Relaxed And Multiregion Relaxed Magnetohydrodynamics

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Abstract

Ideal magnetohydrodynamics (IMHD) is strongly constrained by an infinite number of microscopic constraints expressing mass, entropy and magnetic flux conservation in each infinitesimal fluid element, the latter preventing magnetic reconnection. By contrast, in the Taylor-relaxed equilibrium model [1] all these constraints are relaxed save for global magnetic flux and helicity.
A Lagrangian [2] is presented that leads to a new variational formulation of magnetized fluid dynamics, relaxed magnetohydrodynamics (RxMHD), all static solutions of which are Taylor equilibrium states. By postulating that some long-lived macroscopic current sheets can act as barriers to relaxation, separating the plasma into multiple relaxation regions, a further generalization, multiregion relaxed magnetohydrodynamics (MRxMHD) [3], is developed.
These concepts are illustrated using a simple two-region slab model similar to that proposed by Hahm and Kulsrud [4] — the formation of an initial shielding current sheet after perturbation by boundary rippling is calculated using MRxMHD and the final island state, after the current sheet has relaxed through a reconnection sequence [5], is calculated using RxMHD.

[1] J.B. Taylor, Rev. Mod. Phys. 58, 741 (1986)
[2] cf. e.g. R.L. Dewar, Phys. Fluids 13, 2710 (1970) for the IMHD Lagrangian
[3] S.R. Hudson, R.L. Dewar, G. Dennis , M.J. Hole, M. McGann, G. von Nessi and S. Lazerson, Phys. Plasmas 19, 112502 (2012)
[4] T.S. Hahm and R.M. Kulsrud, Phys. Fluids 28, 2412 (1985)
[5] R.L. Dewar, A. Bhattacharjee, R.M. Kulsrud and A.M. Wright, Phys. Plasmas 20, 082103 (2013)

Generalization of Taylor Relaxation

- This presentation answers in the affirmative the question "Is there a reduced magneto-hydro *dynamics* that leads to Taylor's relaxed *equilibrium* states in the static limit?" by using Hamilton's Principle to derive self-consistent dynamics from an RxMHD Lagrangian.
- A question for the future is: "Is there a natural helicity transport mechanism via reconnection across toroidal current sheets that relaxes rotational transforms to irrational values?"

Hamilton's Action Principle in domain Ω: $\delta S = 0$

$S = \int dt \int_\Omega \mathcal{L}\, d^3x$ denotes the *action*. Its *first variation* is:

$$\delta S = \int dt \int_\Omega \delta\mathcal{L}\, d^3x + \epsilon \int dt \int_{\partial\Omega} \mathcal{L}\, \boldsymbol{\xi}\cdot\mathbf{n} dS$$

$\delta\mathcal{L}$ is $O(\epsilon)$ *Eulerian* variation of action density $\mathcal{L}$,
$\epsilon\boldsymbol{\xi}$ is *Lagrangian* displacement of fluid element positions $\mathbf{r}$ on boundary $\partial\Omega$

MHD Lagrangian density is

$$\mathcal{L}_{\mathrm{MHD}} = \frac{\rho\mathbf{v}\cdot\mathbf{v}}{2} - \frac{p}{\gamma - 1} - \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is velocity, ρ is mass density, p is pressure and $\mathbf{B}$ is magnetic field

Holonomic constraints

- IMHD = Ideal MHD ($\rho, \mathbf{B}$ and p *holonomically* constrained, i.e. *locally* "frozen in" to fluid elements):
$$\delta\rho = -\epsilon\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}),\ \delta p = -\epsilon(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p + \gamma p\boldsymbol{\nabla}\cdot\boldsymbol{\xi}),\ \delta\mathbf{B} = \boldsymbol{\nabla}\times\delta\mathbf{A}$$
$$\delta\mathbf{A} = \epsilon\boldsymbol{\xi}\times\mathbf{B} + \boldsymbol{\nabla}\delta\chi$$
- RxMHD = Relaxed MHD (only ρ holonomically constrained — no effect on static equilibrium — magnetic helicity and entropy constrained only *globally*):
$$\delta\rho = -\epsilon\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi})$$
- MRxMHD = Multi-Relaxed MHD (multiple RxMHD regions Ω_i separated by current sheet transport barriers $\partial\Omega_i$, with holonomic constraints on either side, ±, of $\partial\Omega_i$ to keep $\mathbf{B}$ tangential to the current sheets):
$$\delta\rho = -\epsilon\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}) \text{ in } \Omega_i,\ \delta\mathbf{A}_{\rm tgt} = (\epsilon\boldsymbol{\xi}\times\mathbf{B} + \boldsymbol{\nabla}\delta\chi)_{\rm tgt} \text{ on } \partial\Omega_i^{\pm}$$

