Temperature Contours and Ghost Surfaces for Chaotic Magnetic Fields

2008 International Sherwood Fusion Theory Conference
and Center for Extended Macroscopic Modeling
2008, Boulder, CO.

Stuart Hudson

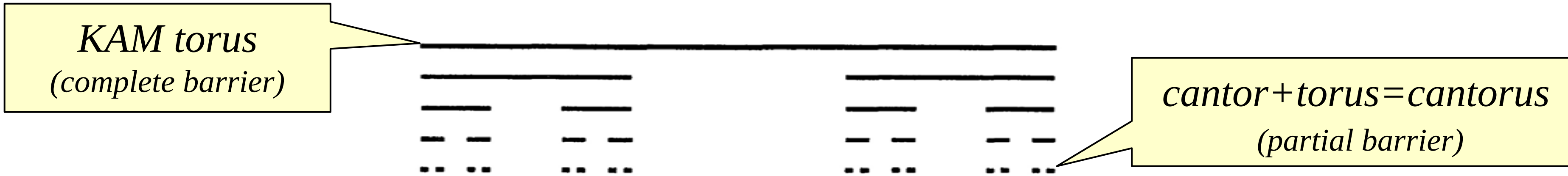
After destruction of the KAM surfaces, the cantori severely inhibit field line flow, and thus present barriers to heat transport in chaotic fields.

Ghost surfaces are constructed from the cantori by "filling in the holes", and coincide closely with isotherms.

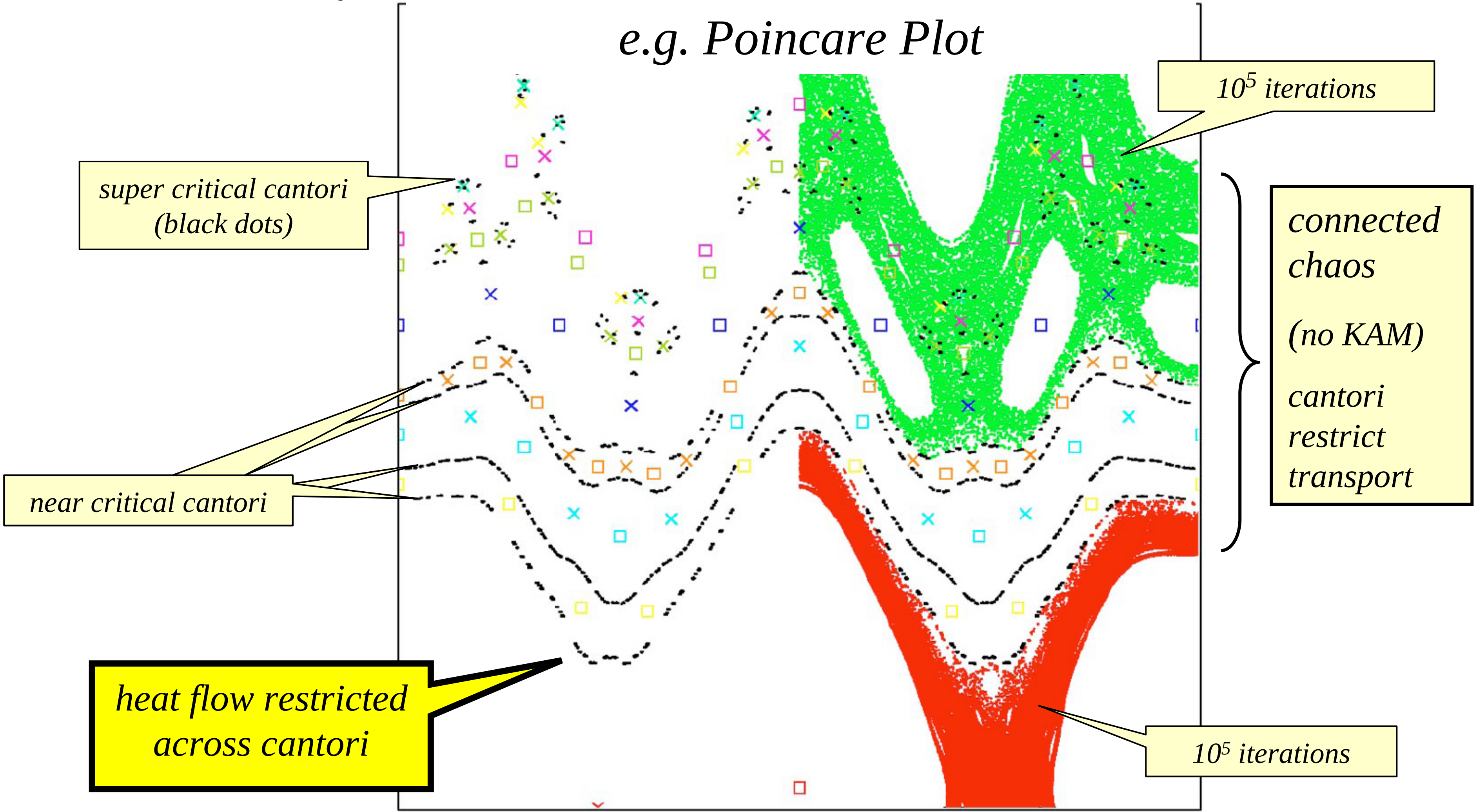
Anisotropic transport is dominated by structure of magnetic field

