Plasma Relaxation Dynamics Moderated by Current Sheets

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Generalizations of Taylor Relaxation

This presentation

- Shows there is a reduced magneto-hydro-*dynamics* that leads to Taylor's relaxed *equilibrium* states in the static limit by using Hamilton's Principle to derive self-consistent dynamics from a *relaxed MHD* (RxMHD) Lagrangian.
- Calculates the modulated current sheet driven by a resonant perturbation at a rational surface by treating the plasma as *two* relaxation regions – 2-region example of *multi-relaxed MHD* (MRxMHD)

Hamilton's Action Principle in domain Ω: $\delta S = 0$

$S = \int dt \int_\Omega \mathcal{L}\, d^3x$ denotes the *action*. Its *first variation* is:

$$\delta S = \int dt \int_\Omega \delta\mathcal{L}\, d^3x + \epsilon \int dt \int_{\partial\Omega} \mathcal{L}\, \boldsymbol{\xi}\cdot\mathbf{n}\, dS$$

$\delta\mathcal{L}$ is $O(\epsilon)$ *Eulerian* variation of action density $\mathcal{L}$,
$\epsilon\boldsymbol{\xi}$ is *Lagrangian* displacement of fluid element positions $\mathbf{r}$ on boundary $\partial\Omega$

MHD Lagrangian density is

$$\mathcal{L}_{\mathrm{MHD}} = \frac{\rho\mathbf{v}\cdot\mathbf{v}}{2} - \frac{p}{\gamma - 1} - \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is velocity, ρ is mass density, p is pressure and $\mathbf{B}$ is magnetic field

Constraints: Holonomic

- IMHD = Ideal MHD ($\rho, \mathbf{B}$ and p *holonomically* constrained, i.e. *locally* "frozen in" to fluid elements):

$$\delta\rho = -\epsilon\nabla\cdot(\rho\boldsymbol{\xi}),\ \delta p = -\epsilon(\boldsymbol{\xi}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{\xi}),\ \delta\mathbf{B} = \nabla\times\delta\mathbf{A}$$

$$\delta\mathbf{A} = \epsilon\boldsymbol{\xi}\times\mathbf{B} + \nabla\delta\chi$$

- RxMHD = Relaxed MHD (only ρ holonomically constrained — no effect on static equilibrium — magnetic helicity and entropy constrained only *globally*):

$$\delta\rho = -\epsilon\nabla\cdot(\rho\boldsymbol{\xi})$$

- MRxMHD = Multi-Relaxed MHD (multiple RxMHD regions Ω_i separated by current sheet transport barriers $\partial\Omega_i$, with holonomic constraints on either side, ±, of $\partial\Omega_i$ to keep $\mathbf{B}$ tangential to the current sheets):

$$\delta\rho = -\epsilon\nabla\cdot(\rho\boldsymbol{\xi}) \text{ in } \Omega_i,\ \delta\mathbf{A}_{\rm tgt} = (\epsilon\boldsymbol{\xi}\times\mathbf{B} + \nabla\delta\chi)_{\rm tgt} \text{ on } \partial\Omega_i^{\pm}$$

Constraints: Global

- IMHD = Ideal MHD (none — mass, entropy and magnetic flux and helicity within Ω all automatically conserved as a consequence of the holonomic constraints):
- RxMHD = Relaxed MHD (mass and flux automatic, entropy and magnetic helicity are constrained globally within Ω using Lagrange multipliers τ and μ respectively):

$$\mathcal{L} = \mathcal{L}_{\mathrm{MHD}} + \tau \frac{\rho \ln(Cp/\rho^{\gamma})}{\gamma - 1} + \mu \frac{\mathbf{A}\cdot\mathbf{B}}{2\mu_0}$$

where γ and C are thermodynamic gas constants.

