Multi-volume, partially-relaxed MHD equilibria

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Outline

- We need an equilibrium code that can handle chaotic fields

Toroidal magnetic confinement depends on flux surfaces

Transport in magnetized plasma dominately parallel to **B**

Confinement depends on existence of toroidal *flux* surfaces

* for axisymmetric magnetic fields

$\rightarrow$ nested family of flux surfaces is guaranteed

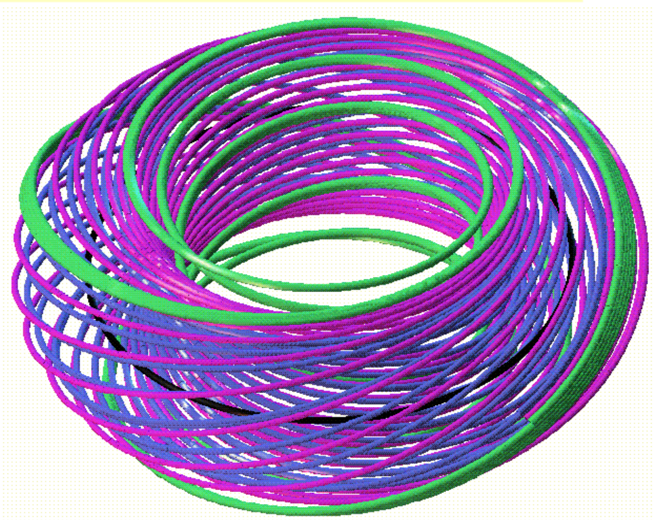
magnetic field is called integrable

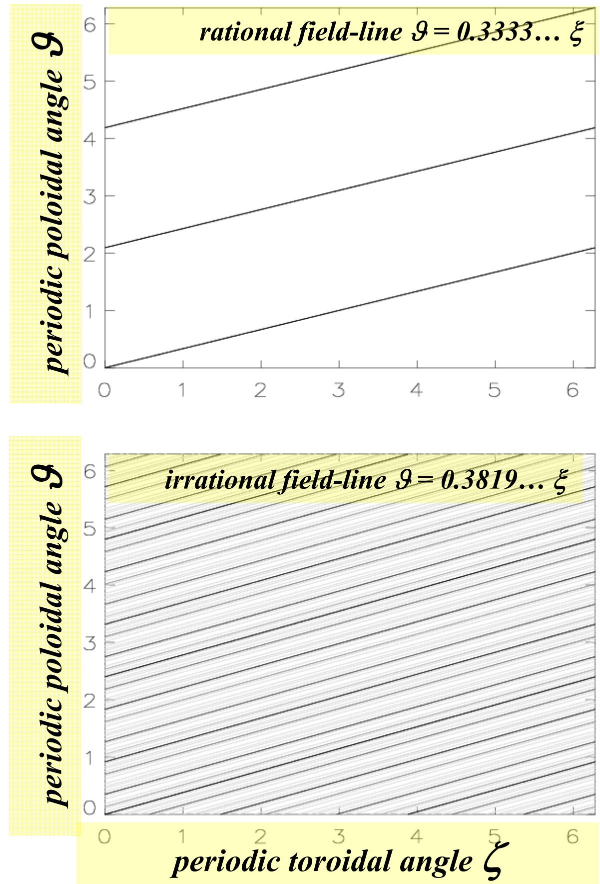
$\rightarrow$ rational field-line $\equiv$ periodic trajectory

family of periodic orbits ≡ rational flux surface

$\rightarrow$ irrational field-lines cover *irrational* flux surface

magnetic field lines wrap around toroidal "flux" surfaces





straight field line flux coordinates,

$$\mathbf{B}\cdot\nabla\psi = 0$$

$$\mathbf{B} = \nabla\psi\times\nabla\vartheta + \iota(\psi)\nabla\zeta\times\nabla\psi$$

$$\sqrt{g}\,\mathbf{B}\cdot\nabla \equiv \partial_\zeta + \iota\,\partial_\vartheta$$

magnetic differential equation, $\mathbf{B}\cdot\nabla\sigma = s$,

is singular at rational surfaces, $(m\,\iota - n)\sigma_{m,n} = i(\sqrt{g}s)_{m,n}$

