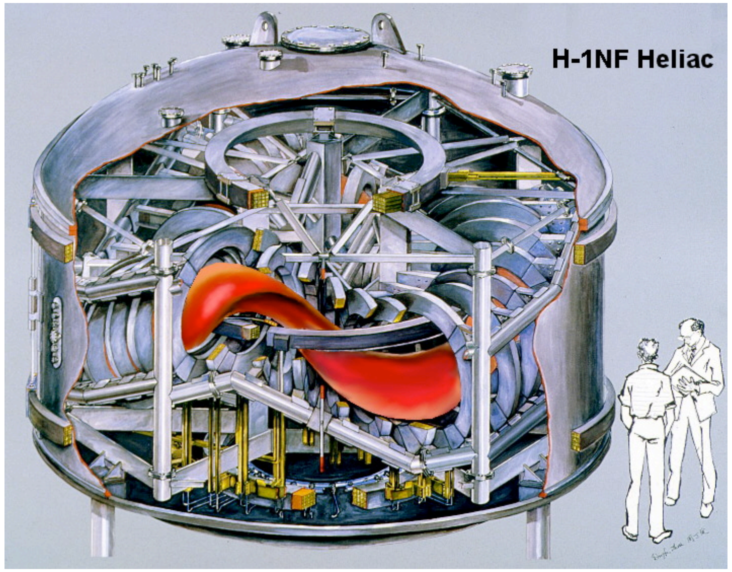
"Well, what <u>do</u> you say to a person who tells you he's working on a doughnut-shaped energy field?"

3D Toroidal plasma equilibrium

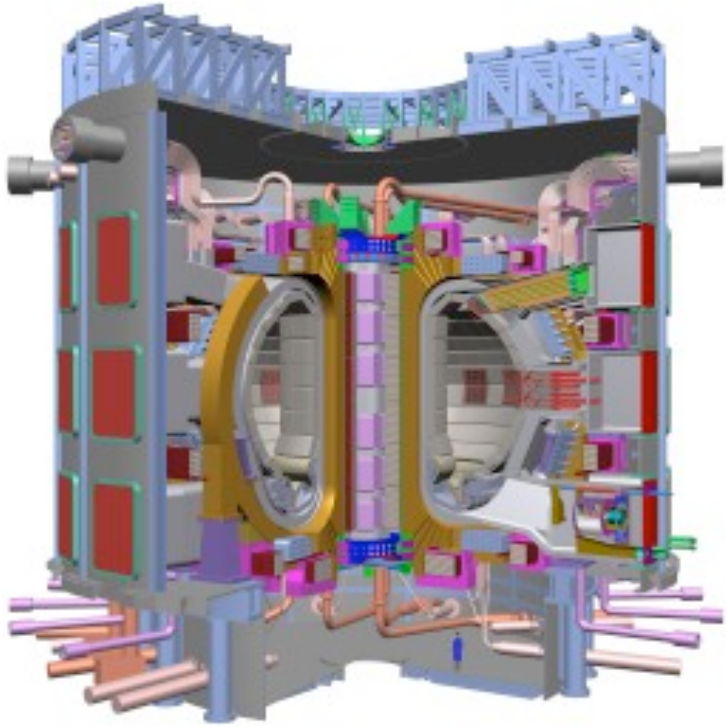
Good model for toroidal fusion plasma steady state is force balance for total pressure *p* combined with Ampère's law relating magnetic field **B** and current density **J**:

$$\nabla p = \mathbf{J} \times \mathbf{B}, \qquad \nabla \times \mathbf{B} = \mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0$$

EG Stellarators—intrinsically 3D, i.e. no continuous symmetry:



and Tokamaks, (*also* 3D due to coil ripple or instabilities):



Holy Grail — find constructive solution of 50-year-old problem

Toroidal Containment of a Plasma

Harold Grad Phys. Fluids **10**, 137 (1967)

Courant Institute of Mathematical Sciences, New York University, New York, New York

(Received 5 July 1966; final manuscript received 10 October 1966)

The question of plasma containment in a torus is much more complicated than in an open-ended mirror system. Serious questions arise of the nonexistence of flux surfaces, of noncontained particle drifts, and of nonexistence of self-consistent equilibria at small gyroradius.

