Are ghost-surfaces quadratic-flux minimizing?

(Construction of magnetic coordinates for chaotic magnetic fields)

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Part I. Motivation

→A coordinate framework of almost-invariant surfaces can be used to simplify the description of chaotic fields.

For magnetic fields with a continuously nested family of flux surfaces, the construction of magnetic coordinates greatly simplifies the dynamics.

Example: straight-field line magnetic coordinates are analogous to **action-action** coordinates for **integrable** Hamiltonian systems.

shall borrow extensively from
*Hamiltonian & **Lagrangian** mechanics*

→ We seek to generalize the construction of magnetic coordinates to non-integrable magnetic fields

Coordinates are adapted to the invariant structures of chaotic fields
(e.g. KAM surfaces, cantori, and periodic-orbits)
are called "chaotic-coordinates".

Hamiltonian chaos theory provides a solid understanding about the destruction of surfaces

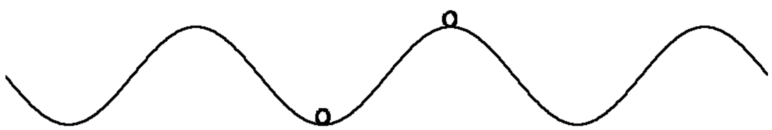
1. Poincaré-Birkhoff Theorem

magnetic field-line action $= \int_{curve} \mathbf{A} \cdot \mathbf{dl}$

curves that extremize the action integral are field lines

For every rational, $\omega = n / m$, where n, m are integers,

- a periodic field-line that is a *minimum* of the action integral will exist
- a . . . . *saddle* . . . . will exist



2. Aubry-Mather Theorem

For every $\omega \neq n / m$,

- there exists an "irrational" field-line that is a *minimum* of the action integral

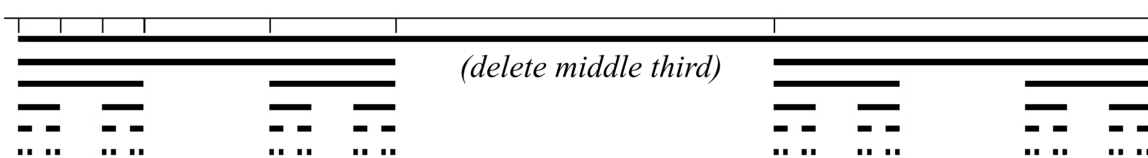
3. Kolmogorov-Arnold-Moser Theorem

- if ω is *very* irrational then the Aubry-Mather field line will cover a surface, called a KAM surface

  ω is very irrational if there exist an r, k such that $|\omega - n / m| > r\, m^{-k}$, for all integers n, m

  Diophantine condition

- if not, the Aubry-Mather field line will cover a Cantor set, called a cantorus

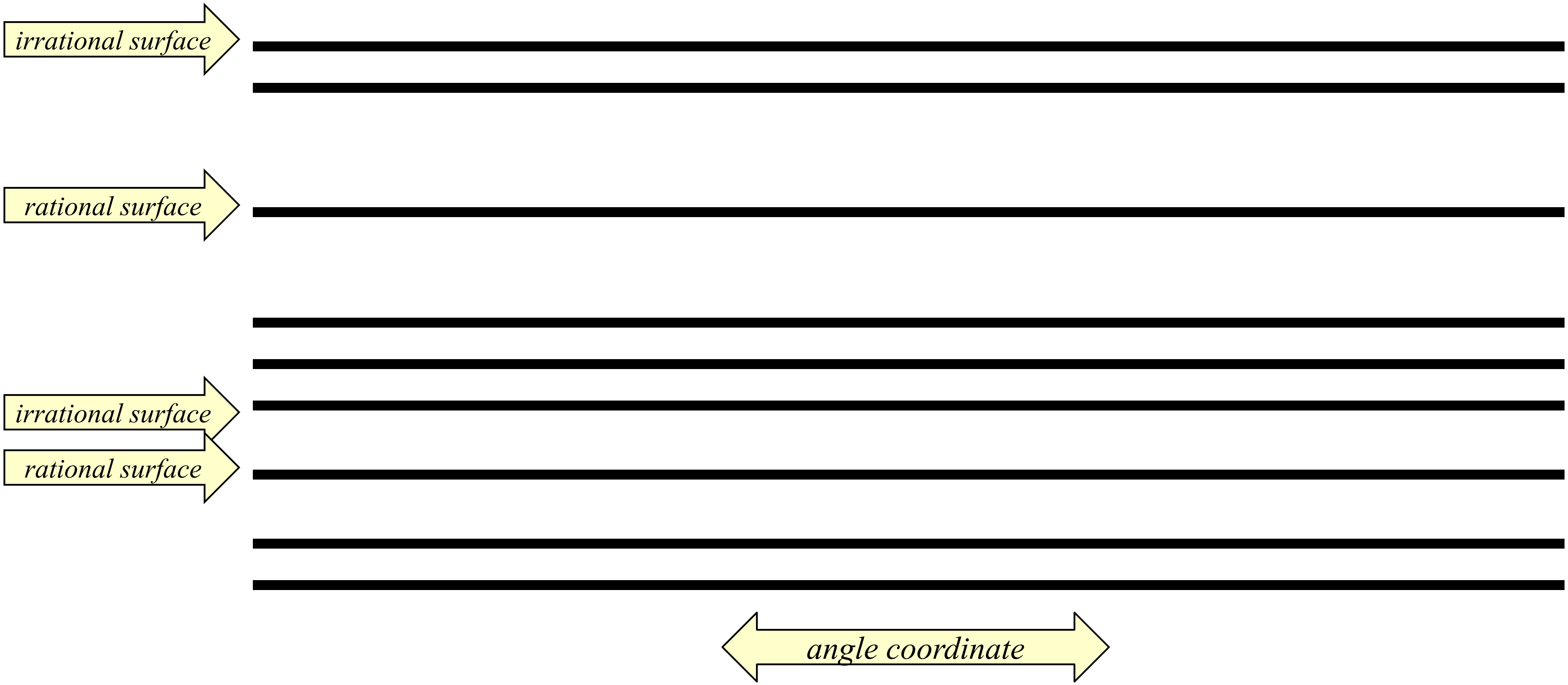


4. Greene's residue criterion

- *the existence of a KAM surface is related to the stability of the nearby Poincaré-Birkhoff periodic orbits*

