Magnetic islands and singular currents at rational surfaces in three-dimensional MHD equilibria

J. Loizu[1,2], S. Hudson[2], A. Bhattacharjee[2], P. Helander[1]

[1] Max-Planck-Institut für Plasmaphysik, Euratom Association, D-17491 Greifswald, Germany

[2] Princeton Plasma Physics Laboratory, Princeton New Jersey, USA

1. Motivation and Summary

- Ideal MHD predicts **singular currents** forming at rational surfaces in 3D equilibria [1]:
  - A Pfirsch-Schlüter $1/x$-**current**, which arises as a result of finite pressure gradient.
  - A $\delta(x)$-**current** which presumably prevents the formation of islands.

$$\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp, \quad \mathbf{j}\times\mathbf{B} = \nabla p, \quad \nabla\cdot\mathbf{j} = 0\ ,\ \nabla\cdot\mathbf{B} = 0 \quad \Longrightarrow \quad \mathbf{j}_\perp = (\mathbf{B}\times\nabla p)/B^2, \quad \mathbf{B}\cdot\nabla u = -\nabla\cdot\mathbf{j}_\perp$$

Magnetic coordinates (ψ,θ,ϕ) $\xRightarrow{\text{ideal MHD}}$ $\sqrt{g}\,\mathbf{B}\cdot\nabla \equiv \iota\partial_\theta + \partial_\phi$

Fourier decomposition $u = \sum_{m,n} u_{mn} e^{i(m\theta - n\phi)}$ $\Longrightarrow$ $(\iota m - n)u_{mn} = i(\sqrt{g}\,\nabla\cdot\mathbf{j}_\perp)_{mn}$

Equation type $xf(x) = h(x)$, $x \equiv \iota m - n$, $h(x) \sim p'$ $\Longrightarrow$ $u_{mn}(x) = h(x)/x + \hat{j}_{mn}\delta(x)$

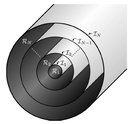
Pfirsch-Schlutter — Delta current

- Singular currents are **critical for the stability** of 3D MHD equilibria [2].
- We provide **the first numerical proof** of their mutual existence [3]:
  - An **analytical, slab, linear model** based on the *multiregion, relaxed MHD* (**MRxMHD**) theory
    (1) describes the formation of magnetic islands at resonant rational surfaces,
    (2) retrieves the ideal MHD limit where magnetic islands are shielded,
    (3) computes the subsequent formation of singular currents.
  - A **numerical implementation** of MRxMHD, the Stepped-Pressure Equilibrium Code (**SPEC**),
    (1) agrees with the analytical model for small perturbations in slab geometry,
    (2) can compute 3D equilibria in cylindrical and toroidal configurations.
- Results encourage the use of MRxMHD to compute three-dimensional MHD equilibria with magnetic islands and singular currents in **cylindrical and toroidal** configurations.

2. MRxMHD theory

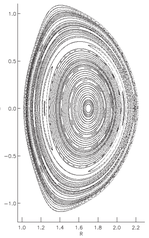
Taylor's theory	MRxMHD	Ideal MHD
Fewer constraints →		More constraints
Helicity is conserved globally	Helicity is conserved discretely	Helicity is conserved locally
$F = W + \frac{\mu}{2}\Big(\underbrace{\int_V \mathbf{A}\cdot\mathbf{B}\,dV}_{H} - H_0\Big)$	$F = \sum_{l=1}^{N}\Big[W_l + \frac{\mu_l}{2}\big(H_l - H_{l0}\big)\Big]$	$W = \int_V \Big(\frac{p}{\gamma-1} + \frac{B^2}{2}\Big)dV$
Topology: $\mathbf{B}\cdot\nabla\psi\big\vert_{\partial V} = 0$	Topology: $\mathbf{B}\cdot\nabla\psi\big\vert_{\partial V_l} = 0$	Topology: $\mathbf{B}\cdot\nabla\psi = 0$
Given $p, \Delta\psi, H_0$	Given $p_l, \Delta\psi_l, \Delta\psi_{p,l}, H_{l0}$	Given $p(\psi), \psi_p(\psi)$
$\delta F = 0$	$\delta F = 0$	$\delta W = 0$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$\nabla\times\mathbf{B} = \mu\mathbf{B}$	$\nabla\times\mathbf{B} = \mu_l\mathbf{B}$, $[[p + B^2/2]] = 0$	$\mathbf{j}\times\mathbf{B} = \nabla p$
Taylor, 1974 [4]	Dewar, Hole, Hudson, 2006 [5]	Kruskal and Kulsrud, 1958 [6]

- MRxMHD **converges** to ideal MHD for $N \to \infty$ [7]
- Stepped-pressure & **singular currents** possible.



