Temperature Gradients are supported by Cantori in Chaotic Magnetic Fields

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After destruction of the KAM surfaces, the cantori severely inhibit field line flow, and thus present barriers to heat transport in chaotic fields.

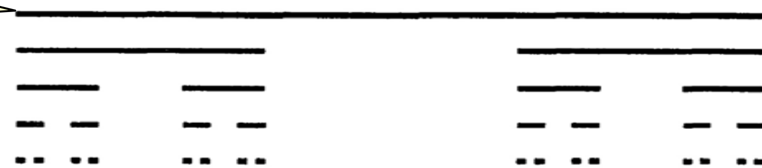
Chaotic magnetic coordinates may be adapted to the cantori.

Anisotropic transport is dominated by structure of magnetic field

- Coordinates adapted to the structure of the magnetic field, magnetic coordinates, provide elegant description of plasma dynamics and can enhance numerical accuracy.
- Magnetic coordinates can be constructed globally when the field lines lie on nested toroidal surfaces.
  - The temperature takes the simple form $T=T(\psi)$, where ψ labels flux surfaces;
- Error fields, non-axisymmetry, instabilities . . . result in partially chaotic fields.
  - The temperature takes the general form $T=T(\psi,\theta,\phi)$;
- For chaotic fields, the *cantori* (broken KAM surfaces) play an important role in restricting heat transport.
  - cantori are invariant sets, with irrational transform; cantorus = KAM surface with gaps/holes;
  - cantori are approximated by high-order minimizing periodic orbits;
- Coordinates adapted to the cantori, called chaotic coordinates, recover a simple description of the temperature.
  - $T \approx T(s)$, where s labels chaotic coordinates;

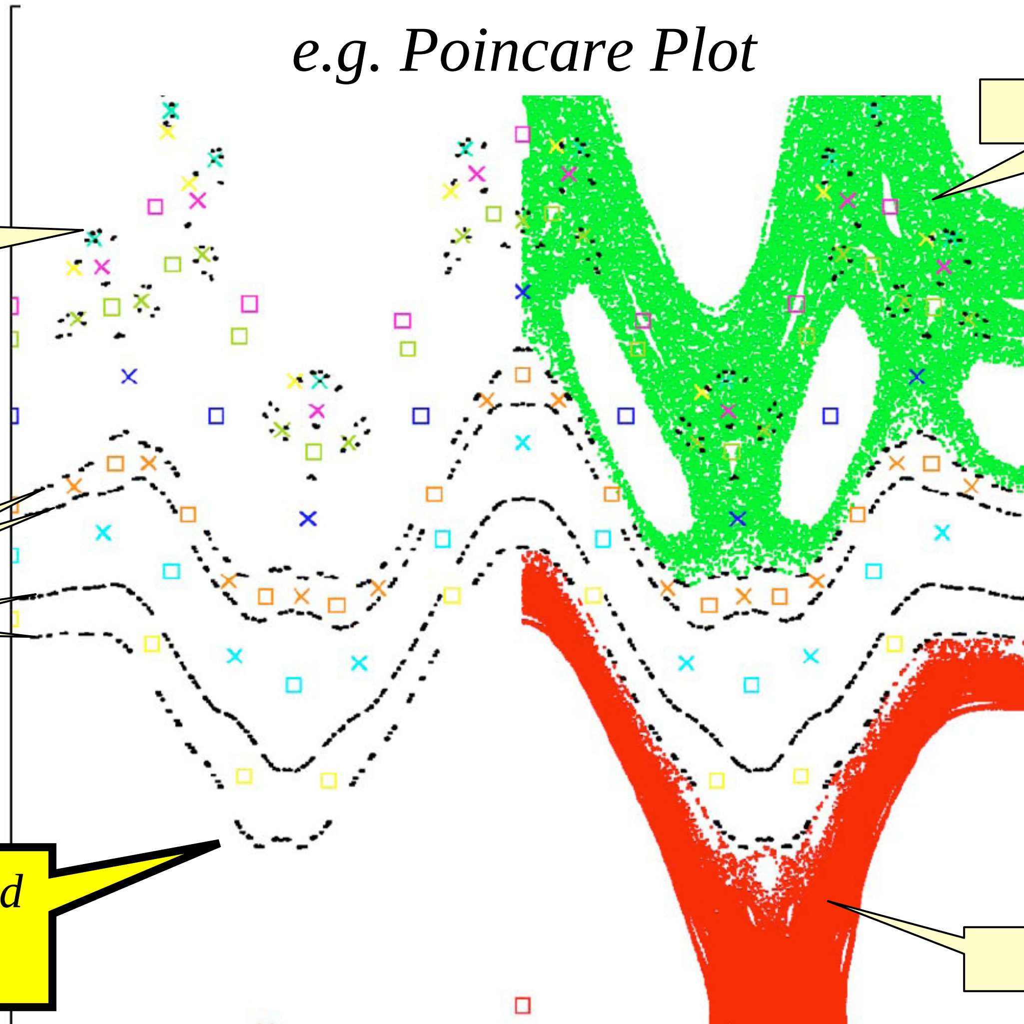
Cantori restrict transport in chaotic regions