- Coordinates adapted to the structure of the magnetic field, magnetic coordinates, provide elegant description of plasma dynamics and can enhance numerical accuracy.
- Magnetic coordinates can be constructed globally when the field lines lie on nested toroidal surfaces.
  - The temperature takes the simple form $T=T(\psi)$, where ψ labels flux surfaces;
- Error fields, non-axisymmetry, instabilities . . . result in partially chaotic fields.
  - The temperature takes the general form $T=T(\psi,\theta,\phi)$;
- For chaotic fields, the *cantori* (broken KAM surfaces) play an important role in restricting heat transport.
  - cantori are invariant sets, with irrational transform; cantorus = KAM surface with gaps/holes;
  - cantori are approximated by high-order minimizing periodic orbits;
- Coordinates adapted to the cantori, called chaotic coordinates, recover a simple description of the temperature.
  - $T\approx T(s)$, where s labels chaotic coordinates;

Cantori restrict transport in chaotic regions



- *cantori are leaky, but can severely restrict transport*



Magnetic field lines (in particular the cantori), are determined from Lagrangian variational principles

Magnetic field lines, $\mathbf{B} = \nabla \times \mathbf{A}$, are stationary curves C of the action integral $S = \int_C \mathbf{A.dl}$,

Canonical form : $\mathbf{A} = \psi \nabla \theta - \chi \nabla \phi$ and $\chi(\psi,\theta,\phi) = \frac{1}{2}\psi^2 + \sum \chi_{m,n}(\psi)\cos(m\theta - n\phi)$.

Setting $\delta S = 0$ gives the field lines $\frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{\partial \chi}{\partial \psi} = \frac{B^\theta}{B^\phi}$ $\qquad$ $\frac{\mathrm{d}\psi}{\mathrm{d}\phi} = -\frac{\partial \chi}{\partial \theta} = \frac{B^\psi}{B^\phi}$

Periodic orbits can be located by

1) field line following;

2) adjusting a *trial curve* to find an extremum of S.

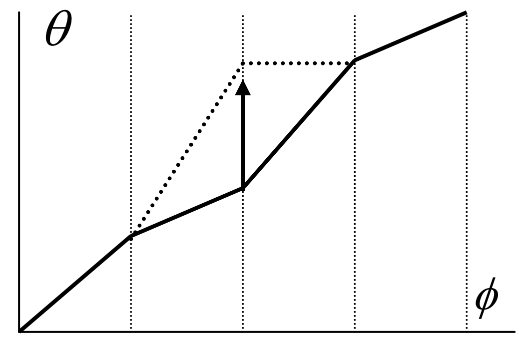
sensitive to exponential increase of integration error in chaotic fields;

robust, efficient;
need to specify trial curve

The simplest representation of a trial curve is piecewise-linear

Trial Curve $\theta(\phi),\ \psi = \dot{\theta}(\phi)$

Linear segment $\phi \in \left[\phi_i, \phi_{i+1}\right],\ \theta = \theta_i + \psi_i(\phi - \phi_i)\ ,\ \psi_{\mathrm{i}} = (\theta_{i+1} - \theta_i)/\Delta\phi$



