Computation of non-axisymmetric equilibria using a partially-relaxed, partially-constrained MHD equilibrium model

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Increasingly it is being realized that a comprehensive understanding of nominally axisymmetric devices requires an understanding of the non-linear, self-consistent plasma response to non-axisymmetric perturbations. In order to construct non-axisymmetric equilibria allowing for topological variations in the field, e.g. magnetic islands and ergodic regions, one must appreciate the fact that non-axisymmetric magnetic fields are generally chaotic. In this case, the commonly used equation of ideal force balance, $\nabla p = \mathbf{j} \times \mathbf{B}$, leads to pathological solutions: any continuous non-trivial pressure that satisfies $\mathbf{B} \cdot \nabla p = 0$ with a chaotic field will have an infinity of discontinuities in the pressure gradient. The perpendicular current $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$ is either zero or discontinuous. This pathological structure causes problems for the so-called Spitzer iterative approach, which is fundamentally ill-posed as it depends on inverting magnetic differential equations, $\mathbf{B} \cdot \nabla (j_\parallel / B) = -\nabla \cdot \mathbf{j}_\perp$, and such equations have a dense set of singularities.

Instead, we have implemented an equilibrium model allowing for chaotic fields by combining elements of Taylor relaxation (that a weakly-resistive plasma will relax, and the topology may break, in order to minimize the plasma energy subject to the constraint of conserved helicity) and ideal constraints. Consider a plasma region comprised of a set of N nested annular regions which are separated by a discrete set of toroidal interfaces, $\mathcal{I}_l$ [Dewar et al., *Entropy*, 2008]. In each volume, $\mathcal{V}_l$, bounded by the $\mathcal{I}_{l-1}$ and $\mathcal{I}_l$ interfaces, the plasma energy, U_l, the helicity, H_l, and the "mass", M_l, are given by the integrals:

$$U_l = \int_{\mathcal{V}_l} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv, \qquad H_l = \int_{\mathcal{V}_l} \mathbf{A} \cdot \mathbf{B} \, dv, \qquad M_l = \int_{\mathcal{V}_l} p^{1/\gamma} \, dv, \tag{1}$$

where $\mathbf{A}$ is the vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$. The equilibrium states that we seek minimize the total plasma energy, subject to the constraints of conserved helicity and mass in each annulus. Such states are extrema of a constrained energy functional, $F = \sum_l (U_l - \mu_l H/2 - \nu_l M)$, where μ_l and ν_l are Lagrange multipliers. Arbitrary variations in the pressure, δp, vector potential, $\delta \mathbf{A}$, and interface geometry, $\delta \boldsymbol{\xi}$ are allowed, except that we assume the magnetic field remains tangential to the interfaces which act as ideal barriers, i.e. on the $\mathcal{I}_l$ we assume that $\delta \mathbf{A} = \delta \boldsymbol{\xi} \times \mathbf{B}$. The Euler-Lagrange equations show that in each annulus the pressure is constant, the magnetic field satisfies $\nabla \times \mathbf{B} = \mu_l \mathbf{B}$, and across each interface the total pressure is continuous, $[[p + B^2/2]] = 0$. The ideal interfaces allow non-trivial "stepped-pressure" profiles to be constructed. The number of interfaces may be made arbitrarily large so that the steps may be made arbitrarily small. Mathematical theorems proving the existence of solutions (under certain conditions) have been presented [Bruno & Laurence, *Commun. Pur. App. Math.*, 1996].

We have implemented this model in the Stepped Pressure Equilibrium Code (SPEC), which uses a mixed Fourier, finite-element representation for the vector potential. Quintic polynomial basis functions give rapid convergence in the radial discretization, and the freedom in the poloidal angle is exploited to minimize a "spectral-width", [Hirshman & Breslau, *Phys. Plasmas*, 1998], giving optimal Fourier resolution. The Beltrami fields in each annulus are constructed in parallel, and a Newton method (with quadratic-convergence) is implemented to adjust the interface geometry to satisfy force-balance. Convergence studies of three-dimensional equilibrium solutions with non-trivial pressure and islands and chaotic fields will be presented. We study the effect of a small resonant perturbation to an otherwise axisymmetric equilibrium. Movies will be shown illustrating a sequence of equilibria with increasing pressure. Near and above the ideal stability boundary, the nonlinear plasma response may be an order of magnitude greater than the applied perturbation, and may be of opposite phase.