Global constraints

- IMHD = Ideal MHD (none — mass, entropy and magnetic flux and helicity within Ω all automatically conserved as a consequence of the holonomic constraints):

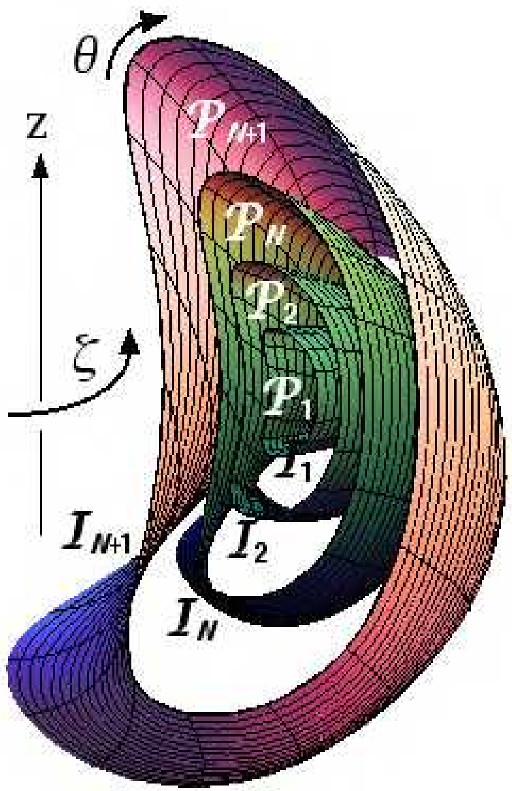
- RxMHD = Relaxed MHD (mass and flux automatic, entropy and magnetic helicity are constrained globally within Ω using Lagrange multipliers τ and μ respectively):

$$\mathcal{L} = \mathcal{L}_{\mathrm{MHD}} + \tau \frac{\rho \ln(Cp/\rho^\gamma)}{\gamma - 1} + \mu \frac{\mathbf{A}\cdot\mathbf{B}}{2\mu_0}$$

  where γ and C are thermodynamic gas constants.

- MRxMHD = Multi-Relaxed MHD (mass and flux automatic, entropy and magnetic helicity are constrained globally within the multiple RxMHD regions Ω_i using Lagrange multipliers τ_i and μ_i giving p and q profile control).

An application of MRxMHD: 3D equilibrium code SPEC



- Relaxed regions $\mathcal{P}_i$, separated by
- nested toroidal transport barrier interfaces $\mathcal{I}_i$, which
- freeze in flux and confine piecewise flat pressures (with pressure jumps ≡ $[[p]]_i$ across current sheets).
- Arbitrarily refinable *as long as magnetic surfaces* $\mathcal{I}_i$ *exist*
- Minimizes energy starting with an initial guess (e.g. a VMEC equilibrium)
- Allows islands and chaos between the toroidal current sheets

MRxMHD equations

- Continuity: $\dfrac{d\rho}{dt} = -\rho\boldsymbol{\nabla}\cdot\mathbf{v}$
- Require Hamilton's Principle: $\delta S = 0$ for all independent variations of $\mathbf{r}$, p and $\mathbf{A}$, where:

$$\delta S = \sum_i \int dt \int_{\Omega_i} \delta\mathcal{L}_i \, d^3x + \epsilon \sum_i \int dt \int_{\partial\Omega_i} \mathcal{L}_i \, \boldsymbol{\xi}\cdot\mathbf{n} dS$$

- Resulting *Euler–Lagrange* equations are:

$\rho\dfrac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla}p$ (momentum equation)

$p = \tau_i\rho$ (isothermal equations of state in each region)

$\boldsymbol{\nabla}\times\mathbf{B} = \mu\mathbf{B}$ (Beltrami equations)

$\left[\!\left[p + \frac{B^2}{2\mu_0} \right]\!\right]_i = 0$ (pressure jump conditions at interfaces)

