Theory and numerics of partially-relaxed, topologically-constrained, MHD equilibria with chaotic magnetic fields

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Motivation & Outline

→ The simplest model of approximating global, macroscopic force balance in toroidal plasma confinement with arbitrary geometry is magnetohydrodynamics (MHD)

→ As toroidal magnetic fields are analogous to 1-1/2 Hamiltonians, and are generally chaotic, we need an MHD equilibrium code that allows for chaotic fields.

→ Existing ideal MHD equilibrium codes with chaotic fields fail to accommodate the fractal structure of Hamiltonian chaos. This leads to an ill-posed numerical algorithm for computing numerically-intractable, pathological equilibria.

→ A new partially-relaxed, topologically-constrained MHD equilibrium model is described and implemented numerically. Results demonstrating convergence tests, benchmarks, and non-trivial solutions are presented.

Ideal-force-balance with *chaotic* field is pathological

ideal MHD theory =

*ideal-force-balance, $\nabla p = \mathbf{j}\times\mathbf{B}$, gives $\mathbf{B}\cdot\nabla p = 0$

→ transport of pressure along field is "infinitely" fast
→ no scale length in ideal MHD
→ pressure adapts exactly to structure of phase space

chaos theory =

*for non-symmetric systems nested family of flux surfaces is destroyed

*islands & irregular field lines appear where transform is rational (n/m), rationals are dense in space

Poincare-Birkhoff theorem $\rightarrow$ periodic orbits, (e.g. stable and unstable) guaranteed to survive into chaos

*irrational surfaces survive if there exists an $r, k \in \Re$ s.t. for all rationals, $|\iota - n/m| > r\, m^{-k}$

i.e. rotational-transform, ι, is *poorly approximated* by rationals,

Diophantine condition
Kolmogorov, Arnold and Moser

ideal MHD theory + chaos theory ≡ pathological equilibrium

*Spitzer iterations are ill-posed

1) $\mathbf{B}_n \cdot \nabla p = 0$ $\rightarrow \nabla p$ is everywhere discontinuous, or zero

2) $\mathbf{j}_\perp = \mathbf{B}_n \times \nabla p / B_n^{\,2}$ $\rightarrow \mathbf{j}_\perp$ is discontinuous or zero

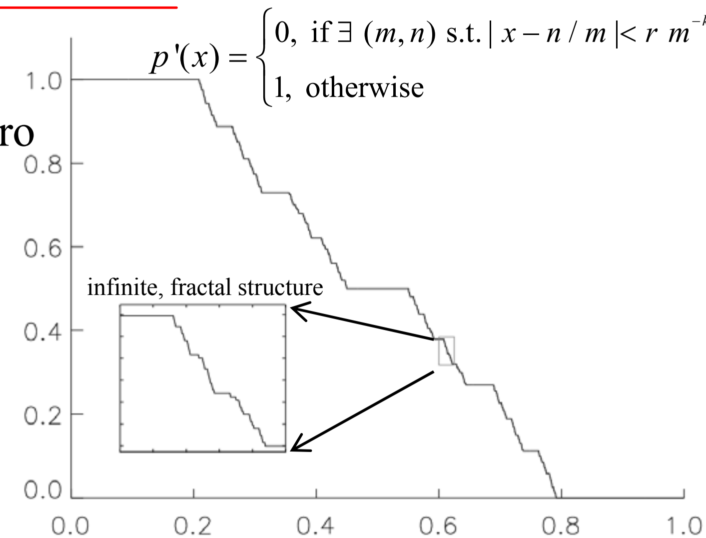
3) $\mathbf{B}_n \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$

→ cannot be inverted to obtain parallel current

$\mathbf{B}\cdot\nabla$ is *densely and irregularly* singular

condition that σ be single valued $\delta\sigma = -\oint_C \nabla \cdot \mathbf{j}_\perp dl / B = 0$

4) $\nabla \times \mathbf{B}_{n+1} = \mathbf{j} \equiv \sigma \mathbf{B}_n + \mathbf{j}_\perp$



To have a well posed equilibrium with chaotic **B** need to extend beyond ideal MHD.

e.g. introduce non-ideal terms, such as resistivity, η, perpendicular diffusion, $\kappa_\perp$, $[HINT,\ M3D, ..]$,

$\rightarrow$ or can relax infinity of ideal MHD constraints

Taylor relaxation: a weakly resistive plasma will relax, *subject to single constraint* of conserved helicity

Taylor relaxation, [Taylor, 1974]

$$\underbrace{W = \int_V \left(p + B^2 / 2 \right) dv}_{\text{plasma energy}}, \qquad \underbrace{H = \int_V \left(\mathbf{A} \cdot \mathbf{B} \right) dv}_{\text{helicity, } \mathbf{B} = \nabla \times \mathbf{A}}$$

Constrained energy functional $F = W - \mu H / 2$, $\mu \equiv$ Lagrange multiplier

Euler-Lagrange equation, for *unconstrained* variations in magnetic field, $\nabla \times \mathbf{B} = \mu \mathbf{B}$

linear force-free field ≡ Beltrami field

But, . . . Taylor relaxed fields have no pressure gradients

Ideal MHD equilibria and Taylor-relaxed equilibria are at opposite extremes . . . .