Simplified Diagram of the structure of integrable fields,

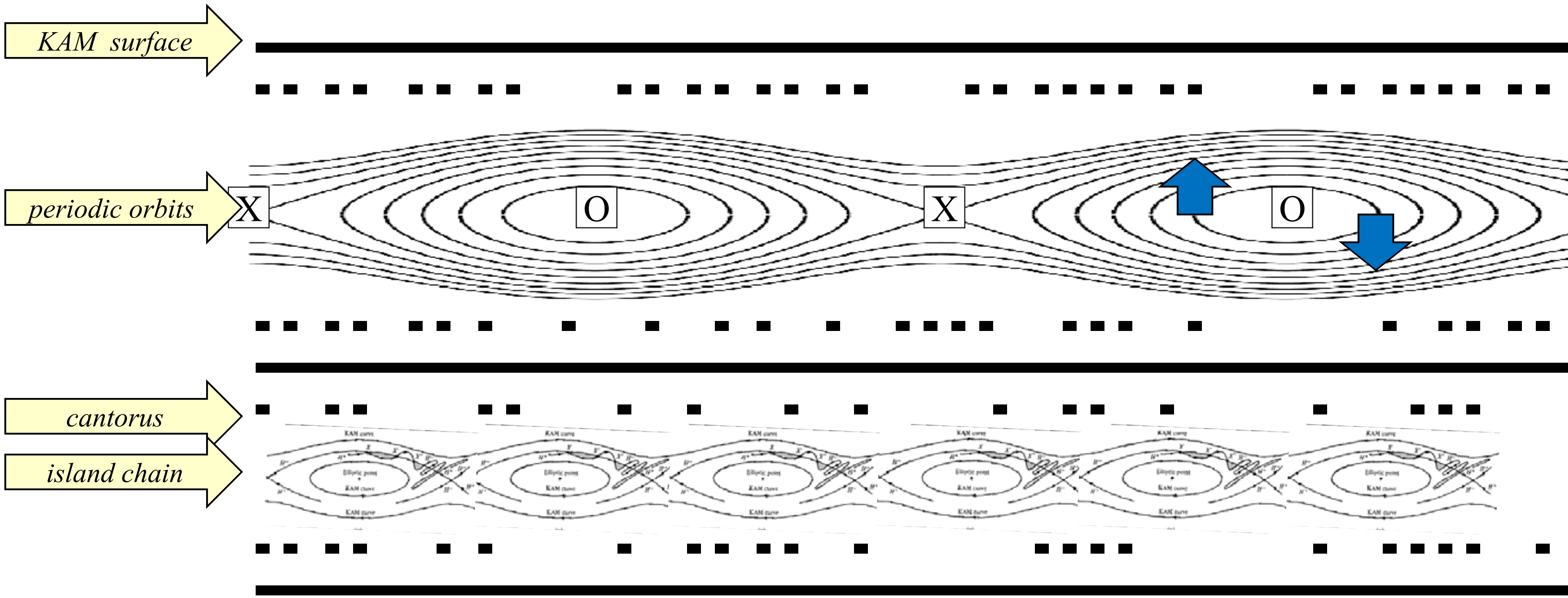
→showing continuous family of invariant surfaces



Action-angle coordinates can be constructed for "integrable" fields

- the "action" coordinate coincides with the invariant surfaces
- dynamics then appears simple

Simplified Diagram of the structure of non-integrable fields, →showing the fractal hierarchy of invariant sets



After perturbation:

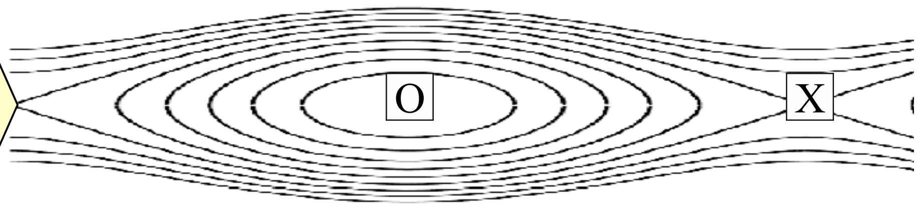
the rational surfaces break into islands, “stable” and “unstable” periodic orbits survive,
some irrational surfaces break into cantori,
some irrational surfaces survive (KAM surfaces), break into cantori as perturbation increases,

→ action-angle coordinates can no longer be constructed globally

Simplified Diagram of the structure of non-integrable fields,

→showing the fractal hierarchy of invariant sets

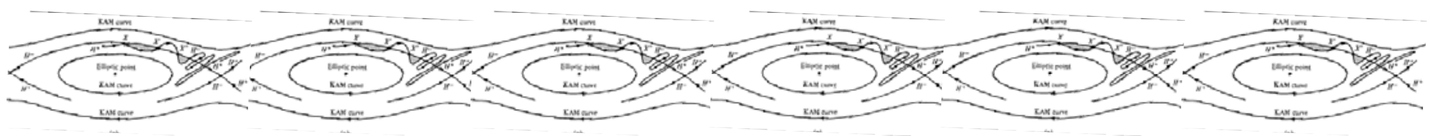
this talk will concentrate on islands and periodic orbits