Project Aims

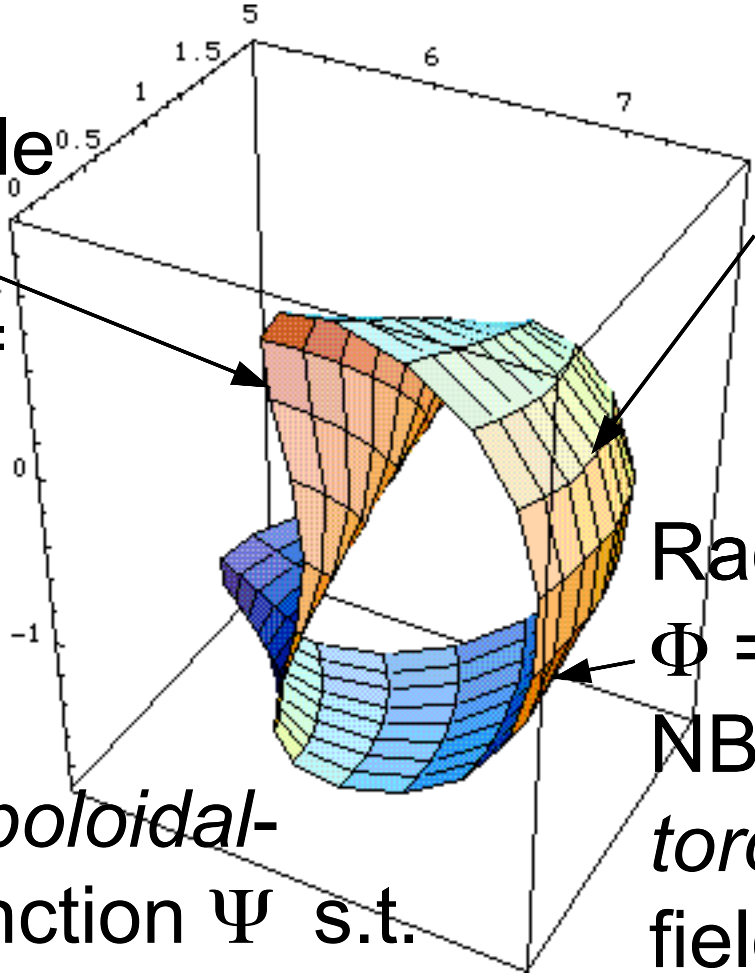
(1) design a convergent algorithm for constructing 3D equilibria,

- Find a mathematically well-posed formulation of problem and implement it numerically with aim to replace current unsatisfactory tools (e.g. VMEC code)
- quantify relationship between magnitude of departure from axisymmetry and existence of 3D equilibria—both fundamental and practical problem
- provide a better computational tool for rapid design and analysis

(2) explore relationship between ideal MHD stability of multiple interface model and *internal transport barrier* formation

Magnetic field **B** in curvilinear toroidal coordinates

Toroidal angle ζ = const curve on Φ = const torus.



Poloidal angle θ = const curve on Φ = const torus.

Radial coordinate Φ = const surface. NB Φ is *average toroidal* flux but field lines do *not* necessarily lie within this torus.

Also define a *poloidal*-flux/stream function Ψ s.t.

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{A} = \Phi \nabla \theta - \Psi(\Phi, \theta, \zeta) \nabla \zeta$$

Field-line flow as a $1^1/_2$ DoF Hamiltonian system

$$\frac{d\theta}{d\zeta} = \frac{\mathbf{B}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\zeta} = \frac{\partial\Psi}{\partial\Phi}$$

$$\frac{d\Phi}{d\zeta} = \frac{\mathbf{B}\cdot\nabla\Phi}{\mathbf{B}\cdot\nabla\zeta} = -\frac{\partial\Psi}{\partial\theta}$$

I.e. magnetic field-line flow is a *non-autonomous Hamiltonian system* with $\Psi(\Phi,\theta,\zeta)$ the Hamiltonian, θ the generalized coordinate, Φ the canonically conjugate momentum, and ζ the “time”.