KAM torus
(complete barrier)



cantor+torus=cantorus
(partial barrier)

- *cantori are leaky, but can severely restrict transport*



super critical cantori
(black dots)

10^5 iterations

connected chaos

(no KAM)

cantori restrict transport

near critical cantori

heat flow restricted across cantori

10^5 iterations

Magnetic field lines (in particular the cantori), are determined from Lagrangian variational principles

Magnetic field lines, $\mathbf{B} = \nabla \times \mathbf{A}$, are stationary curves C of the action integral $S = \int_C \mathbf{A.dl}$,

Canonical form : $\mathbf{A} = \psi \nabla \theta - \chi \nabla \phi$ and $\chi(\psi,\theta,\phi) = \frac{1}{2}\psi^2 + \sum \chi_{m,n}(\psi)\cos(m\theta - n\phi)$.

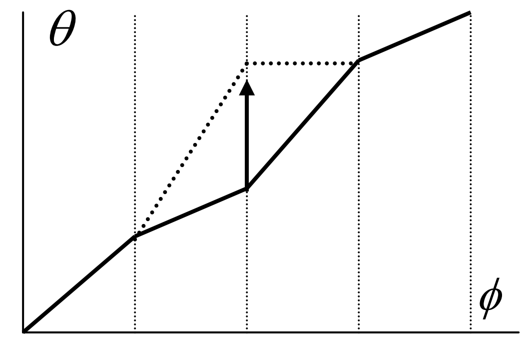
Setting $\delta S = 0$ gives the field lines $\frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{\partial \chi}{\partial \psi} = \frac{B^\theta}{B^\phi}$ $\frac{\mathrm{d}\psi}{\mathrm{d}\phi} = -\frac{\partial \chi}{\partial \theta} = \frac{B^\psi}{B^\phi}$

Periodic orbits can be located by

1) field line following (which is sensitive to exponential increase of integration error);

2) adjusting a *trial curve* to find an extremum of S (which is robust & efficient).

The simplest representation of a trial curve is piecewise-linear



Trial Curve $\theta(\phi),\ \psi = \dot{\theta}(\phi)$

Linear segment $\phi \in [\phi_i, \phi_{i+1}],\ \theta = \theta_i + \psi_i(\phi - \phi_i)\ ,\ \psi_\mathrm{i} = (\theta_{i+1} - \theta_i)/\Delta\phi$

Action integral integrated piecewise analytically

Action $S = \sum_i S_i(\theta_i, \theta_{i+1})\ \ ;\ \ S_i = \int_{\phi_i}^{\phi_{i+1}} (\psi\dot{\theta} - \chi)\,d\phi = \frac{1}{2}\psi_i^2 + \sum_{m,n} \chi_{m,n}(\psi_i)\left[\frac{\sin(m\theta - n\phi)}{m\dot{\theta} - n}\right]_{\phi_i}^{\phi_{i+1}}$

Action Gradient:

To find extremal curves $\frac{\partial S}{\partial \theta_i} = \partial_2 S_{i-1}(\theta_{i-1}, \theta_i) + \partial_1 S_i(\theta_i, \theta_{i+1}) = 0,$ use Newton's method.