Ideal-MHD → imposition of *infinity* of ideal MHD constraints

non-trivial pressure profiles, but structure of field is ***over-constrained***

Taylor relaxation → imposition of *single* constraint of conserved global helicity

structure of field is not-constrained, but pressure profile is trivial, i.e. ***under-constrained***

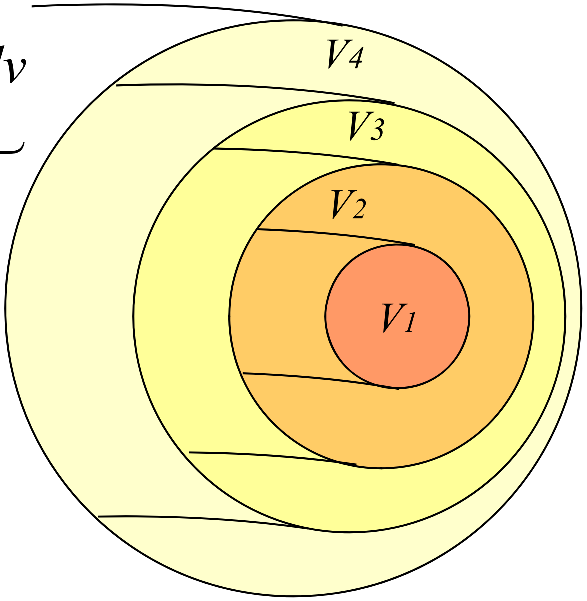
We need something in between . . .

. . . perhaps an equilibrium model with *finitely* many ideal constraints, and *partial* Taylor relaxation?

Introducing the multi-volume, partially-relaxed model of MHD equilibria with topological constraints

Energy, helicity and mass integrals

$$\underbrace{W_l = \int_{V_l}\left(\frac{p}{\gamma-1}+\frac{B^2}{2}\right)dv}_{\text{plasma energy}}, \qquad \underbrace{H_l = \int_{V_l}(\mathbf{A}\cdot\mathbf{B})\,dv}_{\text{helicity}}, \qquad \underbrace{M_l = \int_{V_l} p^{1/\gamma}\,dv}_{\text{mass}}$$



Multi-volume, partially-relaxed energy principle

* A set of N nested toroidal surfaces enclose N annulur volumes

→ the interfaces are assumed to be ideal, $\delta\mathbf{B} = \nabla\times(\delta\boldsymbol{\xi}\times\mathbf{B})$

* The multi-volume energy functional is

$$\boxed{F = \sum_{l=1}^{N}\left(W_l - \mu_l H_l/2 - \nu_l M_l\right)}$$

→ field remains tangential to interfaces,
→ a finite number of ideal constraints, imposed topologically!

Euler-Lagrange equation for *unconstrained* variations in A

In each annulus, the magnetic field satisfies $\nabla\times\mathbf{B}_l = \mu_l\mathbf{B}_l$

Euler-Lagrange equation for variations in interface geometry

Across each interface, pressure jumps allowed, but total pressure is continuous $[[p+B^2/2]]=0$

→ an analysis of the force-balance condition is that the interfaces must have strongly irrational transform

ideal interfaces coincide with KAM surfaces

Topological constraints : pressure gradients coincide with flux surfaces

The ideal interfaces are chosen to coincide with pressure gradients

$\rightarrow$ parallel transport dominates perpendicular transport,

$\rightarrow$ simplest approximation is $\mathbf{B} \bullet \nabla p = 0$

$\rightarrow$ structure of B and structure of the pressure are intimately connected;

$\rightarrow$ pressure gradients **must** coincide with KAM surfaces $\equiv$ ideal interfaces

$\rightarrow$ cannot apriori specify pressure without apriori constraining structure of the field;

A fixed boundary equilibrium is defined by :

(i) given pressure, $p(\psi)$, and rotational-transform profile, $\iota(\psi)$

(ii) geometry of boundary;

(a) only stepped pressure profiles are consistent (numerically tractable) with chaos and $\mathbf{B} \bullet \nabla p = 0$

(b) the computed equilibrium magnetic field must be consistent with the input profiles

(a) + (b) = where the pressure has gradients, the magnetic field must have flux surfaces.

Existence of Three-Dimensional Toroidal MHD Equilibria with Nonconstant Pressure

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We establish an existence result for the three-dimensional MHD equations

$$
\begin{aligned}
(\nabla \times \mathbf{B}) \times \mathbf{B} &= \nabla p \\
\nabla \cdot \mathbf{B} &= 0 \\
\mathbf{B} \cdot n|_{\partial T} &= 0
\end{aligned}
$$

with $p \neq$ const in tori T *without symmetry*. More precisely, our theorems insure the existence of sharp boundary solutions for tori whose departure from axisymmetry is sufficiently small; they allow for solutions to be constructed with an arbitrary number of pressure jumps.