2. Problems with 3D equilibria

Prob. I — they are generically *non-integrable*

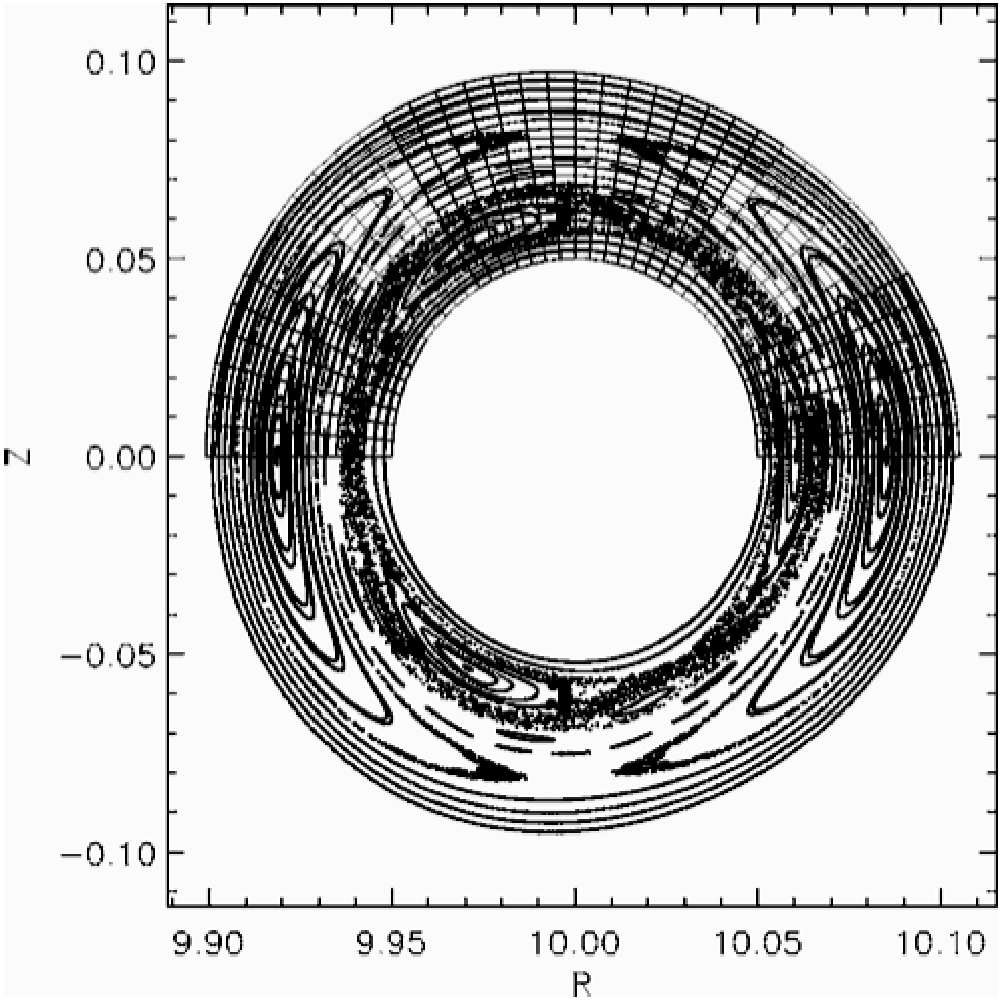
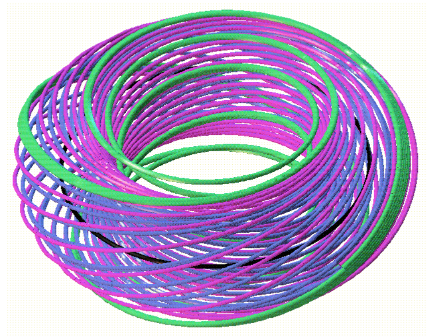


FIG. 7. Poincaré plot of Beltrami field: perturbation of outer boundary δ $=0.0030$, with Fourier resolution $M=7$, $N=2$. Shown in the upper half plot is the coordinates.

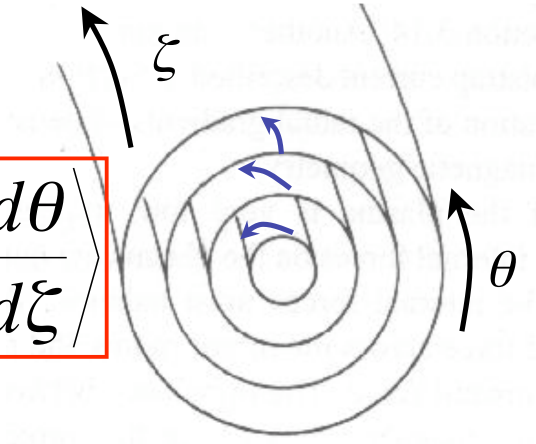
But *design* so there exist some invariant tori (magnetic surfaces)

- **B** field lines are everywhere tangential to a *magnetic surface*, winding around in a helical fashion



- Define winding number or *rotational transform*:

$$\boxed{\iota = \lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2}\frac{d\theta}{d\zeta}d\zeta \equiv \left\langle \frac{d\theta}{d\zeta}\right\rangle}$$



▪ ι = irrational: **B** ergodically passes through all points in magnetic surface

▪ ι = rational (m/n) : **B** lines close on each other.

(In tokamaks, often define $q = 1/\iota$)

Prob. II: 3D MHD equilibria have *current singularities* if $\nabla p \neq 0$

Current perp.to **B** is OK: $\mathbf{J}\times\mathbf{B} = \nabla p \;\Rightarrow \mathbf{J}_\perp = \mathbf{B}\times\nabla p / B^2$

Problem is parallel current: $\mathbf{J}_{||} \equiv \sigma\mathbf{B}$

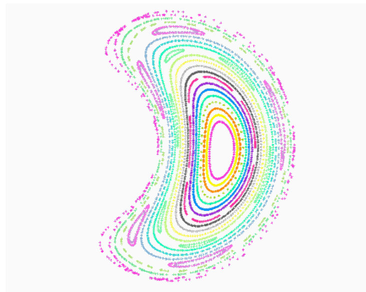
$$\nabla\cdot\mathbf{J} = 0 \;\Rightarrow\; \mathbf{B}\cdot\nabla\sigma = -\nabla\cdot\mathbf{J}_\perp = -\nabla\cdot\left(\mathbf{B}\times\nabla p/B^2\right)$$

Formal solution is: $\boxed{\sigma = \lambda - \left(\mathbf{B}\cdot\nabla\right)^{-1}\nabla\cdot\left(\mathbf{B}\times\nabla p/B^2\right)}$

where λ is an arb. const. along field lines (e.g. over *finite volume* in a chaotic region, filled ergodically by single **B**-line).