Hessian cyclic tridiagonal;
easily inverted O(N);
initial guess by tracking;
2nd order convergence;

(periodic orbit constraint $\phi_0 = 0,\ \phi_N = 2\pi q,\ \theta_N = \theta_0 + 2\pi p$)

$$\begin{pmatrix} \delta\ \partial_0 S \\ \delta\ \partial_1 S \\ . \\ . \\ \delta\ \partial_{N\text{-}1} S \end{pmatrix} = \begin{pmatrix} \partial^2_{0,0}S & \partial^2_{0,1}S & 0 & 0 & \partial^2_{0,N-1}S \\ \partial^2_{1,0}S & \partial^2_{1,1}S & \partial^2_{1,2}S & 0 & 0 \\ 0 & . & . & . & 0 \\ 0 & 0 & . & . & . \\ \partial^2_{N-1,0}S & 0 & 0 & \partial^2_{N-1,N-2}S & \partial^2_{N-1,N-1}S \end{pmatrix} \begin{pmatrix} \delta\theta_0 \\ \delta\theta_1 \\ . \\ . \\ \delta\theta_{N-1} \end{pmatrix}$$

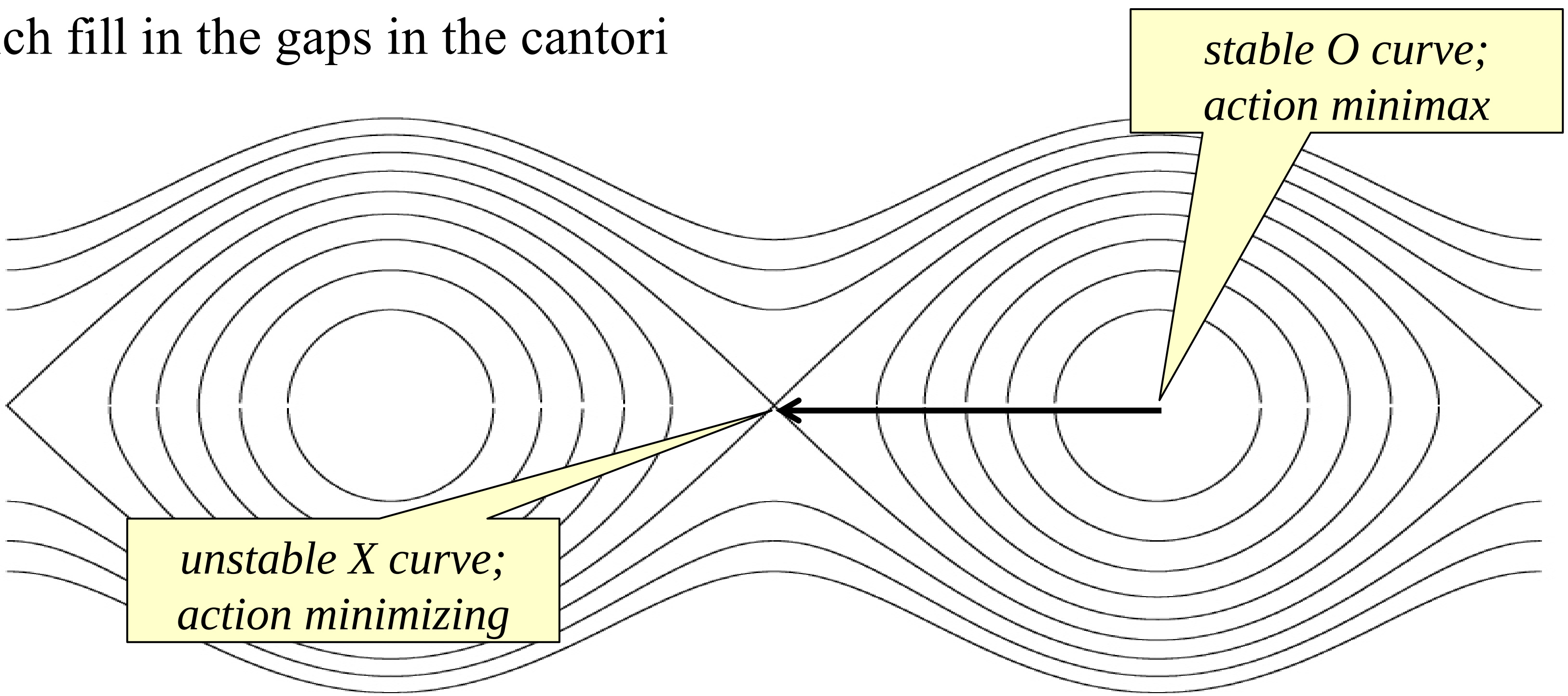
Ghost surfaces are constructed by deforming the stable periodic orbit into the unstable periodic orbit