- MRxMHD = Multi-Relaxed MHD (mass and flux automatic, entropy and magnetic helicity are constrained globally within the multiple RxMHD regions Ω_i using Lagrange multipliers τ_i and μ_i giving p and q profile control).

MRxMHD equations

- Continuity: $\dfrac{d\rho}{dt} = -\rho \boldsymbol{\nabla}\cdot\mathbf{v}$
- Require Hamilton's Principle: $\delta S = 0$ for all independent variations of $\mathbf{r}$, p and $\mathbf{A}$, where:

$$\delta S = \sum_i \int dt \int_{\Omega_i} \delta\mathcal{L}_i \, d^3x + \epsilon \sum_i \int dt \int_{\partial\Omega_i} \mathcal{L}_i \, \boldsymbol{\xi}\cdot\mathbf{n} dS$$

- Resulting *Euler–Lagrange* equations are:

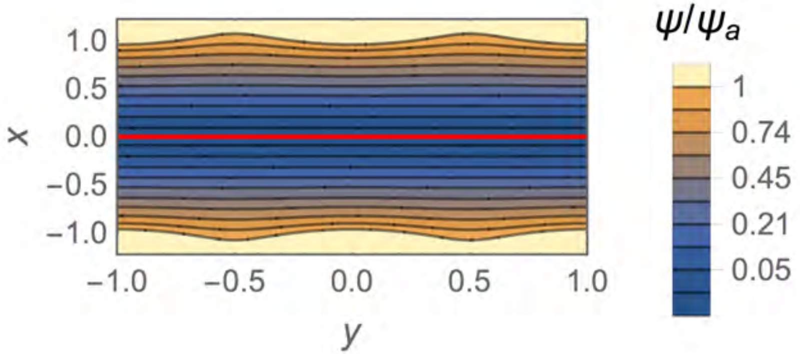
$\rho \dfrac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla} p$ (momentum equation)

$p = \tau_i \rho$ (isothermal equations of state in each region)

$\boldsymbol{\nabla}\times\mathbf{B} = \mu\mathbf{B}$ (Beltrami equations)

$\left[\!\left[p + \frac{B^2}{2\mu_0} \right]\!\right]_i = 0$ (pressure jump conditions at interfaces)

Hahm-Kulsrud Rippled Slab Model



- Simple slab model for resonant current sheet formation near $x = 0$ in response to symmetrical periodic perturbation at boundaries $x = \pm a$
- Hahm & Kulsrud, Phys. Fluids 1985, found 2 solutions:

- shielding current sheet on $x = 0$ (shown in red)

$$\psi = aB_y^a \left[\frac{x^2}{2a^2} + \frac{\alpha}{\sinh(ka)} |\sinh(kx)| \cos(ky) \right]$$

- island with no current sheet

$$\psi = aB_y^a \left[\frac{x^2}{2a^2} + \frac{\alpha}{\cosh(ka)} \cosh(kx) \cos(ky) \right]$$

where $B^a{}_y$ is |unperturbed poloidal field| at boundaries and $\alpha \ll 1$

2-region MRxMHD HKT model

HK-style model is natural application of MRxMHD because:

- Linearity of Beltrami equation leads to easily solvable, *linear GS equation (Poisson in small-μ limit.)*
- Symmetry about, and straightness of, current sheet at x = 0: gives most geometrically simple *2-region* geometry

Relaxation scenario:

- Switch-on: *ripple* on upper and lower boundaries slowly increased from zero (plane slab) to final amplitude
- A *shielding current* sheet at $x = 0$ resonance develops
- Kruskal-Kulsrud damping: evolution through *equilibria*
- Connect equilibrium sequence by *helicity conservation*

Grad-Shafranov-Beltrami equations

Grad-Shafranov equation for force-free field in slab geometry:

$$\mathbf{B} = \nabla z \times \nabla \psi + F(\psi)\nabla z \qquad \nabla^2\psi + FF' = 0$$

$\nabla \times \mathbf{B} = \mu \mathbf{B}$ (Beltrami equation) is satisfied by requiring:

$\nabla^2 \psi = \mu F$ with $F(\psi) = C - \mu\psi$, giving $(\nabla^2 + \mu^2)\psi = C$

General Solution: $\psi = \overline{\psi} + \dfrac{\overline{F}}{B_0}\psi_0(x|\mu) + \widehat{\psi}(x, y)$

where $\overline{\psi}$ is cross-sectional average of ψ, $\psi_0(x|\mu) \equiv \dfrac{B_0}{\mu}(1 - \cos \mu x)$

is plane slab solution, $\overline{F}$ is the cross-sectional average of B_z ,

and $\widehat{\psi}$ obeys a *homogeneous* Beltrami equation: $(\nabla^2 + \mu^2)\widehat{\psi} = 0$

with boundary conditions such that ψ is constant on boundary and on cuts.

Extension of HK shielding solution

Helicity conservation requires *three extensions of HK solution*

Instead of the HK harmonic component ψ_1 we use ansatz

$$\widehat{\psi}(x,y) \equiv \frac{2\alpha\psi_a}{\sinh k_1 a}\Bigg(|\sinh k_1 x| \cos ky + \gamma_{\mathrm{S}}\frac{k_1}{\mu}|\sin \mu x|\Bigg) - \overline{\psi}\cos\mu x$$

where:

1. $\widehat{\psi}$ is a solution of the *Beltrami equation* $(\nabla^2 + \mu^2)\widehat{\psi} = 0$
   It is only *harmonic* in the *small-μ limit.* Likewise
   $k_1(\mu) \equiv (k^2 - \mu^2)^{1/2} \to k \text{ only as } \mu \to 0$
2. The term in γ_{S} was introduced in Dewar *et al.* 2013 to allow control of the *total current* in the sheet
3. The term in $\overline{\psi}$ is required for poloidal flux conservation

Slab-Toroidal analogies

poloidal periodicity length:	toroidal periodicity length:	rotational transform (helical frame):