But **B**.∇ is a very singular operator: for general 3D fields, σ *blows up* at each rational magnetic surface unless

$$\nabla\cdot\left(\mathbf{B}\times\nabla p/B^2\right) = 0$$



Proposed solution: *Stepped-pressure Beltrami equilibria*

To ensure a mathematically well-defined $J_{||}$, we set $\nabla p = 0$ over finite regions $\Rightarrow \nabla\times B = \lambda B$, λ = const (*Beltrami field*) separated by assumed *invariant tori*.

Cf. Grad, *Toroidal Containment of plasma,* Phys. Plas. **10** (1967)

> "In order to have a static (3D) equilibrium, $p'(\Phi)$ must be zero in the neighborhood of **<u>every</u>** rational rotational transform, and *flux surfaces must be relinquished*"

- *Pros*
  - Beltrami eqn. is a linear elliptic PDE, solvable by variety of methods even if **B** has chaotic regions
  - Has already been partially investigated mathematically [e.g. Bruno & Laurence, Comm. Pure Appl. Math. **XLIX**, 717 (1996)]
- *Cons* (?)
  - Pressure profile not differentiable (**but** may approximate a smooth profile arbitrarily closely, limited only by existence of invariant tori)

Force balance on invariant tori

Pressure discontinuous: $[[p]] \neq 0$ (where $[[\bullet]]$ is jump across an invariant torus), but *total* pressure, magnetic plus kinetic, is continuous: $[[p+B^2/2]] = 0$

$\Rightarrow$ δ-function ∇p

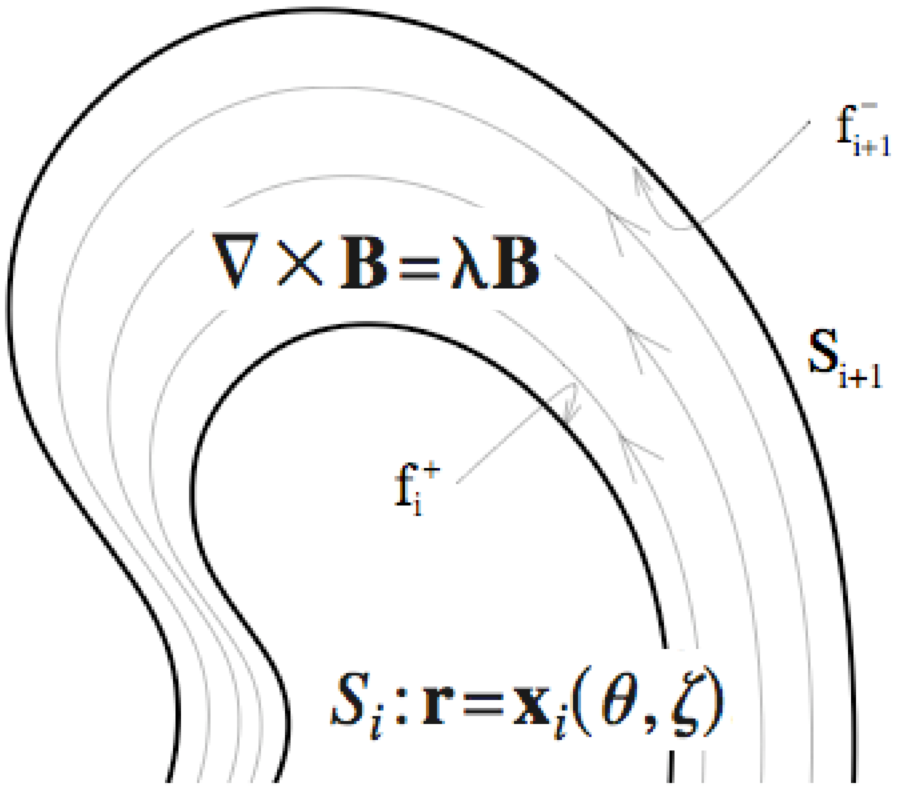
$\Rightarrow$ sheet current $\mathbf{J}_\perp$

$\Rightarrow$ discontinuity in **B** (both magnitude & direction)

$\Rightarrow$ winding number ι not necessarily same on either side of invariant torus (*not* standard KAM problem)

Equilibrium construction method I:

A: solution of 3D Beltrami eqn. by method of lines/shooting method



Coordinates:
Assuming S_i and S_{i+1} are given, construct curvilinear coord. system, s, θ, ζ where θ, ζ are resp. gen. poloidal & toroidal angles and s interpolates between S_i and S_{i+1} —

$$\mathbf{x} = s\mathbf{x}_{i+1} + (1-s)\mathbf{x}_i$$

$$\mathbf{x}_j = R_j(\theta, \zeta)\hat{r} + Z_j(\theta, \zeta)\hat{z}$$

IA contd.: radial integration of poloidal & toroidal components of **B**

Write **B** in covariant form:

$$\mathbf{B} = B_s \nabla s + B_\theta \nabla \theta + B_\zeta \nabla \zeta$$

Can express B_s in terms of B_θ & B_ζ:

$$B_s = \frac{B^s - g^{s\theta} B_\theta - g^{s\zeta} B_\zeta}{g^{ss}}$$ Where (if $\lambda \neq 0$): $$B^s = \frac{\partial_\theta B_\zeta - \partial_\zeta B_\theta}{\lambda \sqrt{g}}$$

So just need to solve for θ & ζ components. Can be done by integrating radially along curves θ, ζ = const :

$$\partial_s B_\theta = \partial_\theta B_s + \lambda \sqrt{g}(g^{s\zeta} B_s + g^{\theta\zeta} B_\theta + g^{\zeta\zeta} B_\zeta)$$

$$\partial_s B_\zeta = \partial_\zeta B_s - \lambda \sqrt{g}(g^{s\theta} B_s + g^{\theta\theta} B_\theta + g^{\zeta\theta} B_\zeta)$$

Construction IA cont.: satisfying tangentiality at S_i & S_{i+1}

B must be tangential to s = 0 and s = 1 surfaces: **n**·**B**= 0

Achieve on s = 0 by using surface magnetic potential f such that **B** = $\nabla_s f$ (exists because **n**· ∇X**B** = 0), where *surface gradient* is defined as $\nabla_s = \nabla - \boldsymbol{n}\frac{\partial}{\partial n}$

Use Fourier representations:

$$f = I\theta - G\zeta + \sum_{m,n} f_{mn} \sin(m\theta - n\zeta)$$

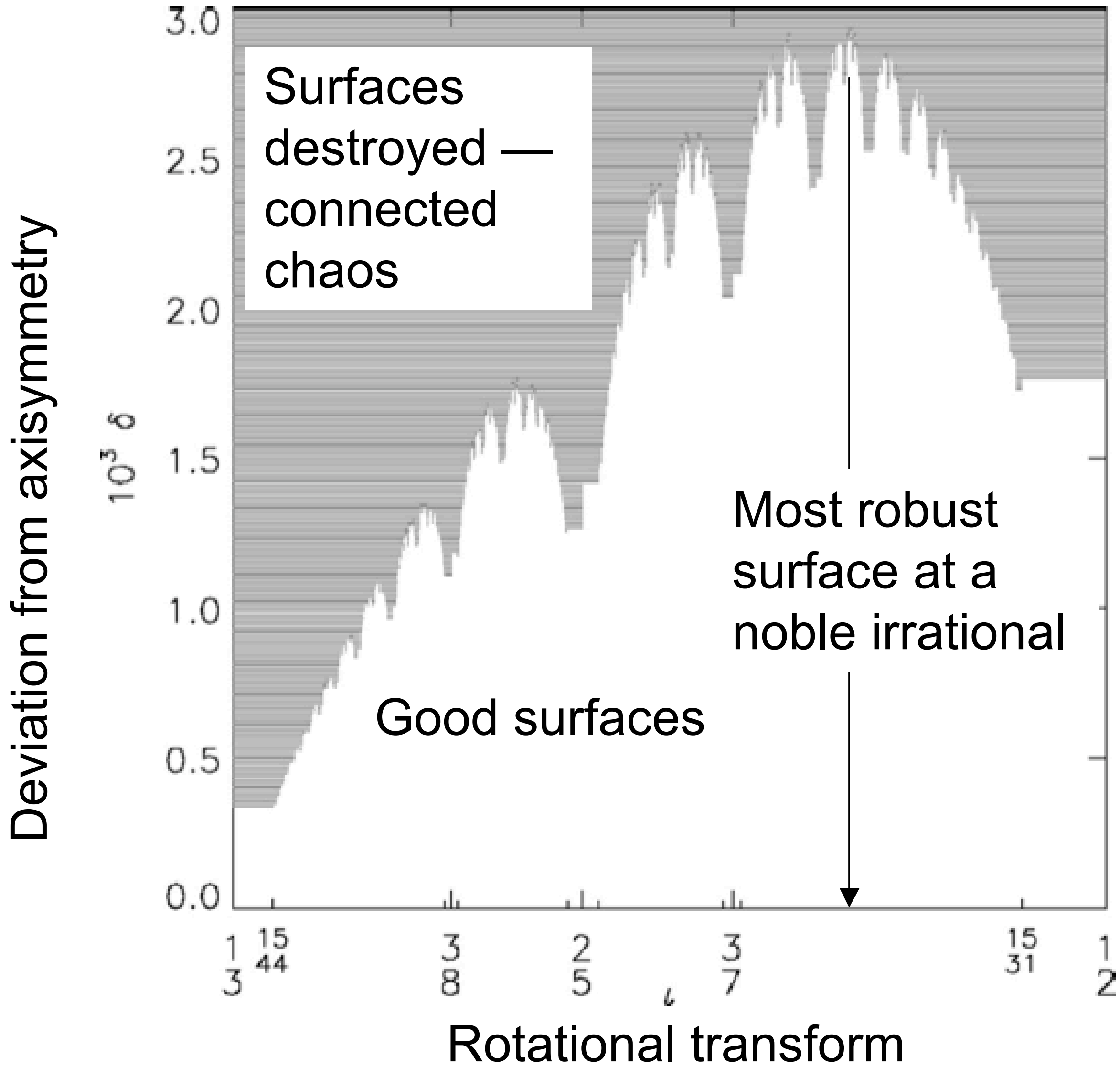
$$B_\theta = \sum_{m,n} B_{\theta,m,n}(s)\cos(m\theta - n\zeta) \qquad B_\zeta = \sum_{m,n} B_{\zeta,m,n}(s)\cos(m\theta - n\zeta)$$