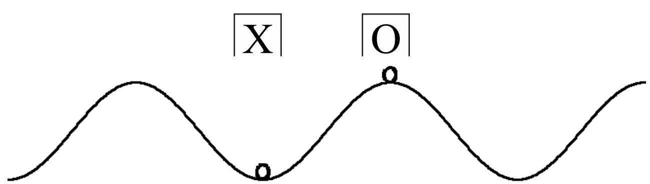


"O" point is a *saddle* of the action

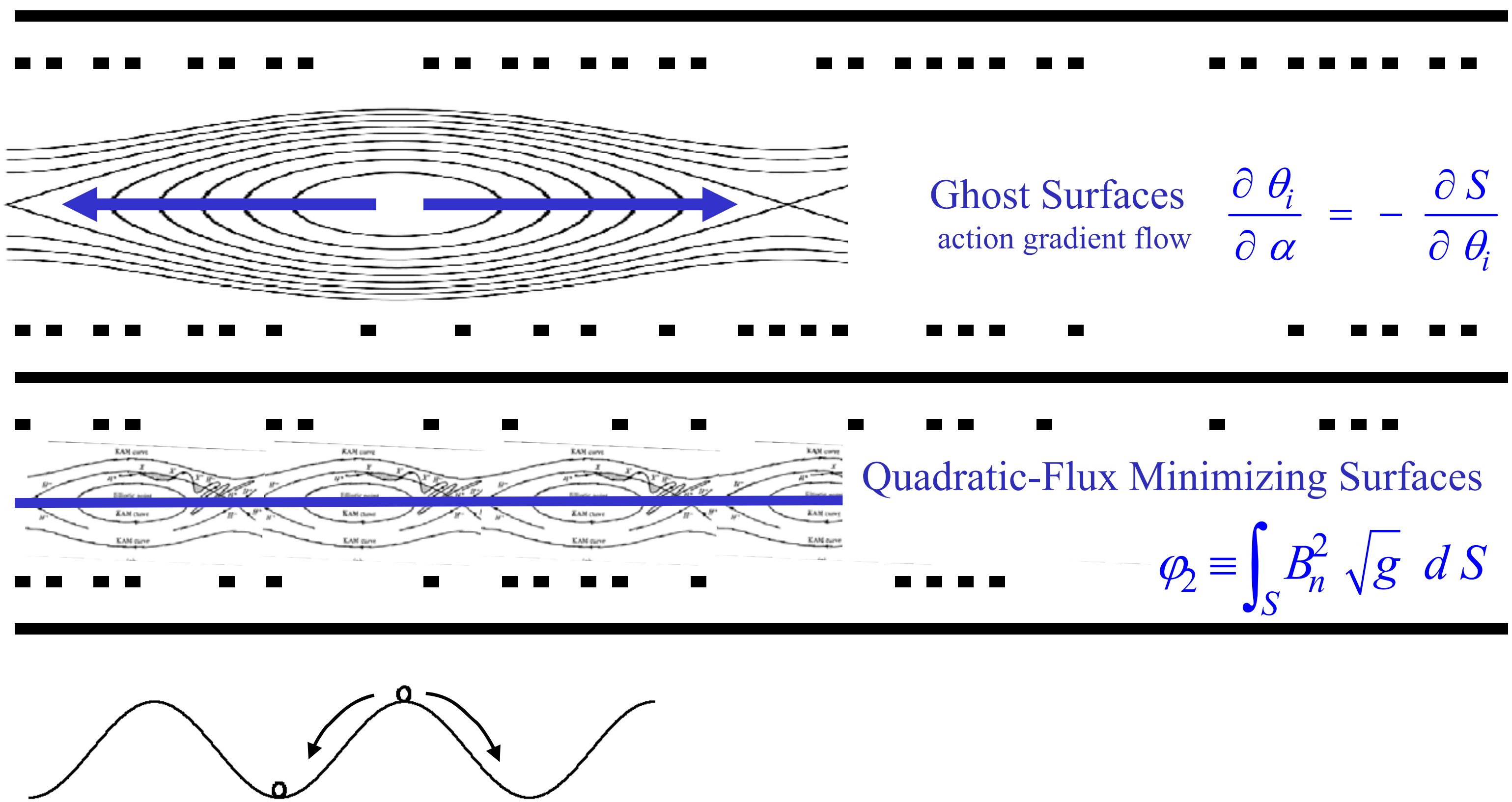
"X" point is a *minimum* of the action

$$S[\,\theta(\phi)\,] = \int \mathbf{A}(\theta,\dot{\theta},\phi)\,.\,\mathbf{dl}$$





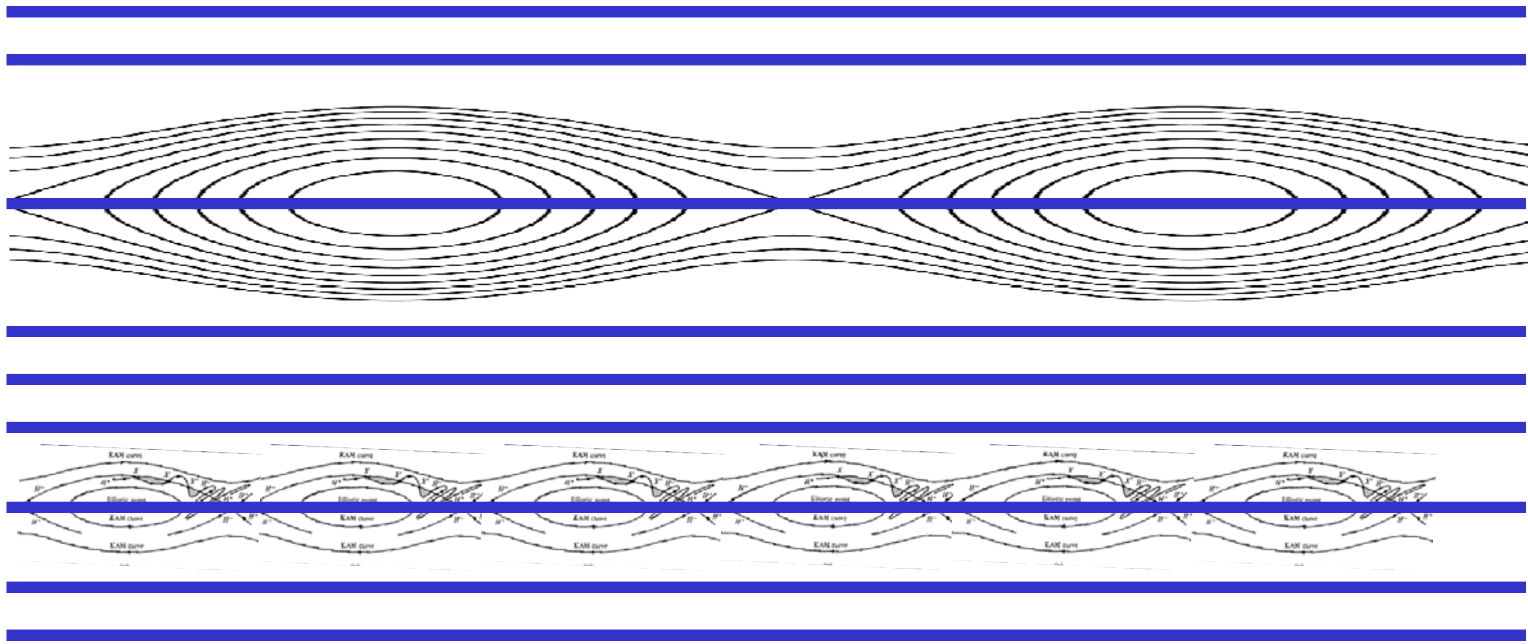
Simplified Diagram of the structure of non-integrable fields, →showing rational, "almost-invariant" surfaces



Ghost surfaces and **Quadratic-Flux Minimizing surfaces** pass <u>through the island chains</u> and connect the O and X periodic orbits.

(These will be described in more detail later.)