3. SPEC code [8]

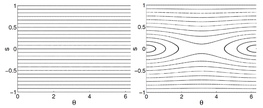
- Computes MRxMHD equilibria.
- Runs slab, cylindrical and toroidal geometries.
- Benchmarked against VMEC in axisymmetry [8].
- Reproduced helical states in RFP [9].
- Calculated DIII-D equilibria with RMPs [8].



4. Analytical linear theory

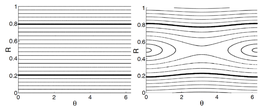
SINGLE RELAXED VOLUME

- Slab torus $\mathbf{r} = \hat{\theta}\hat{\mathbf{i}} + \zeta\hat{\mathbf{j}} + R(s,\theta,\zeta)\hat{\mathbf{k}}$, $s \in [-1,1]$.
- $R(s,\theta,\zeta)$ interpolates boundary interfaces.
- $\nabla\times\mathbf{B} = \mu\mathbf{B}$ with $\mathbf{B}\cdot\nabla s = 0$ at boundaries.
- Unperturbed solution: $\mathbf{B}_u$ depends on $(\mu, \Delta\Psi_p, \Delta\Psi_t)$ **or** $(H, \Delta\Psi_p, \Delta\Psi_t)$ **or** $(\iota^+, \iota^-, \Delta\Psi_t)$
- Perturbed boundary (n=0,m=1): $\mathbf{B}_u + \delta\mathbf{B}$ depends on $(\iota^+, \iota^-, \Delta\Psi_t)$ **and** $(\Delta, R_{0,1}, R_{1,1})$



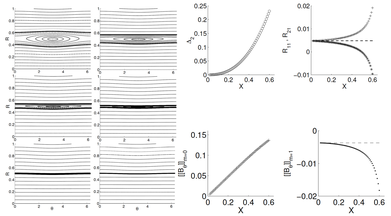
MULTIPLE RELAXED VOLUMES

- $\nabla\times\mathbf{B} = \mu\mathbf{B}$ in each volume $\mathcal{V}_l$, $l = 1\ldots N$.
  - $\mathbf{B}(\iota_l^\pm, \Delta\Psi_{t,l})$ needs interfaces shape $\{\Delta_l, R_{l,1}\}$
- $[[p + B^2/2]] = 0$ across each interface.
  - $N-1$ nonlinear equations $F_l(\Delta_l, \Delta_{l+1}) = 0$
  - $(N-1)^2$ matrix system $\mathcal{M}\mathbf{R} = \mathbf{S}$

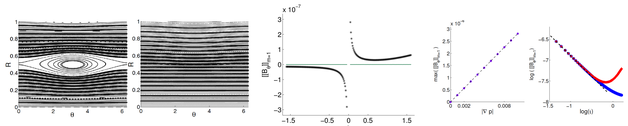


ISLAND SHIELDING AND $\delta(x)$-CURRENT

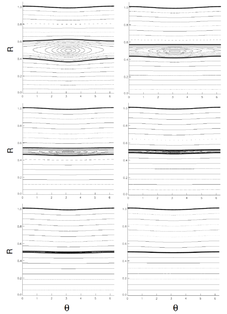
- Squeeze island with N=3.
  - $\iota_2^\pm = \pm X^\alpha$
  - $\Delta\Psi_{t,2} = X^\beta$
  - $X \to 0$
- Prediction $\lim_{X\to 0}[[\mathbf{B}]] \neq 0$
  - $\mathbf{j} = [[\mathbf{B}]]\times\mathbf{n}\,\delta(\iota - \iota_{res})$
  - Recover HKT [10]

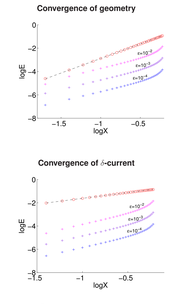


ISLAND SHIELDING AND $1/x$-CURRENT

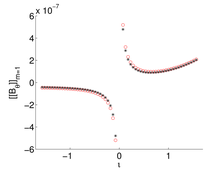


5. SPEC simulations



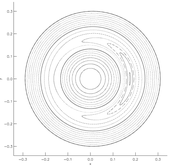


- SPEC captures both δ-current and $1/x$ current.
- Results converge to the linear theory for small boundary perturbation.



6. Outlook

- Cylindrical MRxMHD equilibria are being considered in order to retrieve the Rosenbluth solution for the saturated $m = 1$, $n = 1$, ideal kink mode and the current sheet associated with it [11].
- In the future, SPEC will be used to compute 3D MHD equilibria in tokamaks and stellarators.



7. References

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joaquim.loizu@ipp.mpg.de