Continue **B** from S_i by **shooting method**, satisying **n**·**B**= 0 at S_{i+1} , to required tolerance, by iteratively adjusting f_{mn}

Construction IA conclusion:

- Published in S.R. Hudson, M.J. Hole and R.L. Dewar, Phys. Plasmas **14**, 052505 (2007)
- Have verified above construction of 3D Beltrami fields works (though not elegant) for test problem: Beltrami equation between two specified toroidal surfaces
- Have studied non-standard eigenvalue problem posed by specifying ι on bounding surfaces
- Have used Greene's method for finding KAM surfaces within the region: nest irrational no. by sequence of rationals, using Farey construction, giving rise to sequence of islands. If O-points of islands exist in limit, then KAM surface exists:

KAM invariant tori at irrational ι : fractal dependence on winding no.



Construction method I contd:

B: Hamilton–Jacobi equation for *f*

Can relate surface magnetic potentials $f^{+/-}$ such that $\mathbf{B}^{+/-} = \nabla_s f^{+/-}$ on *either side* of a pressure barrier *S* by solving force balance as a PDE for f^+ given f^- (& vice versa?) :

$$\left(\nabla_s f^+\right)^2 = \left(\nabla_s f^-\right)^2 + [[p]]$$

This is a Hamilton–Jacobi equation — characteristics are Hamiltonian orbits, with a purely *geometric* Hamiltonian. This is a *completely different* problem from the field-line flow problem, yet it leads to the same conclusion: the **rotational transforms on either side of pressure barrier must be irrational.**

Construction II: variational method

A: (Woltjer–)Taylor relaxation principle

Minimize total energy: $W = \int_P \left(\frac{B^2}{2} + \frac{p}{\gamma - 1} \right) d\boldsymbol{\tau}$

Under *constraint of total helicity* : $K_0 = \frac{1}{2} \int_P (\mathbf{A} \cdot \mathbf{B}) d\boldsymbol{\tau}$

Euler–Lagrange eqns. for $\delta F = 0$: $F = W - \lambda K_0$

naturally give Beltrami equation: Lagrange multiplier

$$\nabla \times \mathbf{B} = \lambda \mathbf{B}$$

(To check whether *F* really is a minimum, need second variation — stability criterion similar to resistive stability, since reconnection is allowed.)

Construction method IIA continued:

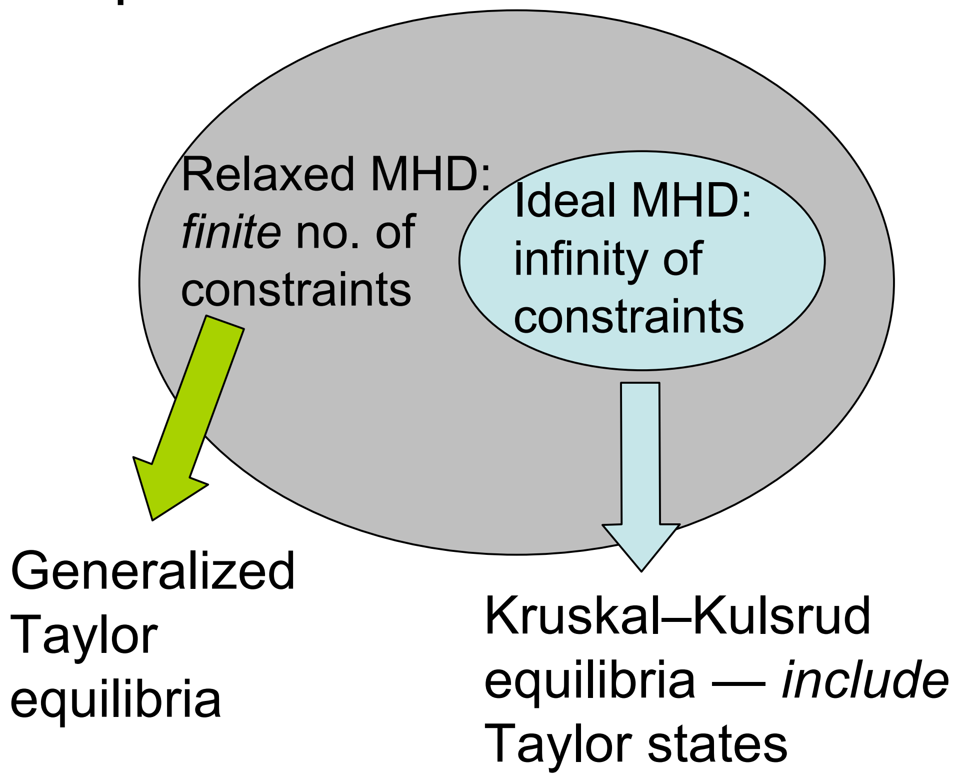
Generalization of Taylor principle — general idea