- Ghost surfaces are constructed by sliding a trial periodic curve

down the gradient flow, $\frac{\partial \theta_i}{\partial \tau} = -\frac{\partial S}{\partial \theta_i}$, from the minimax (O) to the minimum (X)

- As the periodicity $p/q \to \iota$ (irrational), we have *irrational* ghost-surfaces,

which fill in the gaps in the cantori



Steady state heat transport is solved numerically

- heat diffusion equation $\partial_t T = \nabla \cdot \left(\kappa_{\|} \nabla_{\|} T + \kappa_{\perp} \nabla_{\perp} T \right)$ (+ inhomogeneous b.c.)

  solved using operator splitting:
- Parallel : locally, field-alligned coordinates $\mathbf{B} = \nabla \alpha \times \nabla \beta$

$$\nabla_{\|}^2 T = B^{\phi} \frac{\partial}{\partial \phi} \left(\frac{B^{\phi}}{B^2} \frac{\partial T}{\partial \phi} \right)$$

- Perpendicular : symmetric finite-difference

$$\nabla_{\perp}{}^2 T = \sqrt{g}^{-1} \left[\partial_x \left(\sqrt{g} T^x \right) + \partial_y \left(\sqrt{g} T^y \right) \right]$$

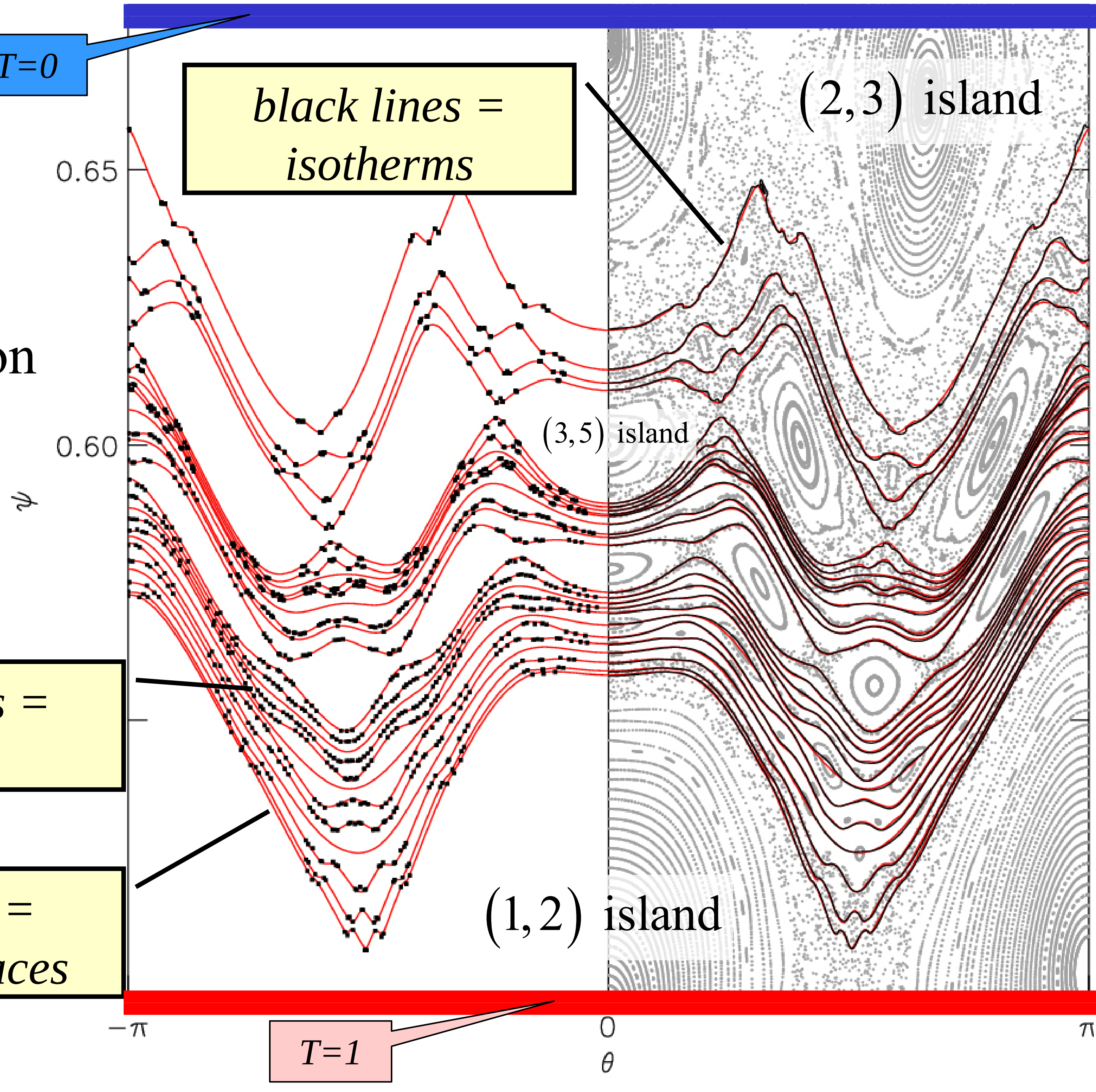
- Sparse linear system solved using BiCGStab