Ideal MHD equilibria are extrema of energy functional

energy functional (*without flow*)

$$W = \int_V (p + B^2/2)\, dv$$

V ≡ global plasma volume

ideal variations

mass conservation } $\partial_t \rho + \nabla \bullet (\rho \mathbf{v}) = 0$

state equation } $d_t(p\rho^{-\gamma}) = 0$

Faraday's law, ideal Ohm's law } $\delta \mathbf{B} = \nabla \times (\delta \boldsymbol{\xi} \times \mathbf{B})$

ideal variations do not allow topology of the field to change
FROZEN FLUX

first variation in plasma energy

$$\delta W = \int_V (\nabla p - \mathbf{j} \times \mathbf{B}) \cdot \delta \boldsymbol{\xi}\, dv$$

Euler Lagrange equation for globally ideally-constrained variations ≡ ideal-force-balance, $\nabla p = \mathbf{j} \times \mathbf{B}$

from $\nabla \bullet (\sigma \mathbf{B} + \mathbf{j}_\perp) = 0$, parallel current must satisfy $\mathbf{B} \bullet \nabla \sigma = -\nabla \bullet \mathbf{j}_\perp$

→ magnetic differential equations are singular at rational surfaces
→ pressure-driven currents have 1/ x type singularity
→ δ - function singular currents shield out islands

Ideal-force-balance with chaotic field is pathological

ideal MHD theory =

*ideal-force-balance, $\nabla p = \mathbf{j} \times \mathbf{B}$, gives $\mathbf{B} \cdot \nabla p = 0$

→ transport of pressure along field is "infinitely" fast
→ no scale length in ideal MHD
→ pressure adapts exactly to structure of phase space

chaos theory =

*islands & chaos appear at all rational surfaces (n/m), and rationals are dense

*irrational surfaces survive if there exists an $r, k \in \Re$ s.t. for all rationals, $|\iota - n/m| > r\, m^{-k}$

i.e. rotational-transform, ι, is *poorly approximated* by rationals,

Diophantine condition
Kolmogorov, Arnold and Moser

ideal MHD theory + chaos theory ≡ pathological equilibrium

*Spitzer iterations are ill-posed

1) $\mathbf{B}_n \cdot \nabla p = 0$ → ∇p is discontinuous or zero

2) $\mathbf{j}_\perp = \mathbf{B}_n \times \nabla p / B_n^{\,2}$ → $\mathbf{j}_\perp$ is discontinuous or zero

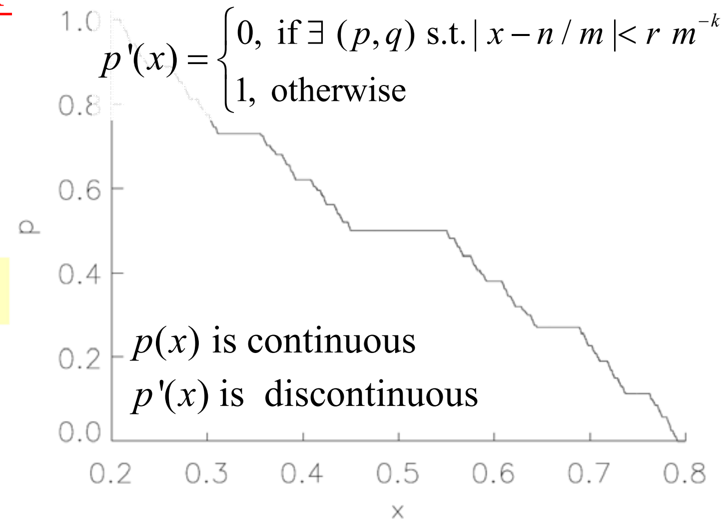
3) $\mathbf{B}_n \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$

→ cannot be inverted to obtain parallel current

$\nabla \cdot \mathbf{j}_\perp$ is not defined

$\mathbf{B} \cdot \nabla$ is *irregularly* and *densely* singular

4) $\nabla \times \mathbf{B}_{n+1} = \mathbf{j} \equiv \sigma \mathbf{B}_n + \mathbf{j}_\perp$



to have a well posed equilibrium with chaotic **B** need to extend beyond ideal MHD

e.g. introduce non-ideal terms, such as resistivity, η, perpendicular diffusion, $\kappa_\perp$, [*Hayashi*, *HINT*], or . . .

→ relax infinity of ideal MHD constraints

Taylor relaxation: a weakly resistive plasma will relax, *subject to single constraint* of conserved helicity

Taylor relaxation

$$\underbrace{W = \int_V \left(p + B^2/2\right) dv}_{\text{plasma energy}}, \qquad \underbrace{H = \int_V \left(\mathbf{A} \cdot \mathbf{B}\right) dv}_{\text{helicity}}$$

Constrained energy functional $F = W - \mu H / 2$, $\mu \equiv$ Lagrange multiplier

Euler-Lagrange equation, for *unconstrained* variations, $\nabla \times \mathbf{B} = \mu \mathbf{B}$

linear force-free field ≡Beltrami field
no pressure gradients

Opposite Extremes . . . .