Simplified Diagram of the structure of non-integrable fields,
→showing coordinate surfaces that pass through islands

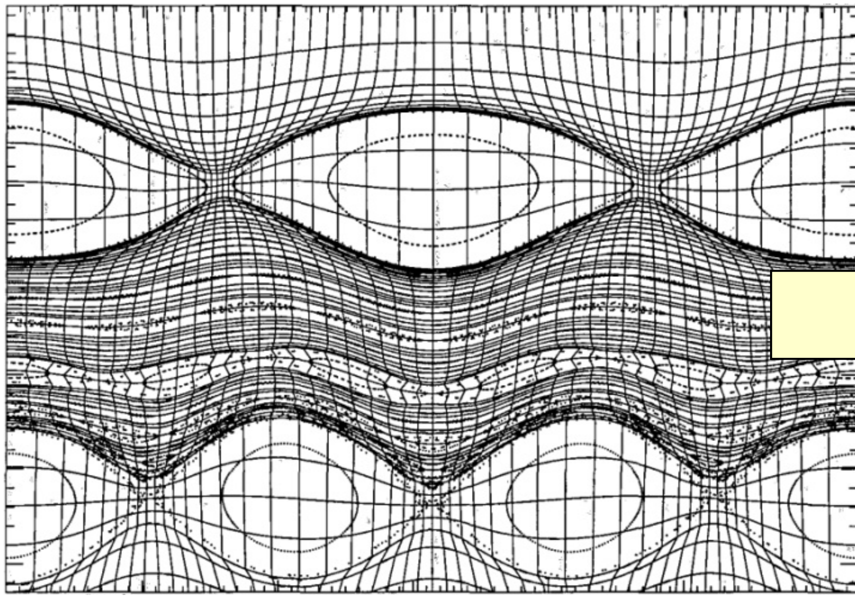


"Chaotic-coordinates" can be constructed

- coordinate surfaces are adapted to the fractal hierarchy of remaining invariant sets
- ghost surfaces ≡ quadratic-flux minimizing surfaces are "almost-invariant"
- dynamics appears "almost-simple"

Chaotic coordinates “straighten out” chaos

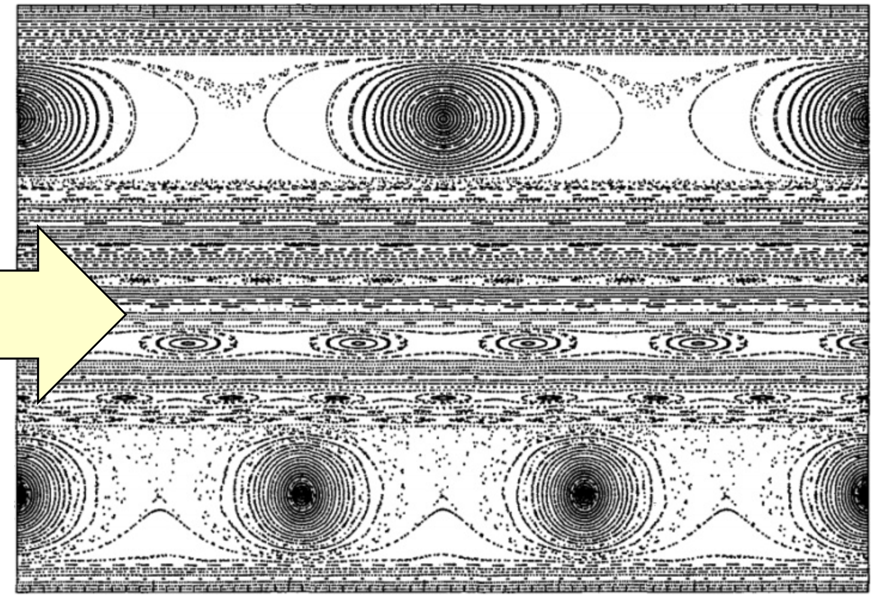
Poincaré plot of chaotic field
(in **action-angle** coordinates of **unperturbed** field)



old radial coordinate →

old angle coordinate →

Poincaré plot of chaotic field
in chaotic coordinates



new radial coordinate →

new angle coordinate →

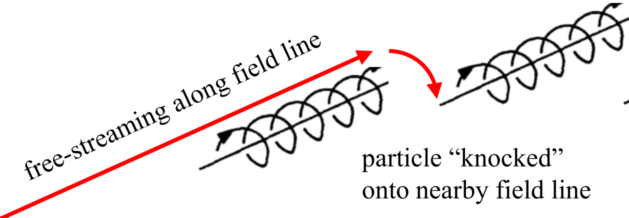
phase-space is partitioned into (1) regular (“irrational”) regions with “good flux surfaces”, temperature gradients
and (2) irregular (“ rational”) regions with islands and chaos, flat profiles

Chaotic coordinates simplify anisotropic transport

The temperature is constant on ghost surfaces, *T=T(s)*

1. Transport *along* the magnetic field is *unrestricted*

→ consider parallel random walk, with **long** steps ≈ *collisional mean free path*

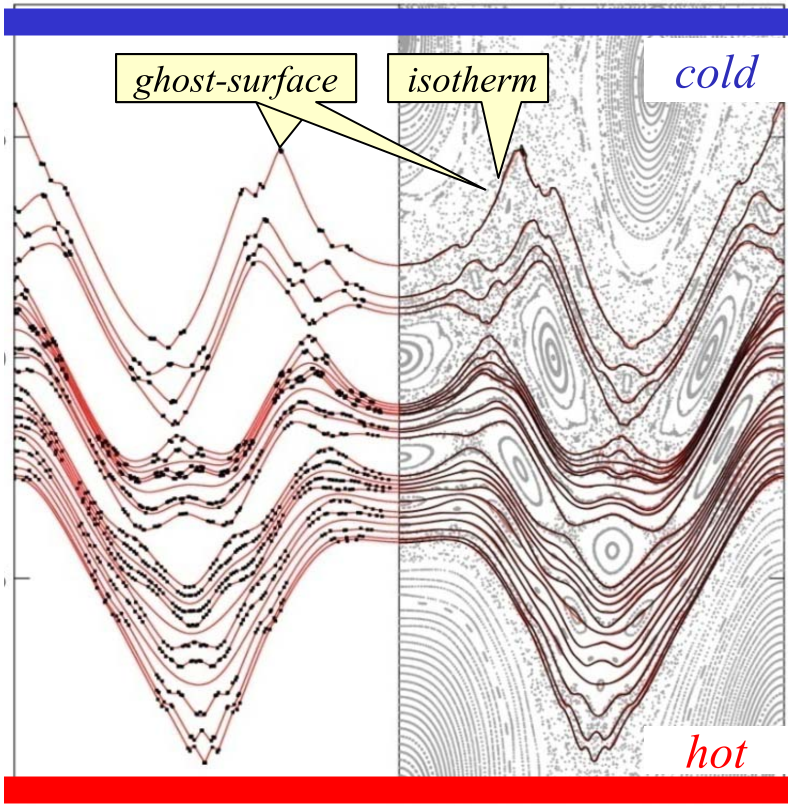


2. Transport *across* the magnetic field is *very small*

→ consider perpendicular random walk with **short** steps ≈ *Larmor radius*

3. Anisotropic diffusion balance $\boxed{\kappa_{\|} \nabla_{\|}^2 T + \kappa_{\perp} \nabla_{\perp}^2 T = 0, \quad \kappa_{\|} \gg \kappa_{\perp}, \quad \kappa_{\perp} / \kappa_{\|} \sim 10^{-10}}$