Cf. A. Bhattacharjee and R.L. Dewar, Phys. Fluids **25**, 887 (1982)
Energy principle with global invariants

Idea: Extremize total energy $W = \int_{P \cup V} \left(\frac{B^2}{2} + \frac{p}{\gamma - 1} \right) d\tau$

subject to *finite* number of ideal-MHD constraints (*unlike* ideal MHD where flux and entropy are “frozen in” to each fluid element — *infinite* no. of constraints).
Require constraints to be a **subset** of the **ideal-MHD** constraints, so generated states are ideal equilibria:

Spaces of allowed variations:

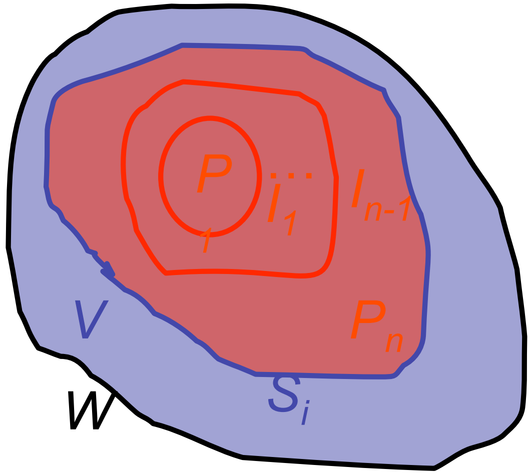


Construction method IIA continued:

Generalization of Taylor principle — specific idea:

Assume invariant tori S_i act as ideal MHD barriers to relaxation, so that Taylor constraints are localized to subregions.

New system comprises:



- N plasma regions P_i in relaxed states.
- Regions separated by ideal MHD barrier S_i.
- Enclosed by a vacuum V,
- Encased in a perfectly conducting wall W

Constraints:

potential energy functionals: $W_i = \int_{R_i} \left(\frac{B_i^2}{2} + \frac{P_i}{\gamma - 1} \right) d\tau^3$

helicity functionals: $K_i = \int_{V_i} \mathbf{A} \cdot \mathbf{B} \; d\tau$

mass/entropy functionals: $M_i = \int_{R_i} P_i^{1/\gamma} d\tau^3$

toroidal and poloidal fluxes: Φ_i and Ψ_i

Construction method II

B Hamiltonian trial function

Use representation

$$\boxed{\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{A} &= \Phi\nabla\theta - \Psi(\Phi,\theta,\zeta)\nabla\zeta\end{aligned}}$$

with Ψ,Φ, and θ as unknown fields (ζ being assumed a prescribed coordinate, say cylindrical polar angle ϕ).

Or, rather, use inverse representation,

$$\phi = \zeta$$

$$\mathbf{r} = R(\Phi,\theta,\zeta)\hat{r}(\phi) + Z(\Phi,\theta,\zeta)\hat{z}$$

Construction method II
B Hamiltonian trial function contd.

- Unlike usual KAM problem, the Hamiltonian Ψ is not prescribed — it is an *unknown* to be solved for.
- Furthermore, Ψ,Φ, and θ are *not unique* because arbitrary canonical transformations can be performed in the regions between the invariant tori, where action-angle coordinates do not in general exist
- Suggests using an extra principle to fix the representation, e.g. *quadratic flux minimization*