Action integral integrated piecewise analytically

$$\text{Action } S = \sum_i S_i(\theta_i, \theta_{i+1}) \ \ ; \ \ S_i = \int_{\phi_i}^{\phi_{i+1}} (\psi\dot{\theta} - \chi) d\phi = \frac{1}{2}\psi_i^2 + \sum_{m,n} \chi_{m,n}(\psi_i) \left[\frac{\sin(m\theta - n\phi)}{m\dot{\theta} - n} \right]_{\phi_i}^{\phi_{i+1}}$$

Action Gradient:

To find extremal curves $\dfrac{\partial S}{\partial \theta_i} = \partial_2 S_{i-1}(\theta_{i-1}, \theta_i) + \partial_1 S_i(\theta_i, \theta_{i+1}) = 0,$ use Newton's method.

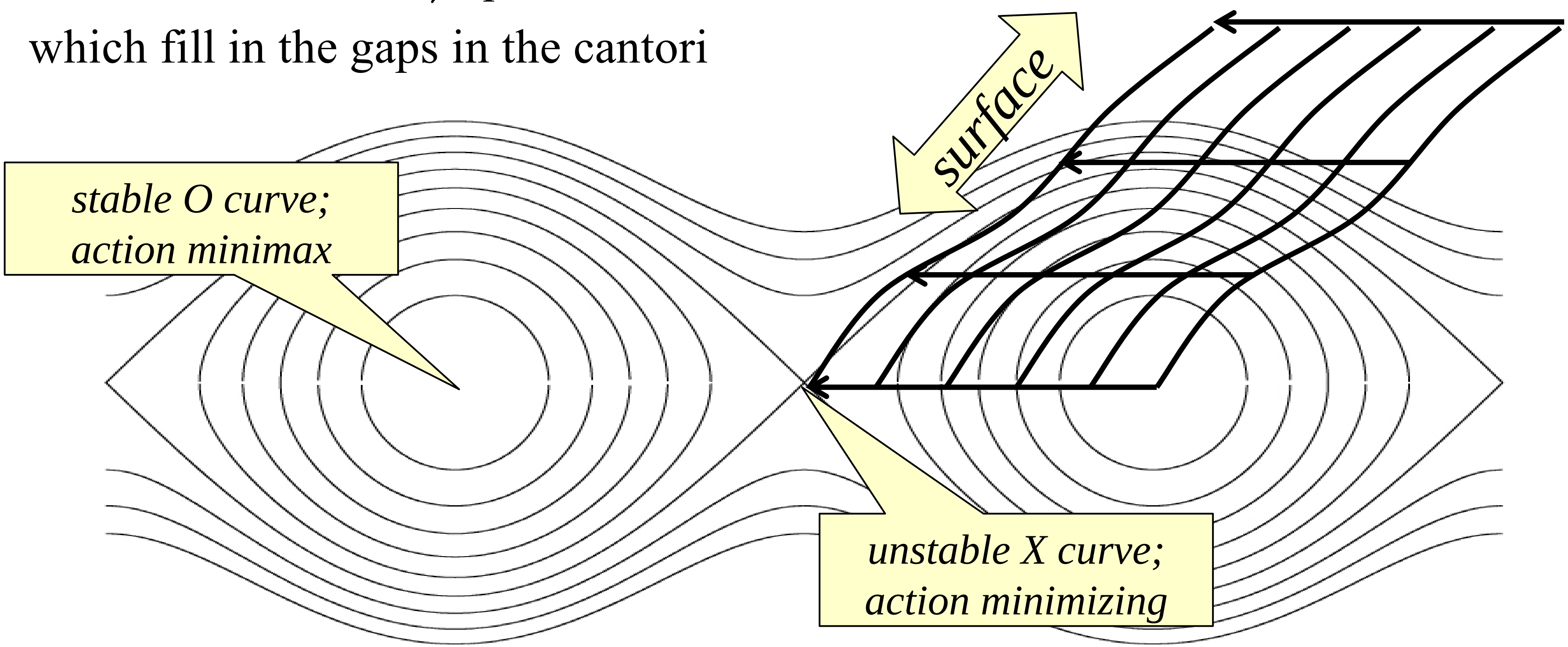
(periodic orbit constraint $\phi_0 = 0,\ \phi_N = 2\pi q,\ \theta_N = \theta_0 + 2\pi p$)