$2^{12} \times 2^{12} = 4096 \times 4096$ grid points (to resolve small structures)



4. Compare solution of numerical calculation to ghost-surfaces

5. The temperature adapts to KAM surfaces, cantori,
and ghost-surfaces!
i.e. T=T(s), where s=const. is a ghost-surface

from $T=T(s,\theta,\phi)$ to $T=T(s)$ is a fantastic simplification, allows analytic solution

$$\frac{dT}{ds} \propto \frac{1}{\kappa_{\|}\, \varphi_2 + \kappa_{\perp}\, G}$$

Part II. Definitions

The action functional is a line integral, $S[C]=\int_C L(\vartheta,\dot{\vartheta},t)dt$, along arbitrary curve, $\vartheta=\vartheta(t)$, where $L\equiv$ Lagrangian,

$$\delta S=\int_C\left[\frac{\partial L}{\partial\vartheta}-\frac{d}{dt}\frac{\partial L}{\partial\dot{\vartheta}}\right]\delta\vartheta\,dt, \qquad \text{action-gradient} \equiv\frac{\delta S}{\delta\vartheta} \equiv \frac{\partial L}{\partial\vartheta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\vartheta}}$$

Ghost surfaces are defined by an action-gradient flow

$$\frac{\partial\theta}{\partial\tau} = -\left(\frac{\delta S}{\delta\theta}\right)$$

infinite dimensional functional derivative

Quadratic-flux minimizing surfaces are defined by a variational principle

quadratic-flux functional $\varphi_2\equiv\dfrac{1}{2}\displaystyle\iint\left(\frac{\delta S}{\delta\theta}\right)^2 d\vartheta\,dt$

Toroidal magnetic fields are a Hamiltonian system →may construct the field-line Lagrangian

1. The magnetic field line Hamiltonian is defined by $\mathbf{B} = \nabla\psi \times \nabla\vartheta + \nabla\chi \times \nabla\zeta$

$\psi \equiv$ "momentum", $\vartheta \equiv$ "position", $\xi \equiv$ "time" $= t$, $\chi(\psi,\vartheta,\zeta) \equiv$ field line Hamiltonian,

2. Can construct the "Action Integral", which is a line integral along an arbitrary curve,

$$S[C] = \int_C \mathbf{A} \bullet d\mathbf{l},$$

which is analogous to $S[C] = \int_C L(\vartheta, \dot{\vartheta}, t)\,dt$, $L \equiv$ Lagrangian,

(will assume that "velocity" is determined by "position", i.e. $\dot{\vartheta} = d\vartheta / dt$,

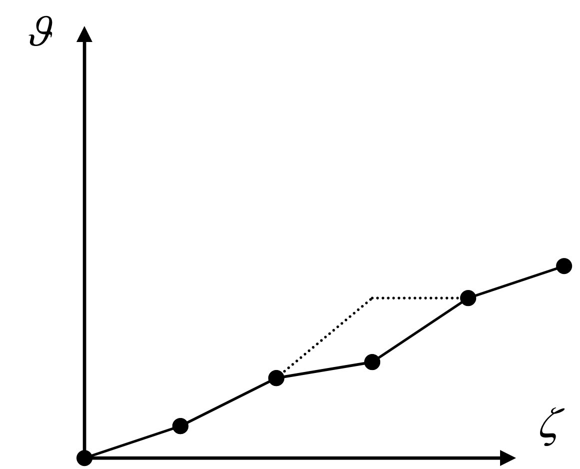
so that action is function of position only, i.e. $S = S[\vartheta(t)]$).

<u>Arbitrary "trial" curves that *extremize* the action integral correspond to "physical" magnetic field lines</u>

4. Shall restrict attention to (discrete) piecewise-linear, periodic curves

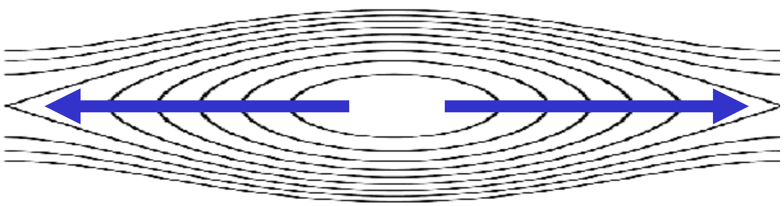
A curve, $\vartheta_i(\zeta)$, is a periodic orbit if it minimizes the action,

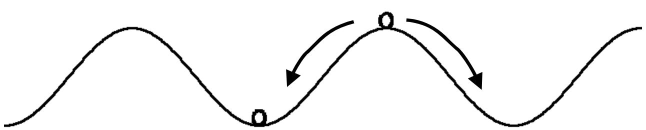
i.e. the action gradient is zero $\left(\dfrac{\delta S}{\delta \vartheta}\right)_i \equiv \dfrac{\partial S}{\partial \vartheta_i} = 0$



Definition of (1) Ghost Surfaces, and (2) Quadratic-Flux Minimizing Surfaces

Ghost Surfaces are defined by an action gradient flow $\frac{\partial \theta_i}{\partial \alpha} = -\frac{\partial S}{\partial \theta_i}$





$\rightarrow$ begin at *"stable"* periodic orbit (closed curve) which is a saddle of the action-integral

$\rightarrow$ give small initial *"push"* in the decreasing direction

$\rightarrow$ allow curve to *"flow"* in direction of steepest descent

$\rightarrow$ curve will finally make it to *"unstable"* periodic orbit which is a minimum of the action-integral

Quadratic-Flux Minimizing Surfaces are surfaces that minimize $\varphi_2 \equiv \frac{1}{2} \iint \left(\frac{\delta S}{\delta \theta} \right)^2 d\vartheta \, d\zeta$

$\rightarrow$ action-gradient is related to "normal field",

$\rightarrow$ for a given magnetic field, **B**, adjust geometry of surface to minimize $\varphi_2 \approx \frac{1}{2} \iint B_n^2 \, d\vartheta \, d\zeta$

(Intuitive definition)

Numerical Evidence:
Ghost-surfaces are almost identical to QFMin surfaces.