Communications on Pure and Applied Mathematics, Vol. XLIX, 717–764 (1996)

→ ***how large the "sufficiently small" departure from axisymmetry can be needs to be explored numerically . . . .***

Extrema of energy functional obtained numerically; introducing the Stepped Pressure Equilibrium Code, SPEC

The vector-potential is discretized

* toroidal coordinates (s,ϑ,ζ), *interface geometry $R_l = \sum_{m,n} R_{l,m,n}\cos(m\vartheta - n\zeta), Z_l = \sum_{m,n} Z_{l,m,n}\sin(m\vartheta - n\zeta)$

* exploit gauge freedom $\mathbf{A} = A_\vartheta(s,\vartheta,\zeta)\nabla\vartheta + A_\zeta(s,\vartheta,\zeta)\nabla\zeta$

* Fourier $A_\vartheta = \sum_{m,n} a_\vartheta(s)\cos(m\vartheta - n\zeta)$

* Finite-element $a_\vartheta(s) = \sum_i a_{\vartheta,i}(s)\varphi(s)$

piecewise cubic or quintic basis polynomials

and inserted into constrained-energy functional.

* derivatives w.r.t. vector-potential $\rightarrow$ linear equation for Beltrami field $\nabla\times\mathbf{B} = \mu\mathbf{B}$

solved using sparse linear solver

* field in each annulus computed independently, distributed across multiple cpus

* field in each annulus depends on enclosed toroidal flux (boundary condition) and

$\rightarrow$ poloidal flux, ψ_P, and helicity-multiplier, μ

adjusted so interface transform is strongly irrational

$\rightarrow$ geometry of interfaces, $\boldsymbol{\xi} \equiv \left\{R_{m,n}, Z_{m,n}\right\}$

Force balance solved using multi-dimensional Newton method.

* interface geometry is adjusted to satisfy force $\mathbf{F}[\boldsymbol{\xi}] \equiv \left\{[[p + B^2/2]]_{m,n}\right\} = 0$

* angle freedom constrained by spectral-condensation, adjust angle freedom to minimize $\sum m^2\left(R_{mn}^2 + Z_{mn}^2\right)$

minimal spectral width [Hirshman, VMEC]

* derivative matrix, $\nabla\mathbf{F}[\boldsymbol{\xi}]$, computed using finite-differences

* quadratic-convergence w.r.t. Newton iterations

future work . . .
1) approximate derivative matrix ~ 2nd variation of energy functional
2) implement pre-conditioner

Numerical error in Beltrami field scales as expected

Scaling of numerical error with radial resolution depends on finite-element basis

$\mathbf{A}=A_\vartheta \nabla\vartheta + A_\zeta \nabla\zeta$, $\mathbf{B}=\nabla\times\mathbf{A}$, $\mathbf{j}=\nabla\times\mathbf{B}$, need to quantify error $\mathbf{j}-\mu\mathbf{B}$

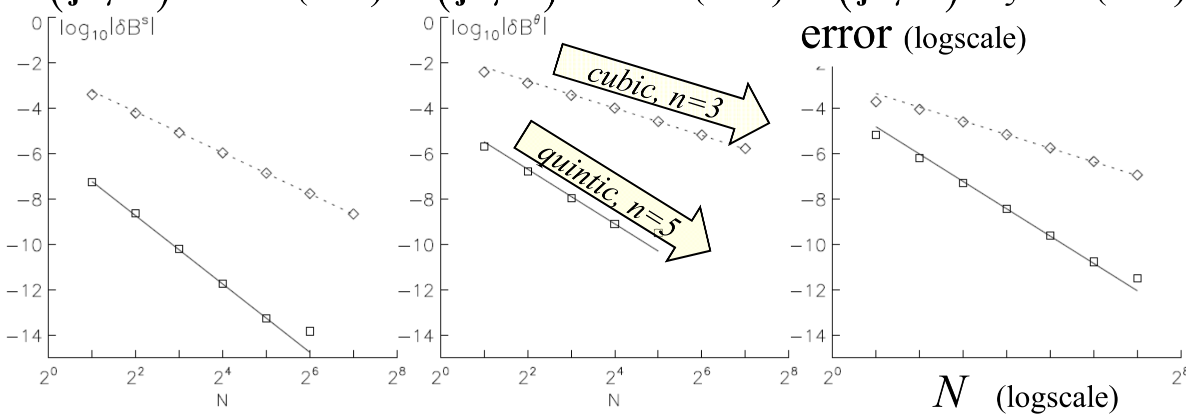
$A_\vartheta, A_\zeta \sim O(h^n)$

h = radial grid size = 1 / N
n = order of polynomial

$$\sqrt{g}B^s = \partial_\vartheta A_\zeta - \partial_\zeta A_\vartheta \sim O(h^n)$$
$$\sqrt{g}B^\vartheta = -\partial_s A_\zeta \sim O(h^{n-1})$$
$$\sqrt{g}B^\zeta = \partial_s A_\vartheta \sim O(h^{n-1})$$

$$\sqrt{g}\,j^s \sim O(h^{n-1})$$
$$\sqrt{g}\,j^\vartheta \sim O(h^{n-2})$$
$$\sqrt{g}\,j^\zeta \sim O(h^{n-2})$$

$(\mathbf{j}-\mu\mathbf{B})\cdot\nabla s \sim O(h^{n-1})$ $(\mathbf{j}-\mu\mathbf{B})\cdot\nabla\vartheta \sim O(h^{n-2})$ $(\mathbf{j}-\mu\mathbf{B})\cdot\nabla\zeta \sim O(h^{n-2})$