Ideal MHD

→ imposition of infinity of ideal MHD constraints (nested flux surfaces)

non-trivial pressure profiles, but structure of field is over-constrained

We need something in between . . . .
perhaps an equilibrium model with *finitely* many constraints, and *partial* Taylor relaxation?

Taylor relaxation

→imposition of single constraint of conserved global helicity

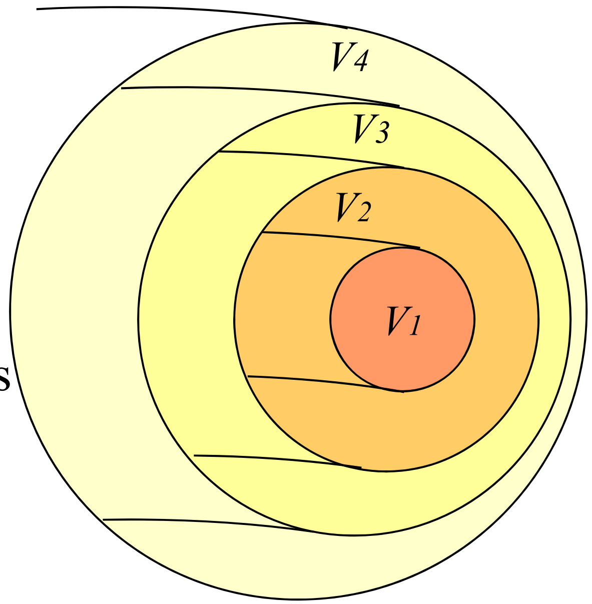
structure of field is not-constrained, but pressure profile is trivial

A multi-volume, partially-constrained model of weakly-resistive MHD equilibria, with topological constraints

Energy, helicity and mass integrals

$$\underbrace{W = \int_V \left(p + B^2/2\right) dv}_{\text{plasma energy}}, \quad \underbrace{H = \int_V \left(\mathbf{A}\cdot\mathbf{B}\right) dv}_{\text{helicity}}, \quad \underbrace{M = \int_V p^{1/\gamma} dv}_{\text{mass}}$$

$\mathbf{B} = \nabla \times \mathbf{A}$



Multi-volume, partially-relaxed energy principle

* A set of N nested toroidal surfaces encloses N annulur volumes

→ the interfaces are assumed to be ideal, $\delta\mathbf{B} = \nabla \times \left(\delta\boldsymbol{\xi} \times \mathbf{B}\right)$

* The multi-volume, partially-constrained, energy functional is

$$F = \sum_{l=1}^{N} \left(W_l - \mu_l H_l / 2 - \nu_l M_l\right)$$

Euler-Lagrange equation for *unconstrained* variations in A

→ field remains tangential to interfaces,
→ a finite number of ideal constraints,
imposed topologically!

In each annulus, the magnetic field satisfies $\nabla \times \mathbf{B} = \mu\mathbf{B}$

Euler-Lagrange equation for variations in interface geometry

Across each interface, the total pressure is continuous $[[p + B^2/2]] = 0$

→ an analysis of the force-balance condition is that the interfaces must have strongly irrational transform

ideal interfaces coincide with KAM surfaces

The equilibrium is defined by pressure & transform profiles, and outermost boundary

The ideal interfaces are chosen to coincide with pressure gradients

$\mathbf{B}\cdot\nabla p = 0$ means that pressure gradients **must** coincide with KAM surfaces $\equiv$ ideal interfaces

A self-consistent model of MHD equilibria with

non-trivial, non-pathological pressure, with $\mathbf{B}\cdot\nabla p = 0$,

chaotic fields, with irregular, chaotic, volumes at every rational

and finite-radial resolution, N radial surfaces

means $\rightarrow$ across the chaotic volumes, the pressure is flat, $\nabla\times\mathbf{B} = \mu\mathbf{B}$,

$\rightarrow$ finite pressure "jumps" at finite set of surfaces,

$\rightarrow$ the "total pressure" $[[p + B^2 / 2]]$ is continuous.