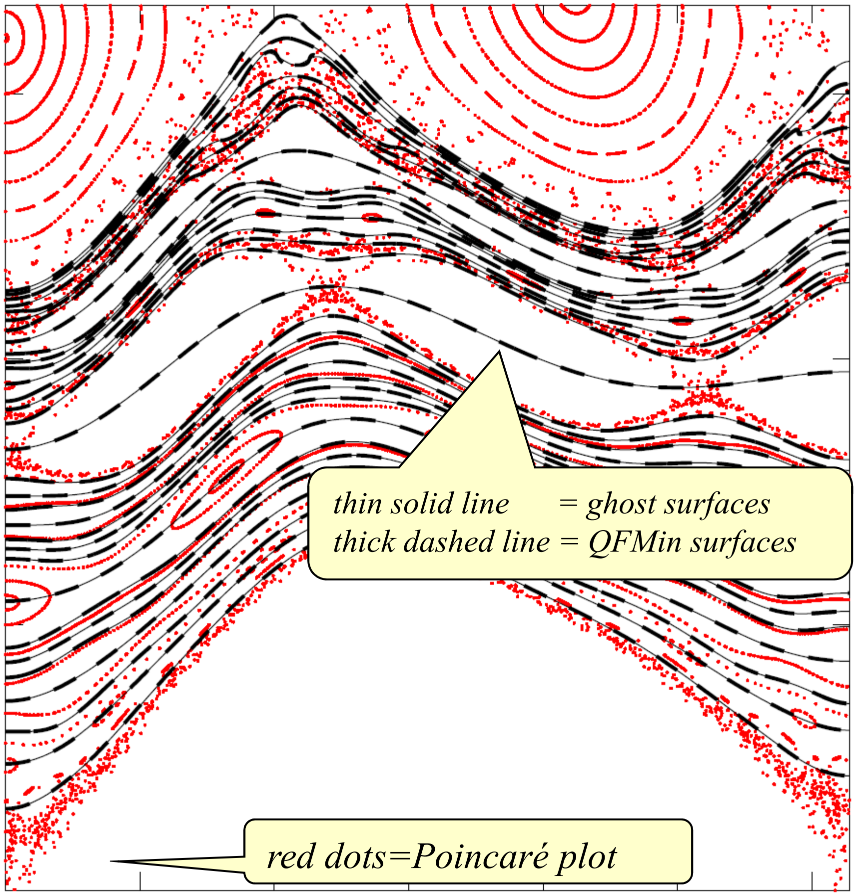
Ghost surfaces are defined by action-gradient flow
QFMin surfaces defined by minimizing quadratic-flux

→ no obvious reason why these different definitions should give the same surfaces

→Numerical evidence suggests that ghost-surfaces and QFMin surfaces are almost the same

→This is confirmed to 1st order using perturbation theory

→ Opens possibility that fast, robust construction of unified almost-invariant surfaces for chaotic coordinate framework



For strong chaos, and high periodicity, discrepancies exist between ghost-surfaces & QFMin surfaces.

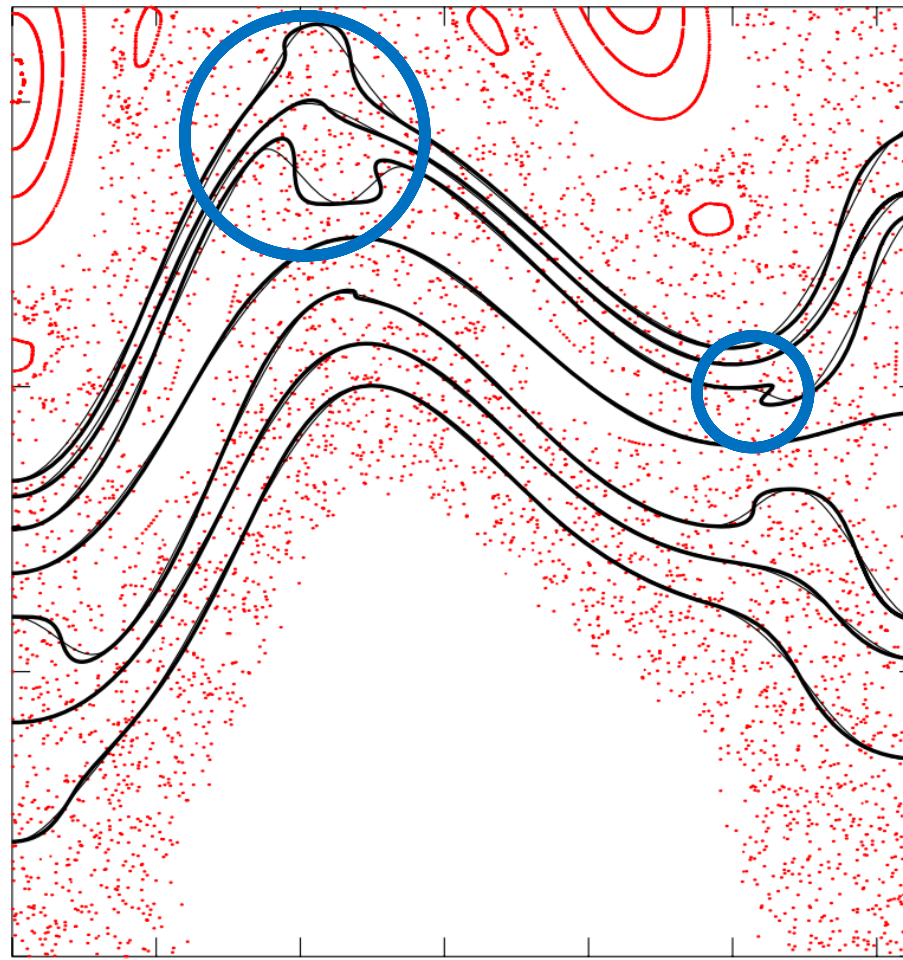
→Ghost-surfaces have better properties
(guaranteed to not intersect, graphs over angle)

but they are more difficult to construct.

→QFMin surfaces have
intuitive definition of being "almost-invariant", and
are easily constructed using the variational principle

but high-periodicity QFMin surfaces become
too deformed in regions of strong chaos

→We want a unified approach that combines
best features of both!



Current research: angle coordinate can be re-defined, so that ghost-surfaces and QFMin surfaces are identical.