Example of chaotic Beltrami field

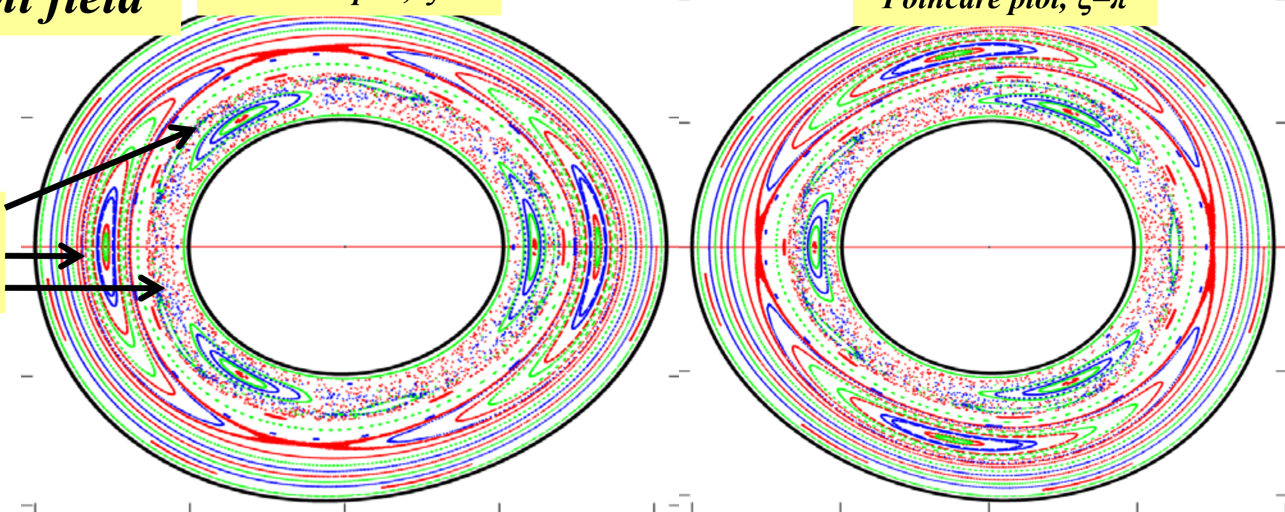
$R = 1.0 + r(\vartheta,\zeta)\cos\vartheta$,
$Z = r(\vartheta,\zeta)\sin\vartheta$,

inner surface
$r = 0.1$

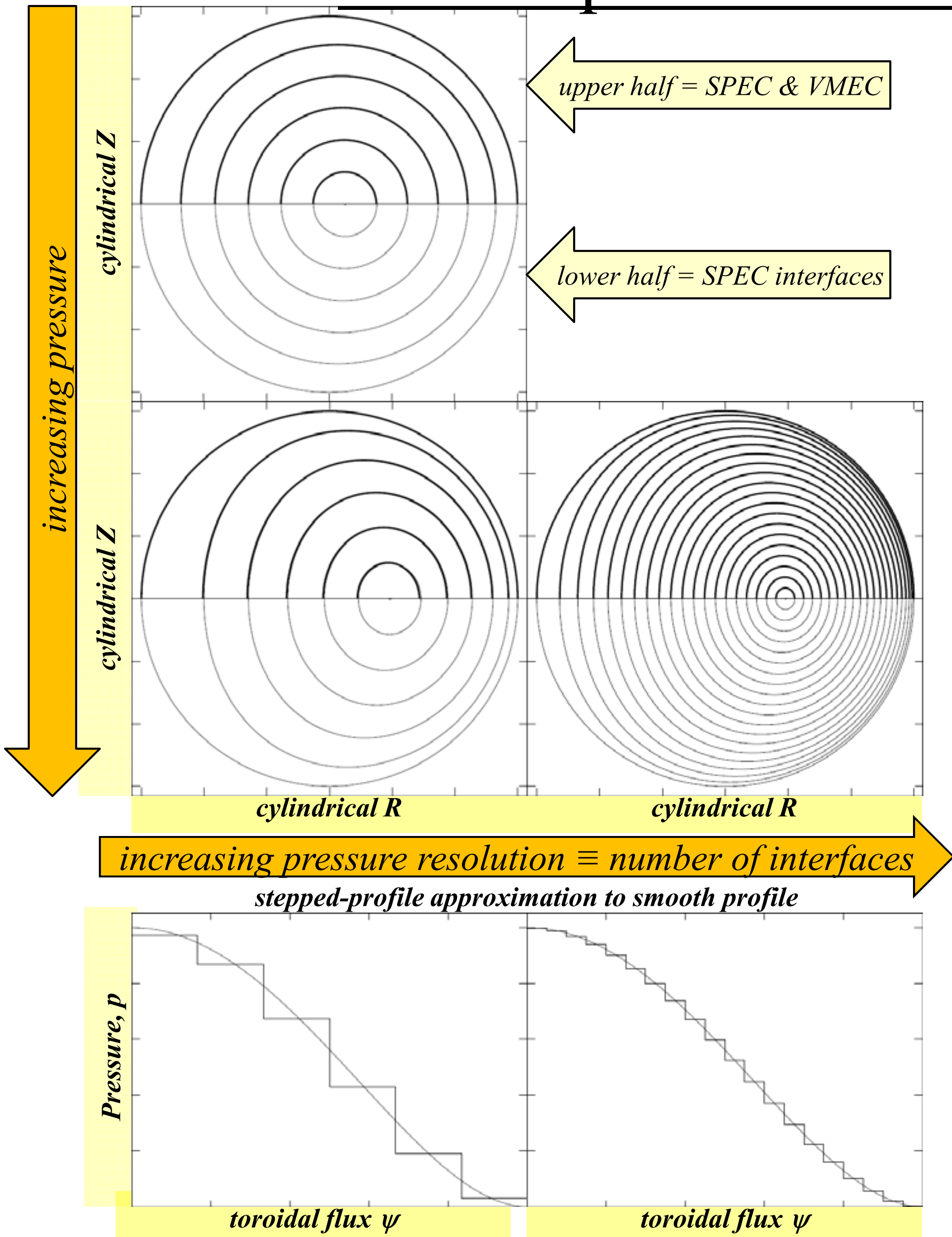
outer interface
$r = 0.2 + \delta\left[\cos(2\vartheta-\zeta)+\cos(3\vartheta-\zeta)\right]$