$$L_{\rm pol} = 2\pi a,\ L_{\rm tor} = 2\pi a R,\ \ ɫ = \frac{R}{a}\tan\mu_0 x$$

E.g. requiring

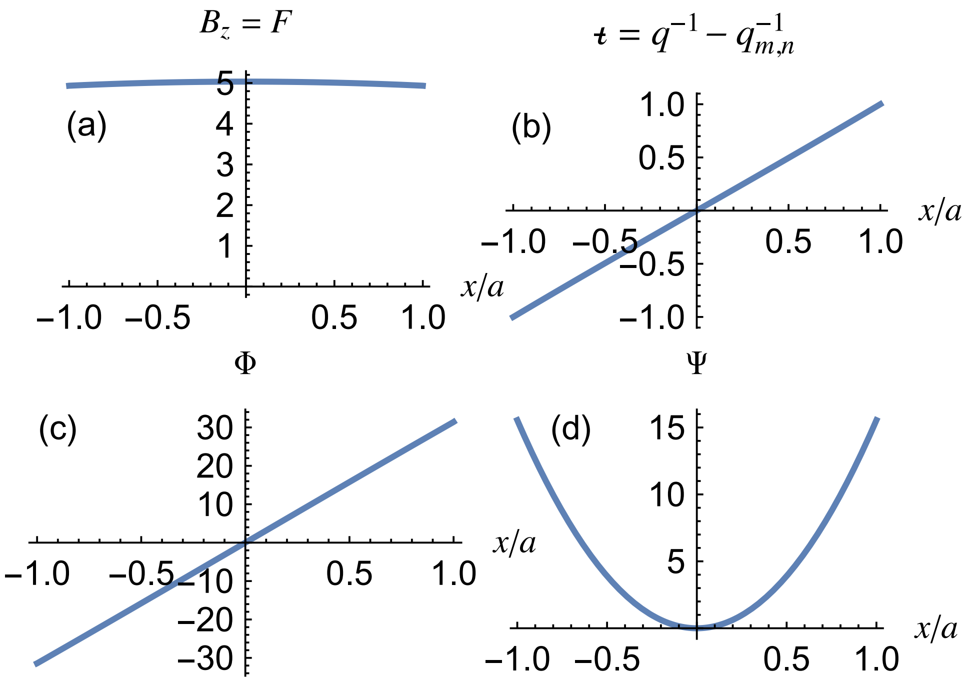
$ɫ = 1$

on boundary

and setting

$\mu_0 a = 1/5$

gives R/a ~ 5:



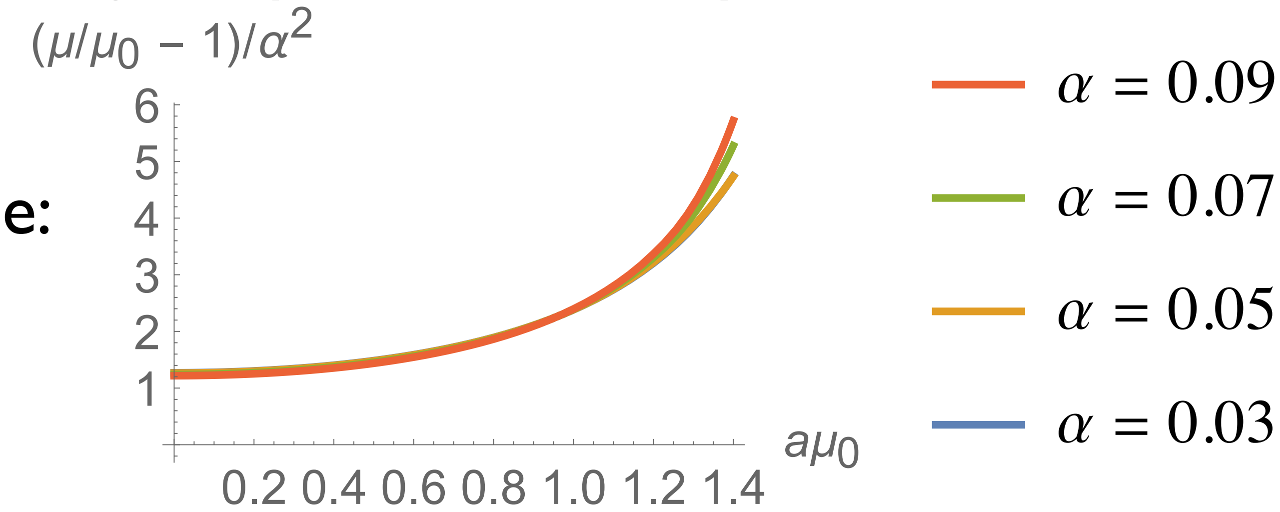
μ is *not* const. wrt. t

- In plane slab, *before* ripple is turned on, the *unperturbed* equilibrium flux function is

$$\psi_0(x|\mu_0) \equiv \frac{B_0}{\mu_0}(1 - \cos \mu_0 x)$$

- As amplitude parameter α is increased from 0, μ must *change* to preserve helicity and fluxes:

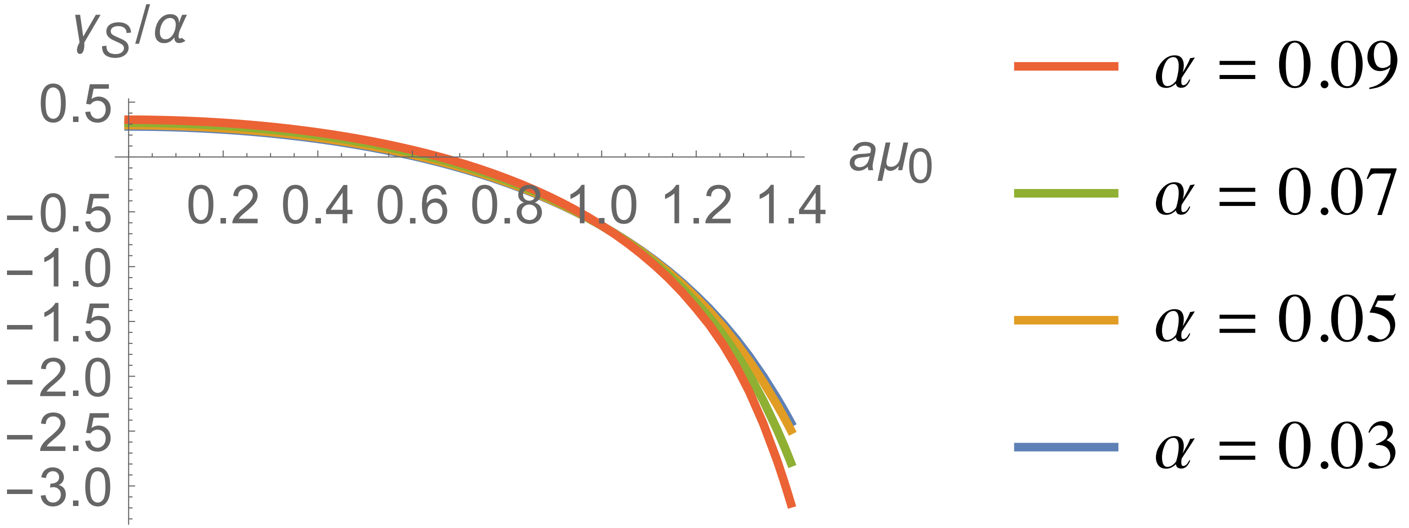
e.g. $m = 2$ case:



Current sheet has a nonlinear d.c. component

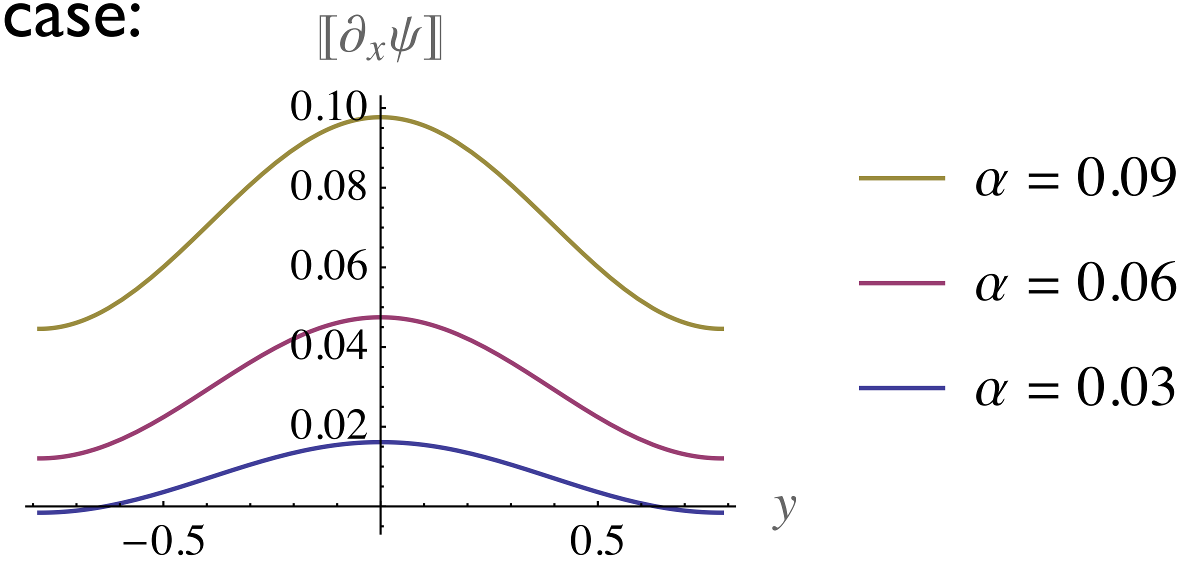
- HK implicitly assumed the total current in the sheet was zero, but MRxMHD switch-on shows there is a *nonzero* total current $J = \dfrac{2\alpha\psi_a k_1 \lambda}{\sinh k_1 a}\gamma_S$ proportional to α^2:

e.g. m = 2 case:



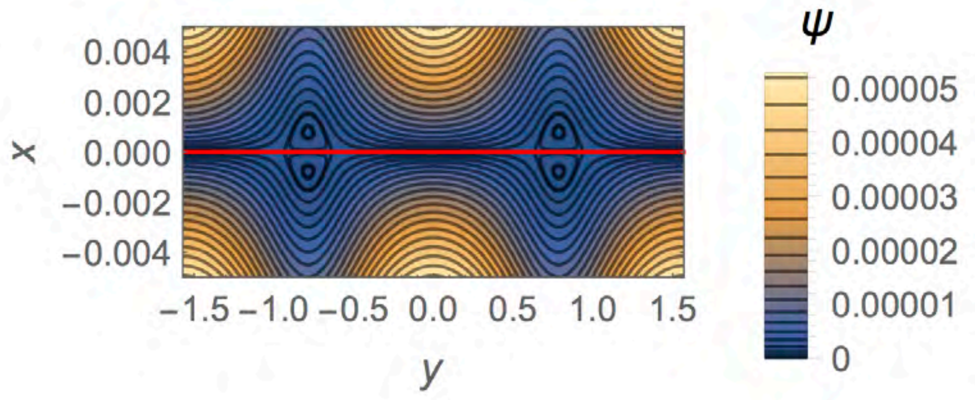
Sheet current: linear ripple + nonlinear d.c.

e.g. m = 4 case:



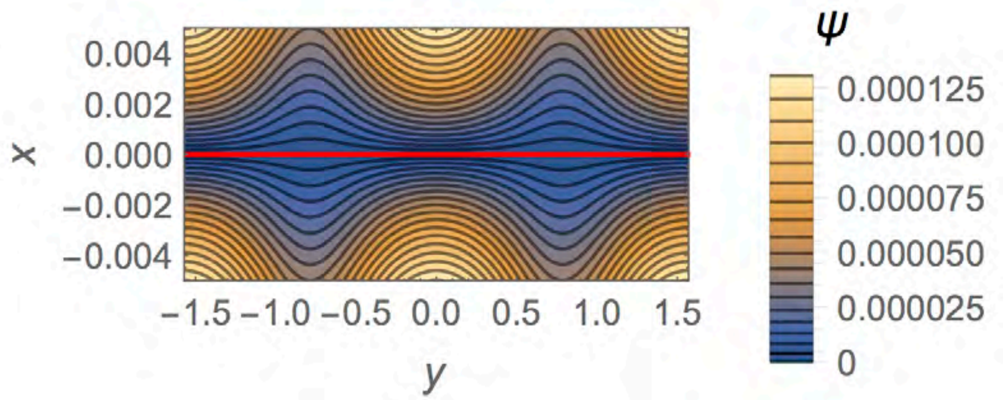
Jump in gradient of ψ, vs. y for $a\mu_0$ = 0.2 and selected values of α, showing current density in both + & – directions wrt. z for the smallest α but only in one direction for larger α, as O(α^2) component begins to dominate.