Introduce new angle, Θ, via angle transformation, $\vartheta = \vartheta(\Theta, \zeta)$

Construction of ghost-surfaces and QFMin surfaces is "angle-dependent"

The action gradient in the new angle is $\dfrac{\delta S}{\delta \Theta} = \dfrac{\partial \vartheta}{\partial \Theta} \dfrac{\delta S}{\delta \vartheta}$

"New" ghost-surfaces are defined $\dfrac{\partial \Theta}{\partial \tau} = -\left(\dfrac{\delta S}{\delta \Theta}\right)$

"New" quadratic-flux functional $\varphi_2 \equiv \dfrac{1}{2}\int\int\left(\dfrac{\delta S}{\delta \Theta}\right)^2 d\Theta dt$

Can choose unique angle transformation that

makes ghost-surfaces and QFMin surfaces identical

Action-gradient minimizing pseudo-orbits and almost-invariant tori
R.L.Dewar, S.R.Hudson & A.M.Gibson
Communications in Nonlinear Science and Numerical Simulations 17(5):2062, 2012

Generalized action-angle coordinates defined on island chains
R.L.Dewar, S.R.Hudson & A.M.Gibson
Plasma Physics and Controlled Fusion 55:014004, 2013

Concluding remarks

→ Straight-field line magnetic coordinates are very useful for describing integrable magnetic fields.

→ The phase space of magnetic fields breaks apart slowly and in a well-defined way with the onset of perturbation.

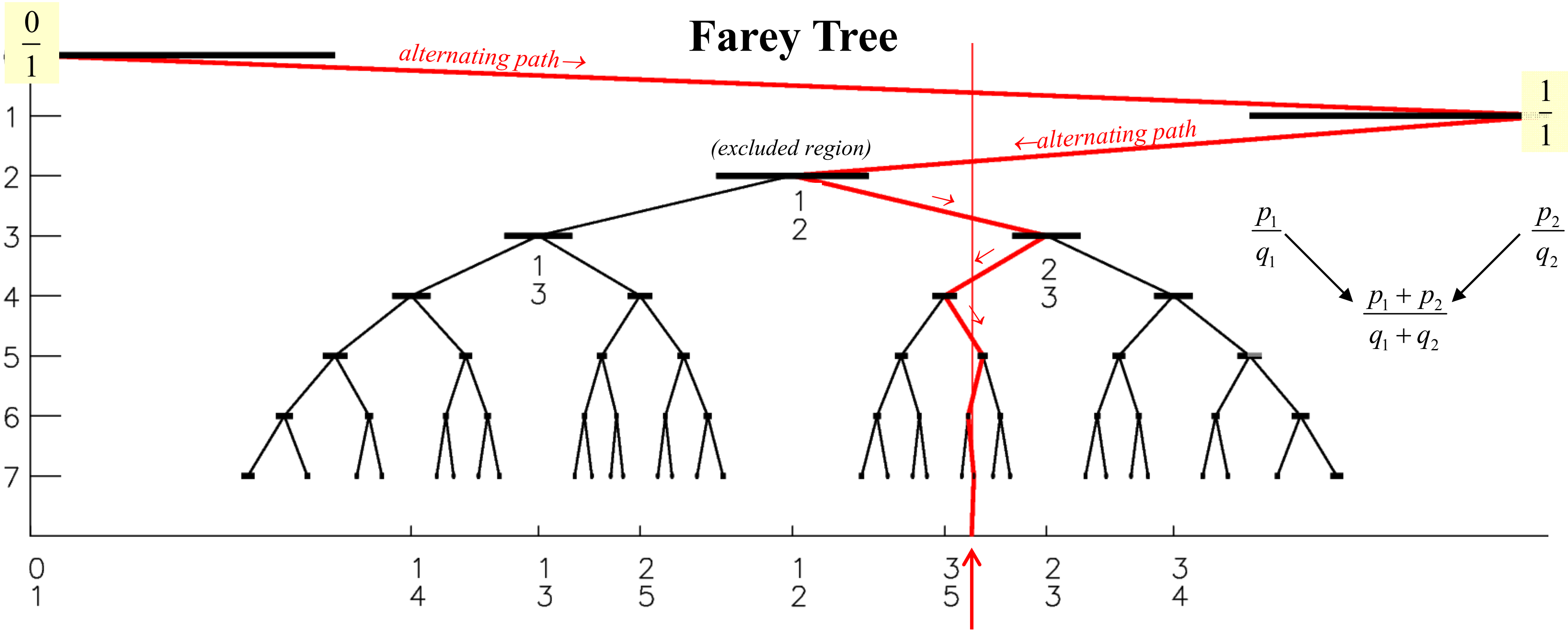
→ This suggests that the construction of “chaotic-coordinates”, which are adapted to the invariant sets of chaotic fields, will similarly be useful.

→ Ghost-surfaces and QFMin surfaces are a natural generalization of flux surfaces.

→ Ghost-surfaces and QFMin surfaces can be unified by an appropriate choice of angle.

WHERE TO START? START WITH CHAOS

The fractal structure of chaos is related to the structure of numbers



islands & chaos emerge at every rational → about each rational n/m, introduce excluded region, width r/m^k

KAM Theorem → flux surface can survive if $|\omega - n/m| > r/m^k$, for all n, m

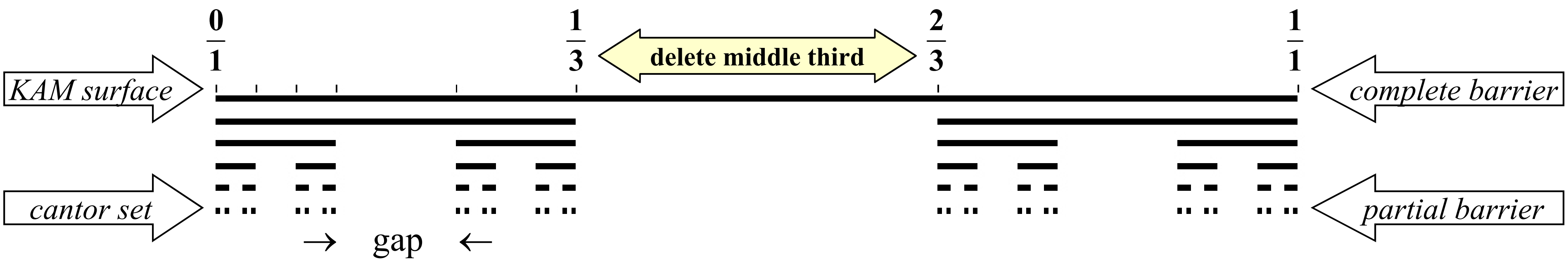
(Kolmogorov, Arnold, Moser) *we say that ω is "strongly - irrational" if ω avoids all excluded regions*

Greene's residue criterion → the most robust flux surfaces are associated with alternating paths

→ Fibonacci ratios $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, ..$

Q) How do non-integrable fields confine field lines?