(m,n)=(3,1) island
+ (m,n)=(2,1) island
= chaos

Poincaré plot, ζ=0 ***Poincaré plot, ζ=π***



stepped-pressure equilibria accurately approximate smooth-pressure *axisymmetric* equilibria



in axisymmetric geometry . . .

→ magnetic fields have family of nested flux surfaces

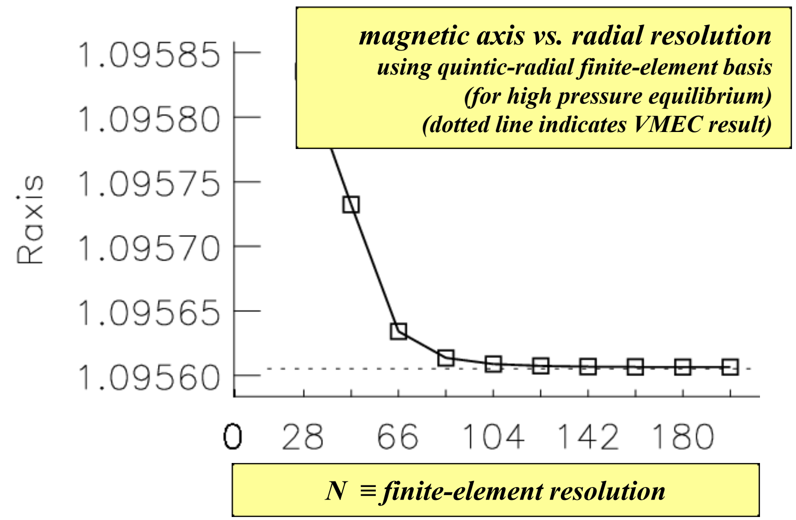
→ equilibria with smooth profiles exist,

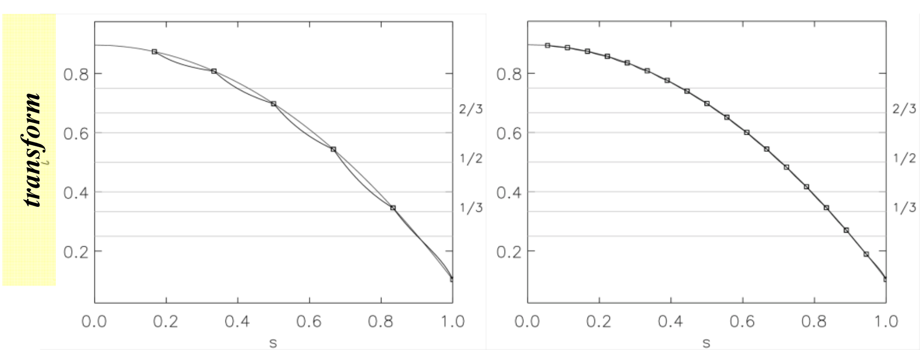
→ may perform benchmarks (e.g. with VMEC)

(arbitrarily approximate smooth-profile with stepped-profile)