The multi-volume, partially relaxed equilibrium model is

1) consistent with weakly resistive plasma dynamics

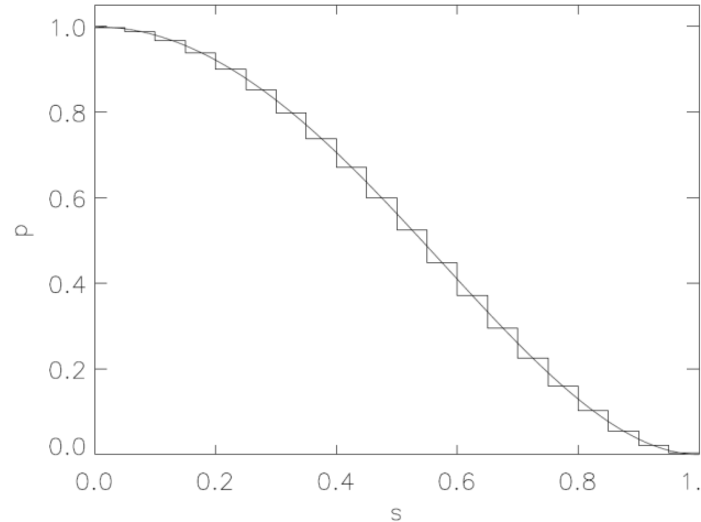
an extension of Taylor relaxation to multiple volumes,
with additional topological constraints to allow for non-trivial pressure

2) consistent with the structure of chaotic fields

does not need to resolve infinite detail of chaos

3) computationally tractable

algorithm does not invert pathological singular operators



Extrema of energy functional obtained numerically

The vector-potential is discretized

* toroidal coordinates (s, ϑ, ζ)
* exploit gauge freedom $\mathbf{A} = A_\vartheta(s,\vartheta,\zeta)\nabla\vartheta + A_\zeta(s,\vartheta,\zeta)\nabla\zeta$
* Fourier $A_\vartheta = \sum_{m,n} a_\vartheta(s)\cos(m\vartheta - n\zeta)$
* Finite-element $a_\vartheta(s) = \sum_i a_{\vartheta,i}(s)\varphi(s)$

piecewise cubic or quintic basis polynomials

and inserted into constrained-energy functional.

* derivatives w.r.t. vector-potential d.o.f. $\rightarrow$ linear equation for Beltrami field

solved using sparse linear solver

* field in each annulus computed independently, distributed across multiple cpus
* field in each annulus depends on
  - $\rightarrow$ poloidal flux, ψ_P, and helicity-multiplier, μ
  - $\rightarrow$ geometry of interfaces

adjusted so interface transform is strongly irrational

Force balance solved using multi-dimensional Newton method.

* geometrical d.o.f. $\equiv$ interface geometry is adjusted to satisfy $[[p + B^2/2]] = 0$
* tangential $\equiv$ angle d.o.f. constrained by spectral-condensation

minimal spectral width [Hirshman, VMEC]

* derivative matrix computed using finite-differences
* quadratic-convergence w.r.t. Newton iterations
* Beltrami fields in each annulus computed in parallel

future work . . .
1) approximate derivative matrix ~ 2nd variation of energy functional
2) implement pre-conditioner

Numerical error in Beltrami field scales as expected

Scaling of numerical error with radial resolution depends on finite-element basis

$\mathbf{A} = A_\vartheta \nabla\vartheta + A_\zeta \nabla\zeta$, $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{j} = \nabla \times \mathbf{B}$, quantify error $\mathbf{j} - \mu\mathbf{B}$