A) Field line transport is restricted by KAM surfaces and cantori

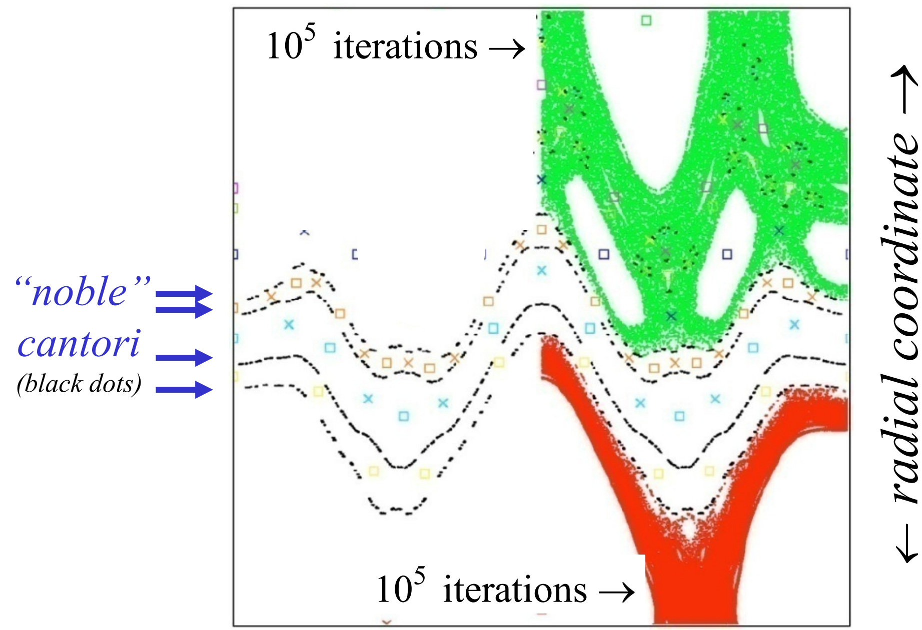


→ *KAM surfaces are closed, toroidal surfaces; and **stop** radial field line transport*

→ *Cantori have many holes, but still **cantori can severely "slow down"** radial field line transport*

→ *Example, all flux surfaces destroyed by chaos, but even after **100 000 transits** around torus the field lines **cannot get past cantori***

Calculation of cantori for Hamiltonian flows
S.R. Hudson, Physical Review E 74:056203, 2006



THEN, ADD PLASMA PHYSICS

Force balance means the pressure is a *“fractal staircase”*

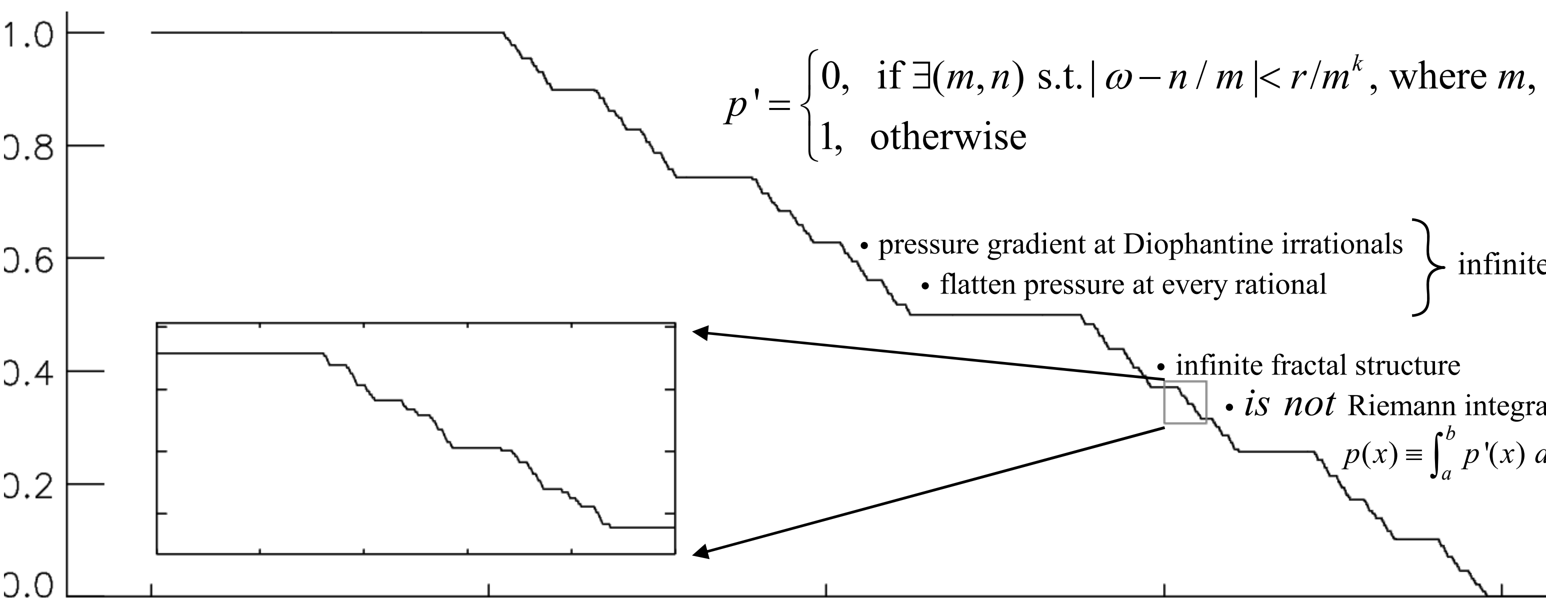
- $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$, implies that $\boldsymbol{B}.\nabla p=0$ i.e. pressure is constant along a field line

- Pressure is flat across the rationals (assuming no “pressure” source inside the islands)
  → islands and chaos at every rational → chaotic field lines wander about over a volume

- Pressure gradients supported on the “most-irrational” irrationals
  → surviving “KAM” flux surfaces confine particles and pressure

Diophantine Pressure Profile

is it pathological?



$$p' = \begin{cases} 0, & \text{if } \exists (m,n) \text{ s.t.} \,|\,\omega - n/m\,| < r/m^k, \text{ where } m,\ n \text{ are integer} \\ 1, & \text{otherwise} \end{cases}$$

- pressure gradient at Diophantine irrationals
- flatten pressure at every rational

} infinitely discontinuous

- infinite fractal structure
- *is not* Riemann integrable

$$p(x) \equiv \int_a^b p'(x)\,dx \neq \lim_{N\to\infty} \sum_{n=1}^{N} p'(x_n)(x_{n+1} - x_n)$$