→ approximation improves as number of interfaces increases

→ location of magnetic axis converges w.r.t radial resolution

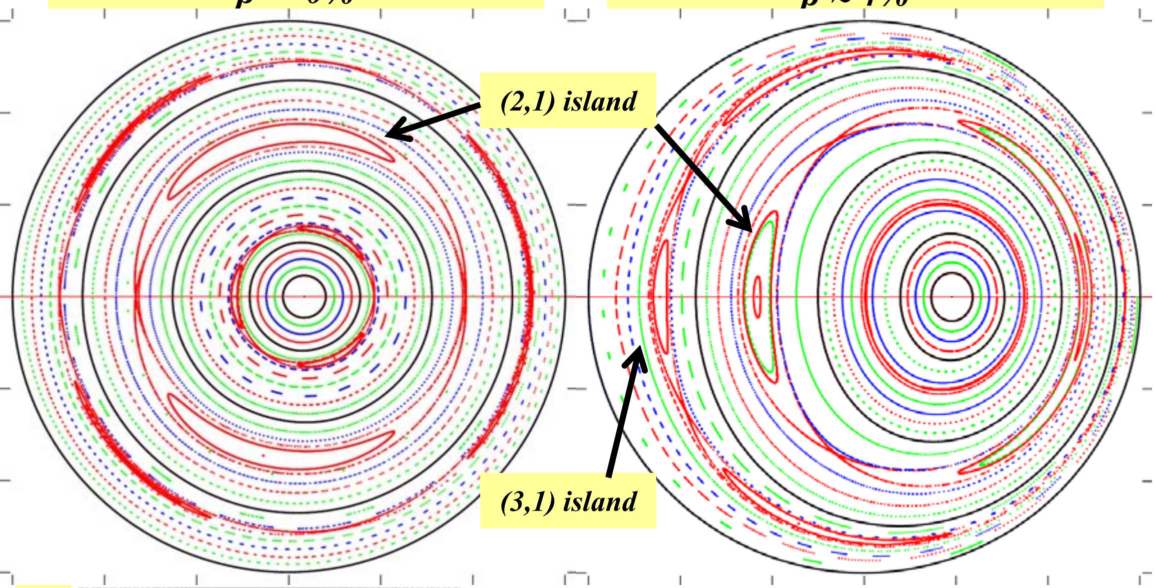




Equilibria with (i) perturbed boundary→chaotic fields, and (ii) pressure are computed .

zero-pressure equilibrium
$\beta = 0\%$

high-pressure equilibrium
$\beta \approx 4\%$

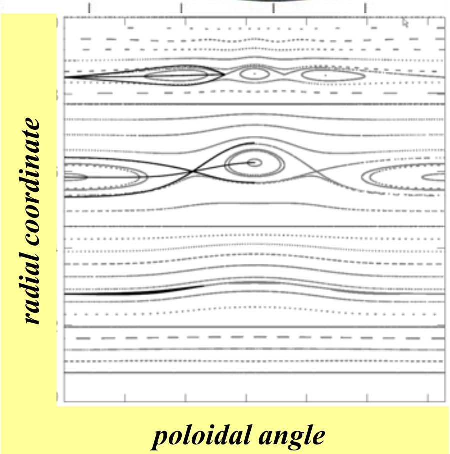


boundary deformation induces islands

$$R = 1.0 + r\cos\vartheta, \; Z = r\sin\vartheta$$

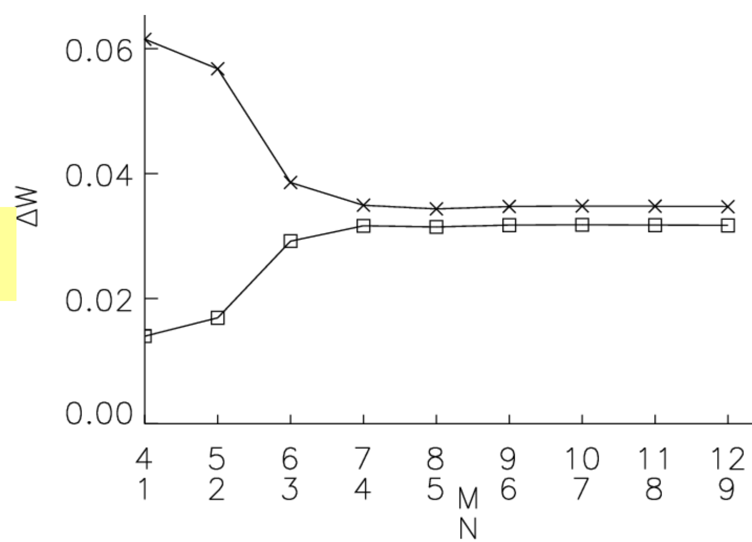
$$r = 0.3 + \delta\cos(2\vartheta - \phi) + \delta\cos(3\vartheta - \phi)$$

Demonstrated convergence with Fourier resolution



← island widths estimated by fitting quasi-rational surface through island chain

Convergence of (2,1) & (3,1) island widths with Fourier resolution for $\beta \approx 4\%$ case



Summary

→ A partially-relaxed, topologically-constrained energy principle has been presented for MHD equilibria with chaotic fields and non-trivial (i.e. non-constant) pressure

→ The model has been implemented numerically

* using a high-order (piecewise quintic) radial discretization
* an optimal (i.e. spectrally condensed) Fourier representation
* workload distrubuted across multiple cpus,
* extrema located using Newton's method with quadratic-convergence

→ Intuitively, the equilibrium model is an extension of Taylor relaxation to multiple volumes

→ The model has a sound theoretical foundation

* solutions guaranteed to exist (under certain conditions)

→ The numerical method is computationally tractable

* does not invert singular operators
* does not struggle to resolve fractal structure of chaos

→ Convergence studies have been performed

* expected error scaling with radial resolution confirmed
* detailed benchmark with axisymmetric equilibria (with smooth profiles)
* that the island widths converge with Fourier resolution has been confirmed