  confirmed 2nd order scaling of error

  temperature flattens when island width exceeds $\Delta \mathrm{w}_{\mathrm{C}} \propto \left(\kappa_{\perp} / \kappa_{\|} \right)^{1/4}$

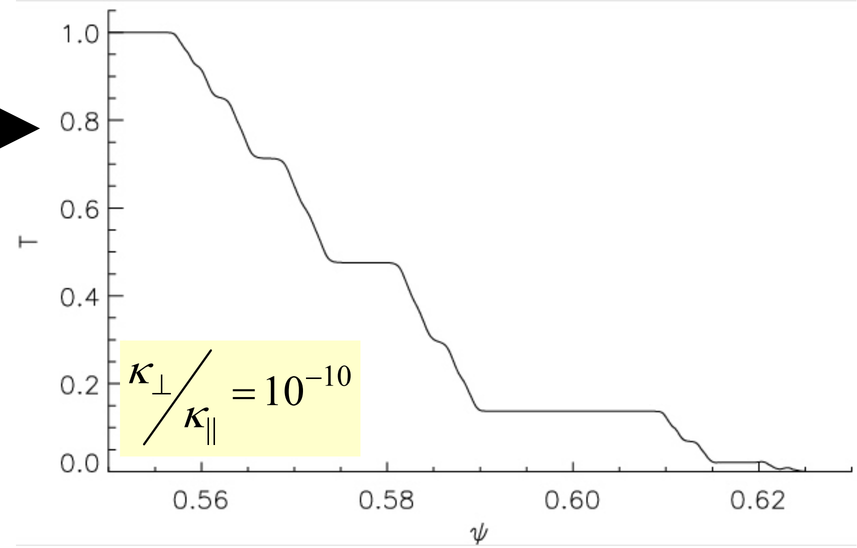
- $\dfrac{\kappa_\perp}{\kappa_\parallel} = 10^{-10}$
- grid resolution