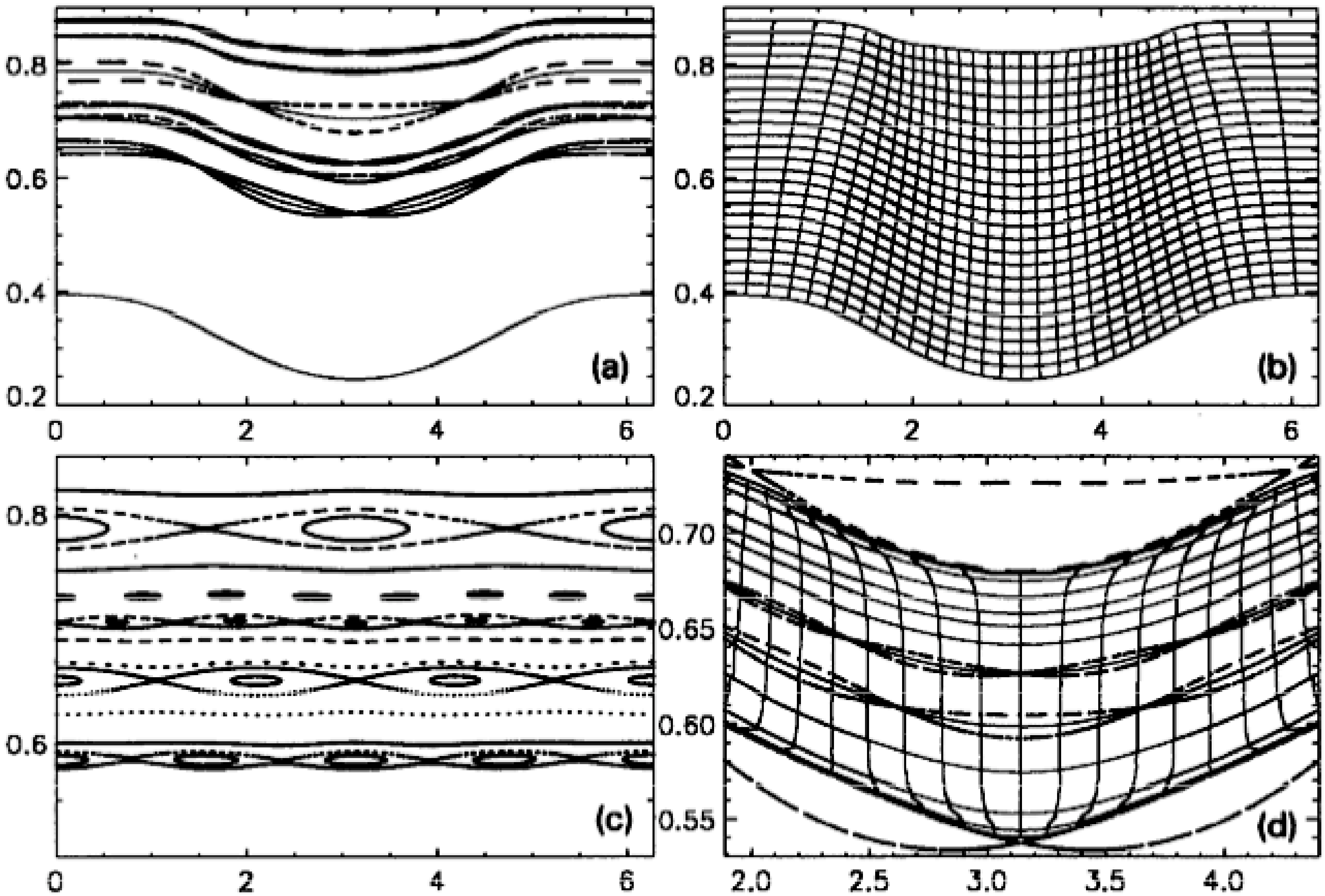
Quadratic flux minimization

- R.L. Dewar & J.D. Meiss Physica D **57**, 476 (1992); R.L. Dewar & A.B. Khorev Physica D **85**, 66 (1995) — area-preserving twist maps
- S.R. Hudson & R.L. Dewar Phys. Lett. A **226**, 85 (1997) — for magnetic fields: define field-line action $S = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$
- Minimize square of "action gradient" $\delta S/\delta\theta$

$$\delta S = \int_{\theta_1} \frac{\delta S}{\delta \theta} \delta\theta \, \mathrm{d}\zeta = \int_{\theta_1} \frac{\boldsymbol{B} \cdot \boldsymbol{n}}{\boldsymbol{n} \cdot \nabla\theta \times \nabla\zeta} \delta\theta \, \mathrm{d}\zeta .$$

Action-angle coordinates for *nonintegrable* field

S.R. Hudson, R.L. Dewar / Physics Letters A 247 (1998) 246–251



Conclusion

- Hamilton-Jacobi equation for force balance across interfaces relates directly to KAM $\Rightarrow$ irrational ι, but defines a direction of "information transfer" $\Rightarrow$ method of lines for Beltrami equation: *inappropriate* for an elliptic problem
- Variational approach is *a priori* more appropriate, but, to relate to KAM, need to find field-line Hamiltonian in inverse representation: Beltrami equation becomes nonlinear and need to fix nonuniqueness, either using Meiss' quadratic flux or minimizing $\int (\nabla\zeta \times \nabla\tilde{\Psi})^2 d^3x$
- Second variation of free energy, $\delta^2 F$, is important both for optimization approach above and for stability calculations (Hole *et al* 2007 — cylindrical studies)