Current reversal causes “half-islands”



α = 0.03

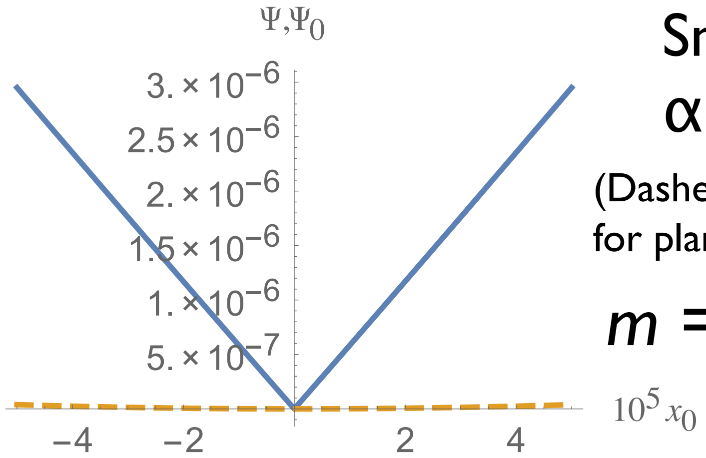
Fully shielded case: Level surfaces of ψ (magnetic surfaces) in the case $a\mu_0 = 0.2$, $m = 4$, showing the occurrence of a small half-island, bisected by the reversed-current section of the current sheet, for boundary ripple amplitude α = 0.03, but not for the greater amplitude α = 0.06, for which there is no current reversal.



α = 0.06

No current reversal — no half-islands

Fluxes and rotational transform I

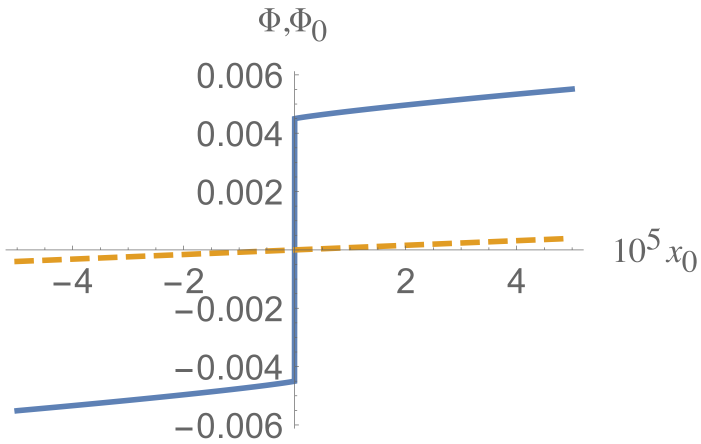


Small α:

α = 0.01

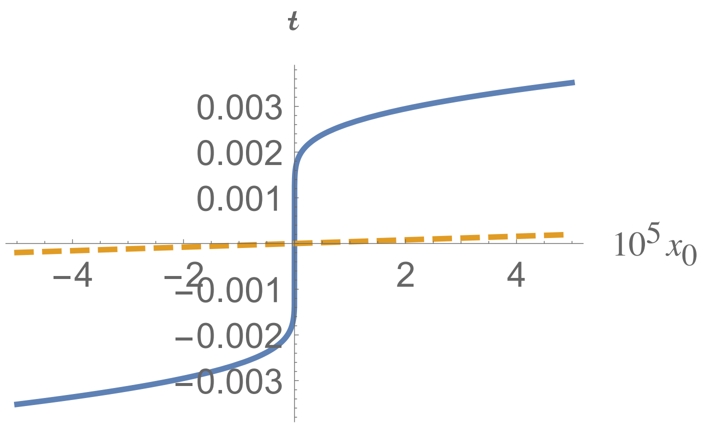
(Dashed curves are for plane slab, α = 0)

m = 4 case:



Above: Poloidal flux as a function of x_0 (= x along y-axis), showing discontinuity in slope at $x = 0$ caused by current sheet

Above: Toroidal flux as a function of x along y-axis, showing jump at $x = 0$ caused by half-island.