Hessian cyclic tridiagonal;
easily inverted O(N);
initial guess by tracking;
2nd order convergence;

$$\begin{pmatrix} \delta\ \partial_0 S \\ \delta\ \partial_1 S \\ . \\ . \\ \delta\ \partial_{N\text{-}1} S \end{pmatrix} = \begin{pmatrix} \partial^2_{0,0} S & \partial^2_{0,1} S & 0 & 0 & \partial^2_{0,N-1} S \\ \partial^2_{1,0} S & \partial^2_{1,1} S & \partial^2_{1,2} S & 0 & 0 \\ 0 & . & . & . & 0 \\ 0 & 0 & . & . & . \\ \partial^2_{N-1,0} S & 0 & 0 & \partial^2_{N-1,N-2} S & \partial^2_{N-1,N-1} S \end{pmatrix} \begin{pmatrix} \delta\theta_0 \\ \delta\theta_1 \\ . \\ . \\ \delta\theta_{N-1} \end{pmatrix}$$

Ghost surfaces are constructed by deforming the stable periodic orbit into the unstable periodic orbit

- Ghost surfaces are constructed by sliding a trial periodic curve down the gradient flow, $\frac{\partial \theta_i}{\partial \tau} = -\frac{\partial S}{\partial \theta_i}$, from the minimax (O) to the minimum (X)
- As the periodicity $p/q \rightarrow \iota$ (irrational), we have *irrational* ghost-surfaces, which fill in the gaps in the cantori



Steady state heat transport is solved numerically

- heat diffusion equation $\partial_t T = \nabla \cdot \left(\kappa_{\|} \nabla_{\|} T + \kappa_{\perp} \nabla_{\perp} T \right)$ (+ inhomogeneous b.c.)

  solved using operator splitting:
- Parallel : locally field alligned coordinates $\mathbf{B} = \nabla\alpha \times \nabla\beta$

$$\nabla_{\|}^{2} T = B^{\phi} \frac{\partial}{\partial\phi} \left(\frac{B^{\phi}}{B^{2}} \frac{\partial T}{\partial\phi} \right)$$

- Perpendicular : symmetric finite-difference

$$\nabla_{\perp}{}^{2} T = \sqrt{g}^{-1} \left[\partial_x \left(\sqrt{g} T^{x} \right) + \partial_y \left(\sqrt{g} T^{y} \right) \right]$$