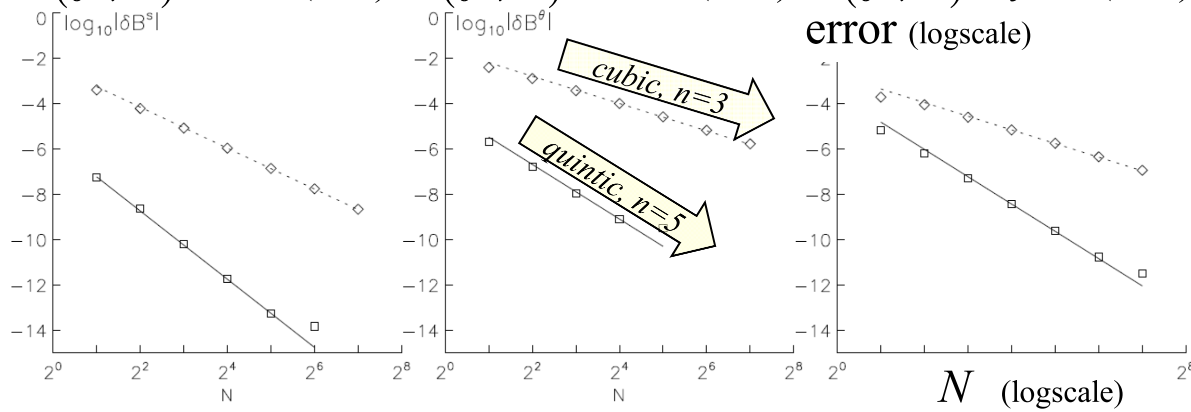
$A_\vartheta, A_\zeta \sim O(h^n)$

h = radial grid size = 1 / N
n = order of polynomial

$$\sqrt{g}B^s = \partial_\vartheta A_\zeta - \partial_\zeta A_\vartheta \sim O(h^n)$$
$$\sqrt{g}B^\vartheta = \quad - \partial_s A_\zeta \sim O(h^{n-1})$$
$$\sqrt{g}B^\zeta = \partial_s A_\vartheta \sim O(h^{n-1})$$

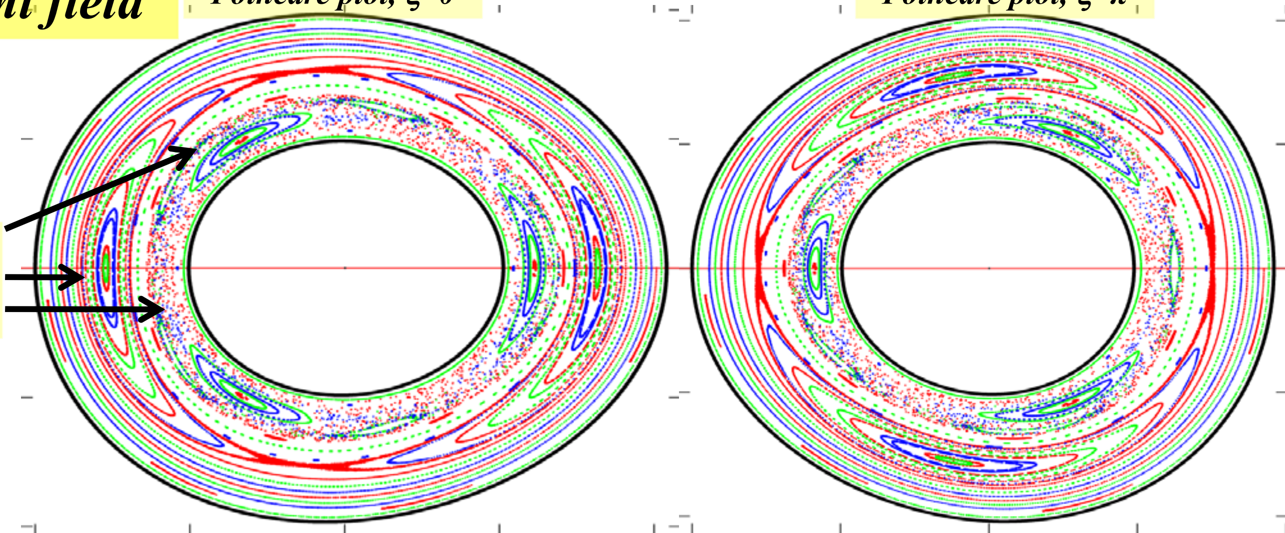
$$\sqrt{g}\, j^s \sim O(h^{n-1})$$
$$\sqrt{g}\, j^\vartheta \sim O(h^{n-2})$$
$$\sqrt{g}\, j^\zeta \sim O(h^{n-2})$$

$(\mathbf{j} - \mu\mathbf{B}) \bullet \nabla s \sim O(h^{n-1})$ $\quad$ $(\mathbf{j} - \mu\mathbf{B}) \bullet \nabla \vartheta \sim O(h^{n-2})$ $\quad$ $(\mathbf{j} - \mu\mathbf{B}) \bullet \nabla \zeta \sim O(h^{n-2})$



Example of chaotic Beltrami field

Poincaré plot, ζ=0 $\quad$ *Poincaré plot, ζ=π*



$$R = 1.0 + r(\vartheta,\zeta)\cos\vartheta,$$
$$Z = \quad r(\vartheta,\zeta)\sin\vartheta,$$

(m,n)=(3,1) island
+ (m,n)=(2,1) island
= chaos

inner surface
$r = 0.1$

outer interface
$r = 0.2 \; + \delta\left[\cos(2\vartheta - \zeta) + \cos(3\vartheta - \zeta)\right]$

Benchmark in axisymmetric geometry

In axisymmetric geometry, equilibria with smooth profiles exist

* in each annulu