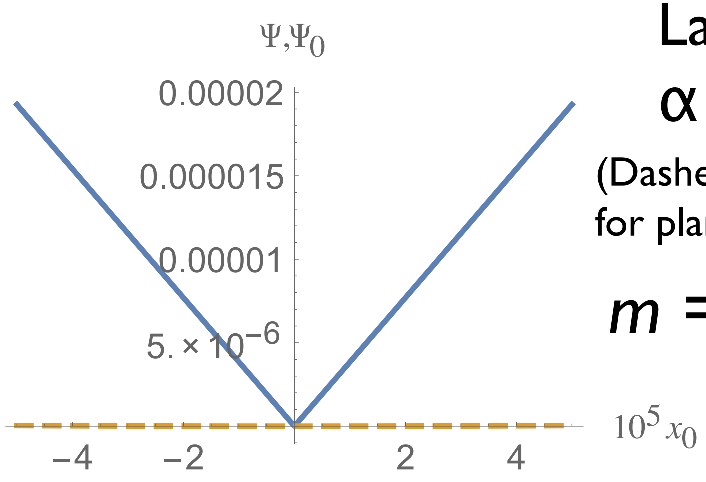


Left: Rotational transform

$$\Psi'(x_0)/\Phi'(x_0)$$

showing smooth quasi-jump across $x_0 = 0$.

Fluxes and rotational transform II

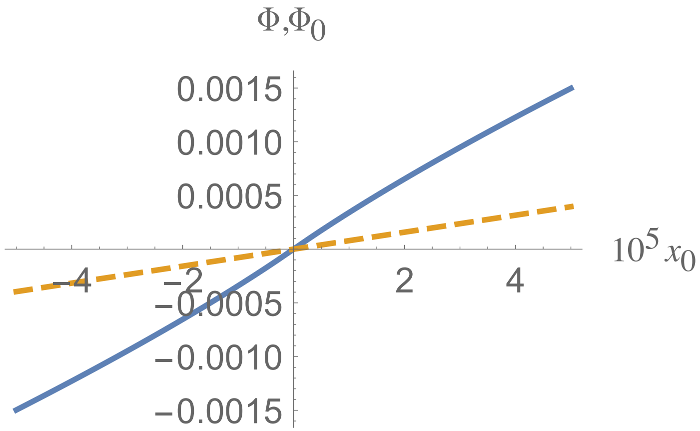


Larger α:

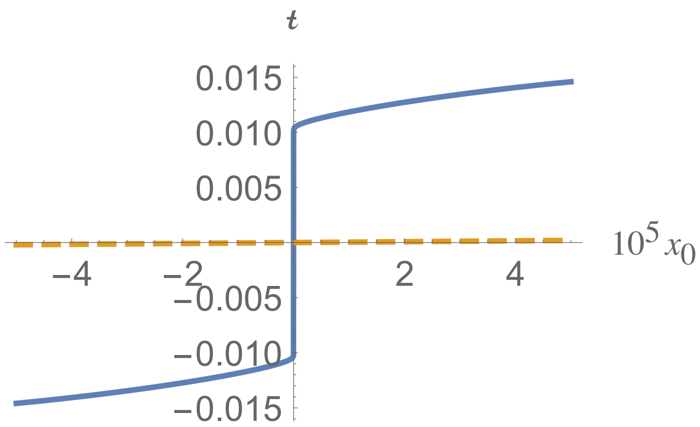
α = 0.04

(Dashed curves are for plane slab, α = 0)

m = 4 case:



Toroidal flux jump has gone as there are no half-islands above a threshold between α = 0.03 and 0.04



There is now a definite jump in rotational transform

Conclusions

- Multi-region generalization of Taylor relaxation has been extended to a self-consistent dynamics through Hamilton’ Principle of Stationary Action.
- A rippled slab model has been used to illustrate the formation of a resonant current sheet as boundary ripple is switched on
- For small ripple amplitudes, current reversal occurs in the current sheet — unperturbed sheared magnetic field exhibits topological change, with small half-islands, locking rotational transform to resonant value
- For larger ripple amplitude rotational transform can jump