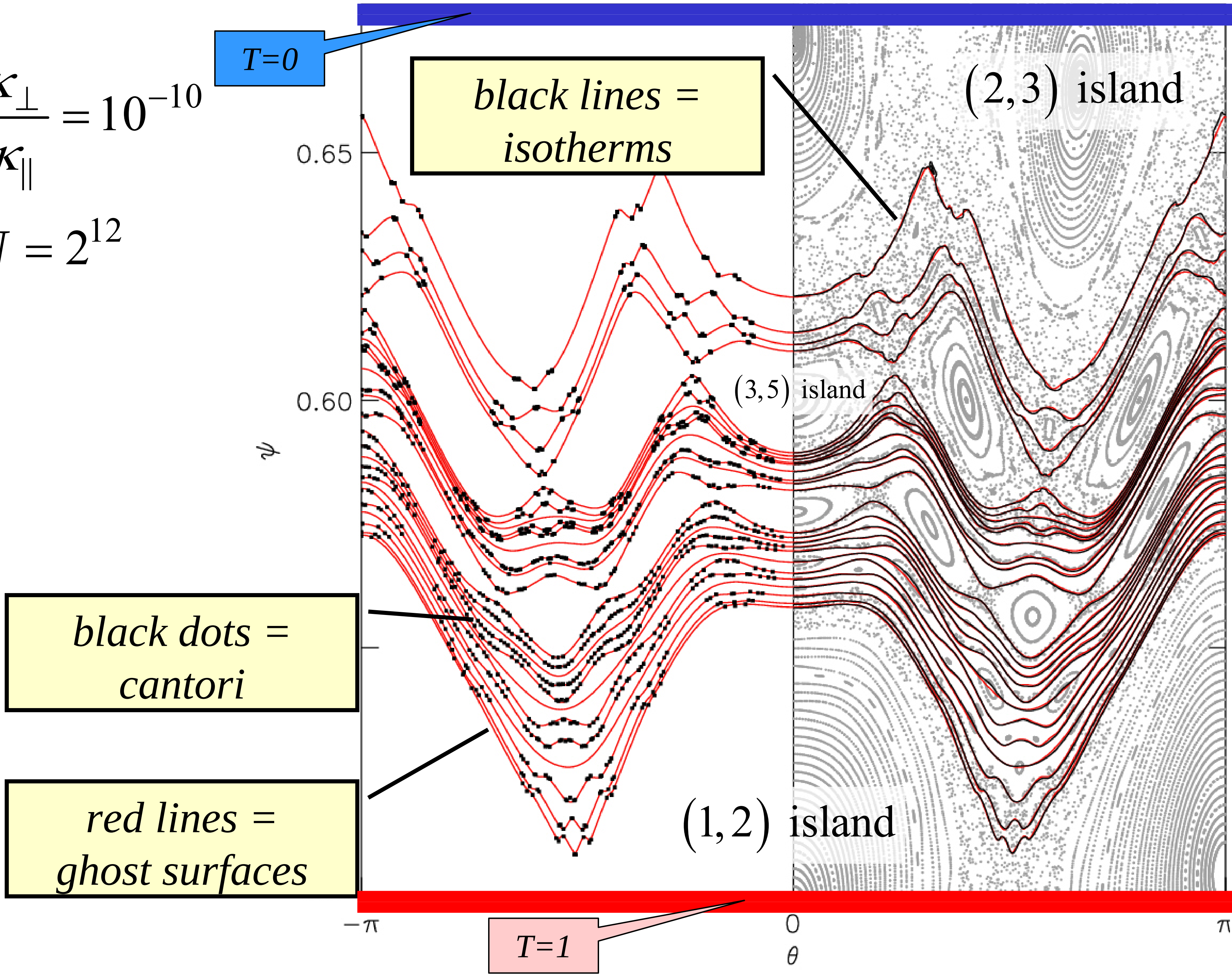
- Sparse linear system solved using BiCGStab

  confirmed 2nd order scaling of error

  scaling of island width $\Delta\text{w} = \left(\kappa_{\perp} / \kappa_{\|} \right)^{1/4}$

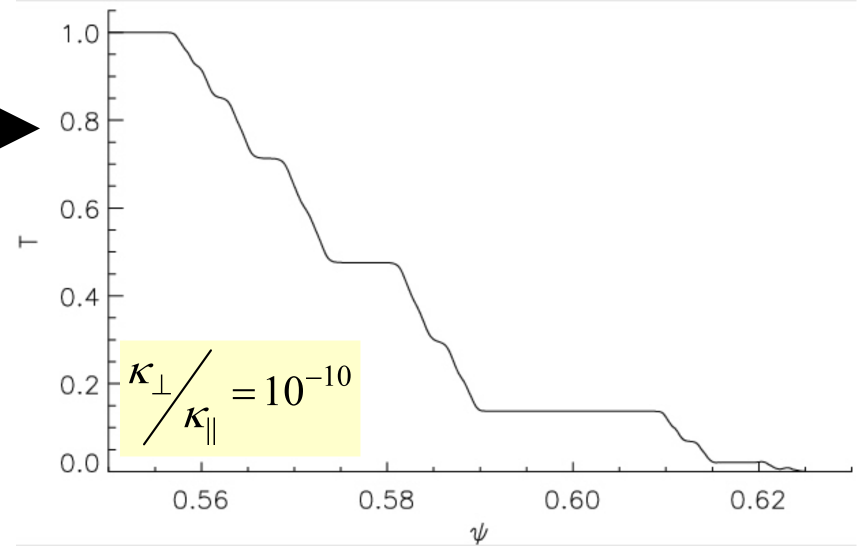
temperature flattens across larger islands