$= 2^{12} \times 2^{12}$

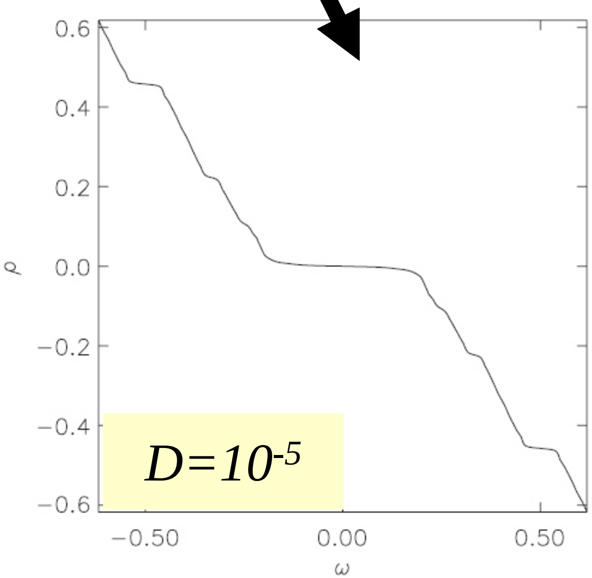


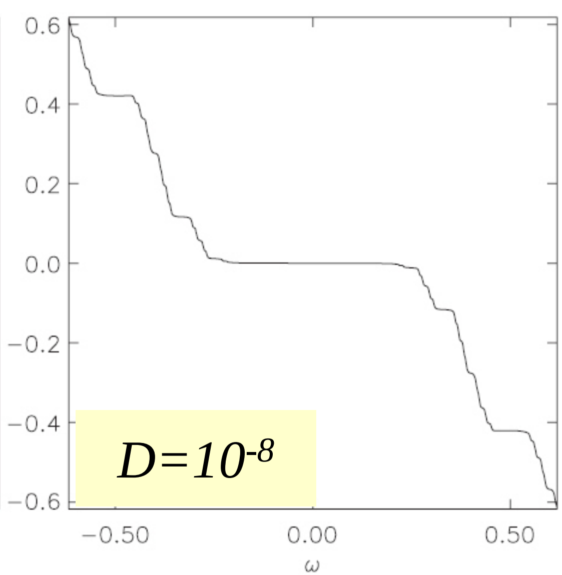
The temperature profile is a smoothed devils staircase for strongly anisotropic systems

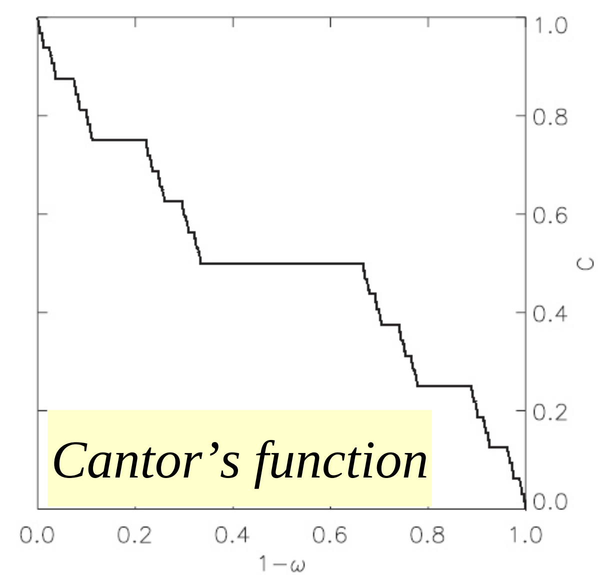
- temperature profile for $\nabla \cdot q = 0$



- temperature profile in chaotic coordinates for $\dfrac{\partial T}{\partial t} + \mathbf{B}.\nabla T = D\nabla^2 T$







CONCLUSIONS

Chaotic coordinates recover a simple description of heat transport in chaotic fields.

- Temperature gradients are supported by cantori.

- Ghost surfaces coincide closely with isotherms

(and ghost surfaces can be constructed extremely quickly compared to

the numerically intensive solution to $\nabla \cdot q = 0$).

- Suggests that the temperature may be written

$$T(s,\theta,\phi) = T_0(s) + \delta T(s,\theta,\phi)$$

where $T_0(s)$ is a smoothed devils staircase, and $|\delta T|$ is small,

$$|\delta T| \sim \mathrm{O}\left(\kappa_\perp / \kappa_\parallel \right)^{??}$$