Toroidal magnetic confinement depends on flux surfaces

Transport in magnetized plasma dominately parallel to **B**

$\rightarrow$ if the field lines are not confined (e.g. by flux surfaces), then the plasma is poorly confined

Axisymmetric magnetic fields possess a continuously nested family of flux surfaces

$\rightarrow$ nested family of flux surfaces is guaranteed if the system has an ignorable coordinate

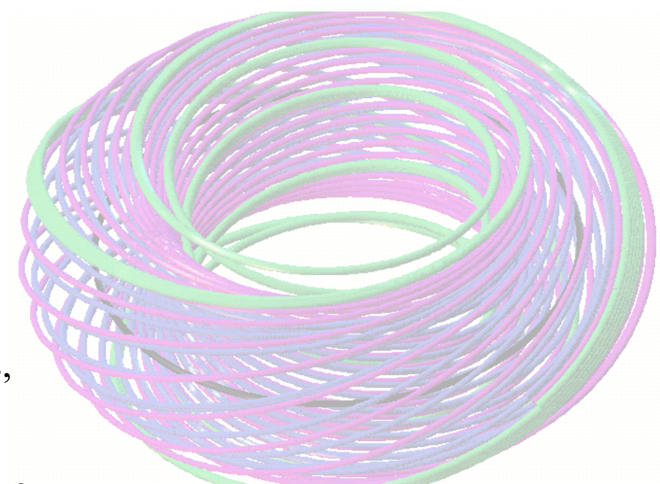
magnetic field is called integrable

$\rightarrow$ rational field-line $\equiv$ periodic trajectory

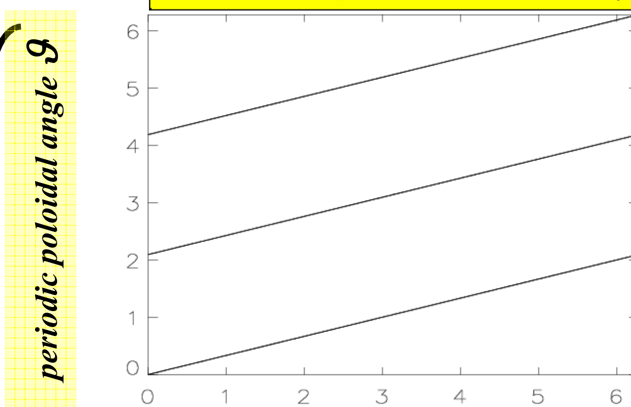
family of periodic orbits ≡ rational flux surface

$\rightarrow$ irrational field-lines cover *irrational* flux surface

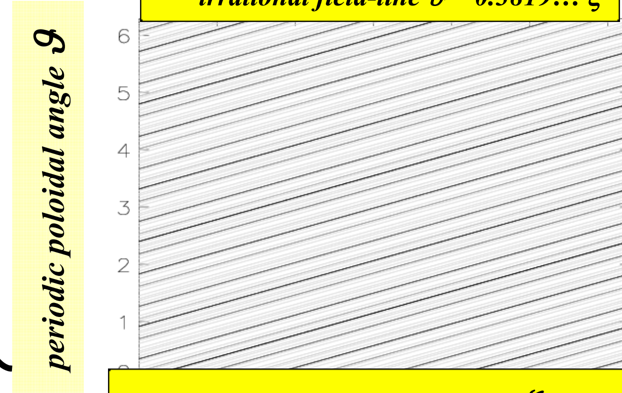
magnetic field lines wrap around toroidal "flux" surfaces



rational field-line $\vartheta = 0.3333\ldots\ \xi$



irrational field-line $\vartheta = 0.3819\ldots\ \xi$



periodic toroidal angle ζ

straight-field-line flux coordinates,

$$\mathbf{B}\bullet\nabla\psi = 0$$

$$\mathbf{B} = \nabla\psi\times\nabla\vartheta + \iota(\psi)\nabla\zeta\times\nabla\psi$$

$$\sqrt{g}\,\mathbf{B}\bullet\nabla \equiv \partial_\zeta + \iota\,\partial_\vartheta$$

magnetic differential equation, $\mathbf{B}\bullet\nabla\sigma = s$,

is singular at rational surfaces, $(m\,\iota - n)\sigma_{m,n} = i(\sqrt{g}\,s)_{m,n}$

Ideal MHD equilibria are extrema of energy functional

The energy functional is

$$W = \int_V (p + B^2/2)\, dv$$

$V \equiv$ global plasma volume

ideal variations

mass conservation $\quad \} \; \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$

state equation $\quad \} \; d_t (p \rho^{-\gamma}) = 0$

Faraday's law, ideal Ohm's law $\quad \} \; \delta \mathbf{B} = \nabla \times (\delta \xi \times \mathbf{B})$

→ideal variations don't allow field topology to change "frozen-flux"

the first variation in plasma energy is

$$\delta W = \int_V (\nabla p - \mathbf{j} \times \mathbf{B}) \cdot \delta \xi \, dv$$

Euler Lagrange equation for globally ideally-constrained variations

ideal-force-balance $\quad \nabla p = \mathbf{j} \times \mathbf{B}$

→ two surface functions, e.g. the pressure, $p(s)$, and rotational-transform ≡ inverse-safety-factor, $\iota(s)$,
and → a boundary surface (. . . for fixed boundary equilibria . . .),
constitute "boundary-conditions" that must be provided to uniquely define an equilibrium solution
. . . . . . The computational task is to compute the magnetic field that is consistent with the given boundary conditions . . .