- $\frac{\kappa_\perp}{\kappa_\parallel} = 10^{-10}$
- $N = 2^{12}$



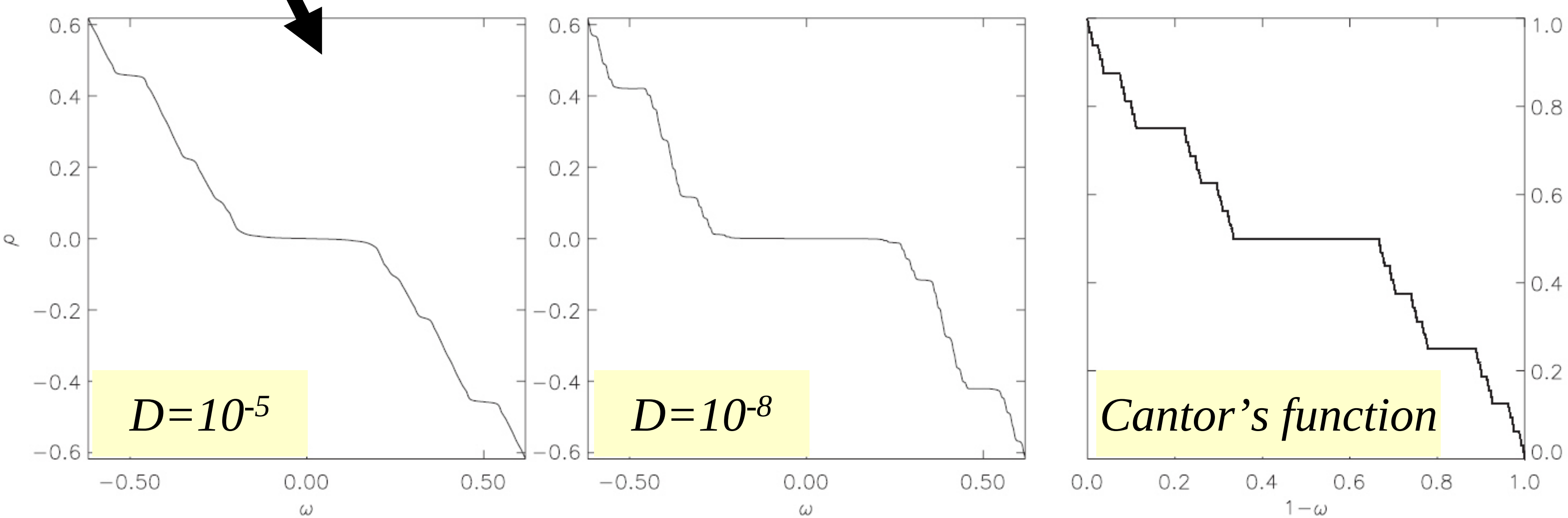
The temperature profile is a smoothed devils staircase for strongly anisotropic systems

- temperature profile for $\nabla \cdot q=0$



- temperature profile in chaotic coordinates

for $\dfrac{\partial T}{\partial t} + \mathbf{B}.\nabla T = D\nabla^2 T$



CONCLUSIONS

Chaotic coordinates recover a simple description of heat transport in chaotic fields.

- Temperature gradients are supported by cantori.

- Ghost surfaces coincide closely with isotherms

(and ghost surfaces can be constructed extremely quickly compared to

the numerically intensive solution to $\nabla \cdot q = 0$).

- Suggests that the temperature may be written

$$T(s,\theta,\phi) = T_0(s) + \delta T(s,\theta,\phi)$$

where $T_0(s)$ is a smoothed devils staircase, and $|\delta T|$ is small,

$$|\delta T| \sim \mathrm{O}\left(\kappa_\perp / \kappa_\parallel \right)^{??}$$