nested flux surface topology maintained by singular currents at rational surfaces

from $\nabla \cdot (\sigma \mathbf{B} + \mathbf{j}_\perp) = 0$, parallel current must satisfy $\mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$, $\quad$ where $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$

$$\sigma_{m,n} = \frac{i(\sqrt{g}\; \nabla \cdot \mathbf{j}_\perp)_{m,n}}{(m\iota - n)} + \delta(m\iota - n)$$

→ magnetic differential equations are singular at rational surfaces (periodic orbits)
→ pressure-driven "Pfirsch-Schlüter currents" have $1/x$ type singularity
→ δ - function singular currents shield out islands

Topological constraints : pressure gradients coincide with flux surfaces

The ideal interfaces are chosen to coincide with pressure gradients

$\rightarrow$ parallel transport dominates perpendicular transport,

$\rightarrow$ simplest approximation is $\mathbf{B}\cdot\nabla p = 0$

$\rightarrow$ pressure gradients **must** coincide with KAM surfaces $\equiv$ ideal interfaces

$\rightarrow$ structure of B and structure of the pressure are intimately connected;

$\rightarrow$ cannot apriori specify pressure without apriori constraining structure of the field;

[next order of approximation, $\mathbf{B}\cdot\nabla p$ is small, e.g. $\partial_t p = \kappa_{\parallel}\nabla_{\parallel}^2 p + \kappa_{\perp}\nabla_{\perp}^2 p = 0$, with $\kappa_{\parallel} \gg \kappa_{\perp}$, e.g. $\kappa_{\perp}/\kappa_{\parallel} \sim 10^{-10}$

*pressure gradients coincide with KAM surfaces, cantori . .

*pressure flattened across islands, chaos with width > $\Delta w_C \sim \left(\kappa_{\perp}/\kappa_{\parallel}\right)^{1/4}$

* anisotropic diffusion equation solved analytically, p' $\propto 1 / \left(\kappa_{\parallel}\varphi_2 + \kappa_{\perp} G\right)$, φ_2 is quadratic-flux across cantori, G is metric term]

$\rightarrow$ where there are significant pressure gradients, there can be no islands or chaotic regions with width $> \Delta wc$

A fixed boundary equilibrium is defined by :

(i) given pressure, $p(\psi)$, and rotational-transform profile, $\iota(\psi)$

(ii) geometry of boundary;

(a) only stepped pressure profiles are consistent (numerically tractable) with chaos and $\mathbf{B}\cdot\nabla p = 0$

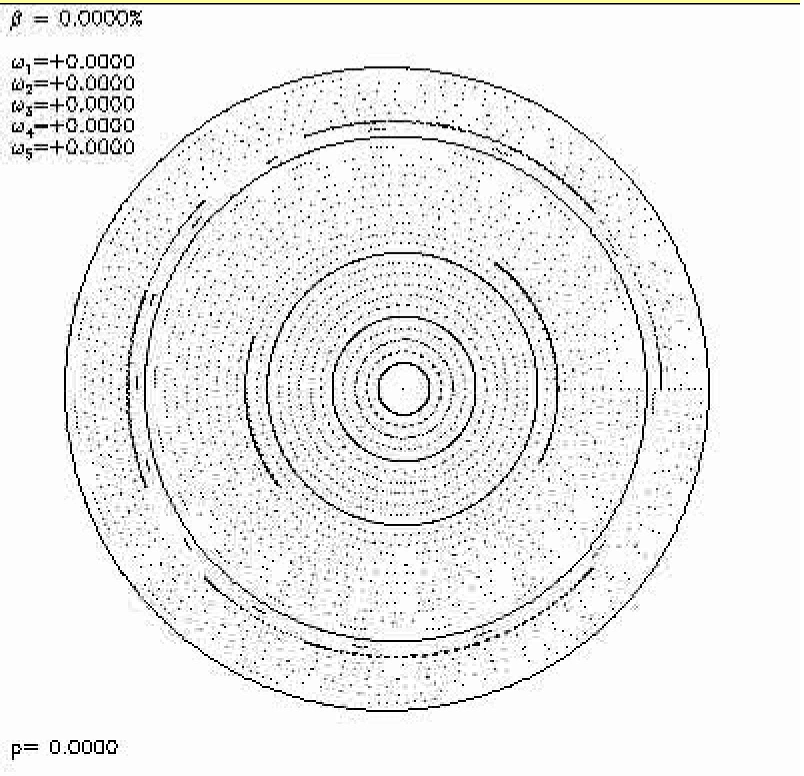
(b) the computed equilibrium magnetic field must be consistent with the input profiles

(a) + (b) = where the pressure has gradients, the magnetic field must have flux surfaces.

$\rightarrow$ non-trivial stepped pressure equilibrium solutions are *guaranteed* to exist

A sequence of equilibria with increasing pressure and perturbed boundary are computed

movie of sequence of equilibria with increasing pressure



phase of islands flips near n=∞ stability boundary