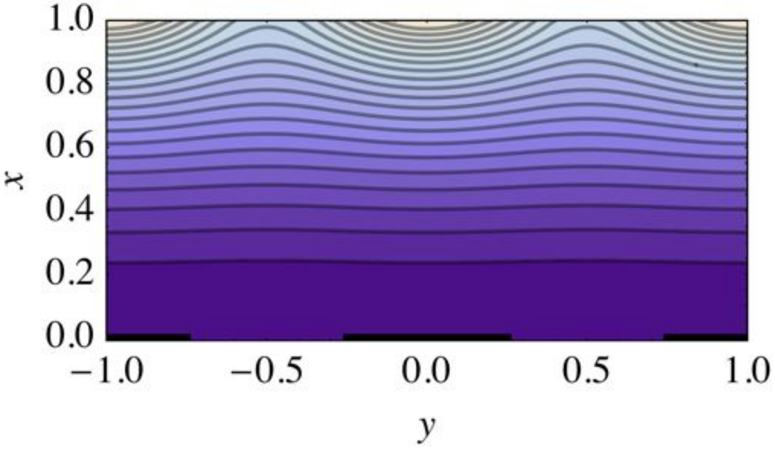
Jump and boundary conditions on a current sheet

- SPEC interfaces must be *current sheets* so a delta function $\mathbf{J}\times\mathbf{B}$ force can balance the ∇p delta function
- *Force balance* criterion is simply $\left[\!\left[p + \frac{B^2}{2} \right]\!\right] = 0$ where $[[p]]$ denotes the *jump*, $p_+ - p_-$, between the two sides, $\pm$, of the interface
- In addition we have *tangentiality*, $\mathbf{B}\cdot\mathbf{n}$ & $\mathbf{J}\cdot\mathbf{n} = 0$, which implies the existence of two 2D *scalar* potentials $f_\pm(\theta,\zeta)$ such that $B_{\pm\theta} = \partial_\theta f_\pm$, $B_{\pm\zeta} = \partial_\zeta f_\pm$. Here ∂_i , $i = \theta,\zeta,$ are the covariant derivatives on the interface, regarded as 2D Riemannian manifold with metric $g_{i,j}$. Force balance gives *Hamilton-Jacobi equation*.

Rotational transform **paradox** in MRxMHD toroidal relaxation

- A KAM argument shows 3-D toroidal *equilibrium* current sheets can in general only exist if rotational transforms on both sides of sheet are strong *irrationals*
- Starting with non-equilibrium tori, relaxation of torus shape *with conserved fluxes & helicities* leads to uncontrolled *change* of rotational transforms — *no apparent relaxation mechanism to reach desired irrationals*
- A possible resolution is to allow helicity transport between regions through partial *reconnection*

Hahm-Kulsrud-Taylor (HKT) Slab Model



- Simple slab model for resonant current sheet formation near $x = 0$ in response to symmetrical periodic perturbation at boundaries $x = \pm a$
- Hahm & Kulsrud, Phys. Fluids 1985, found 2 solutions:

- shielding current sheet on $x = 0$

$$\psi = aB_y^a \left[\frac{x^2}{2a^2} + \frac{\alpha}{\sinh(ka)} |\sinh(kx)| \cos(ky) \right]$$

- island with no current sheet

$$\psi = aB_y^a \left[\frac{x^2}{2a^2} + \frac{\alpha}{\cosh(ka)} \cosh(kx) \cos(ky) \right]$$

where $B^a{}_y$ is |unperturbed poloidal field| at boundaries and $\alpha \ll 1$

Construction essentially based on *linear* Grad-Shafranov (GS) equation

(on *x*,*y* plane cut along current sheets)

$$\mathbf{B} = \nabla z \times \nabla \psi + F(\psi)\nabla z$$

Local force balance implies GS equation (except at cuts)

$$\nabla^2 \psi + \partial_\psi [\mu_0 p(\psi) + \tfrac{1}{2} F(\psi)^2] = 0$$

Choose *p* and *F* profiles such that

$$\mu_0 p(\psi) + \tfrac{1}{2} F(\psi)^2 = \mathrm{const} - \frac{B_y^a}{a}\psi$$

then poloidal stream function ψ obeys (except at cuts) a Poisson eq.

$$\nabla^2 \psi = \frac{B_y^a}{a}$$

with general solution

$$\psi = a B_y^a \left[\frac{x^2}{2a^2} + \alpha \psi_1(x, y) \right]$$

where ψ_1 is any *harmonic function*, i.e. it obeys Laplace's equation

E.g. conformal mapping solutions

The linearity of this GS allows easy analytic solution, For example, the powerful *conformal mapping method*, which relies on fact that real (or imaginary) part of an analytic complex function obeys Laplace's equation:

$$\nabla^2\,\mathrm{Re} f(x+iy) = 0$$

Also $$\nabla^2\,\mathrm{Re}\, g(f(x+iy)) = 0$$

So one harmonic solution can be mapped (conformally) into another simply by composing with a suitably chosen function of a complex variable.

Also, we can introduce *current sheets* by finding a complex function with *branch cuts*:

Equilibrium condition on cuts

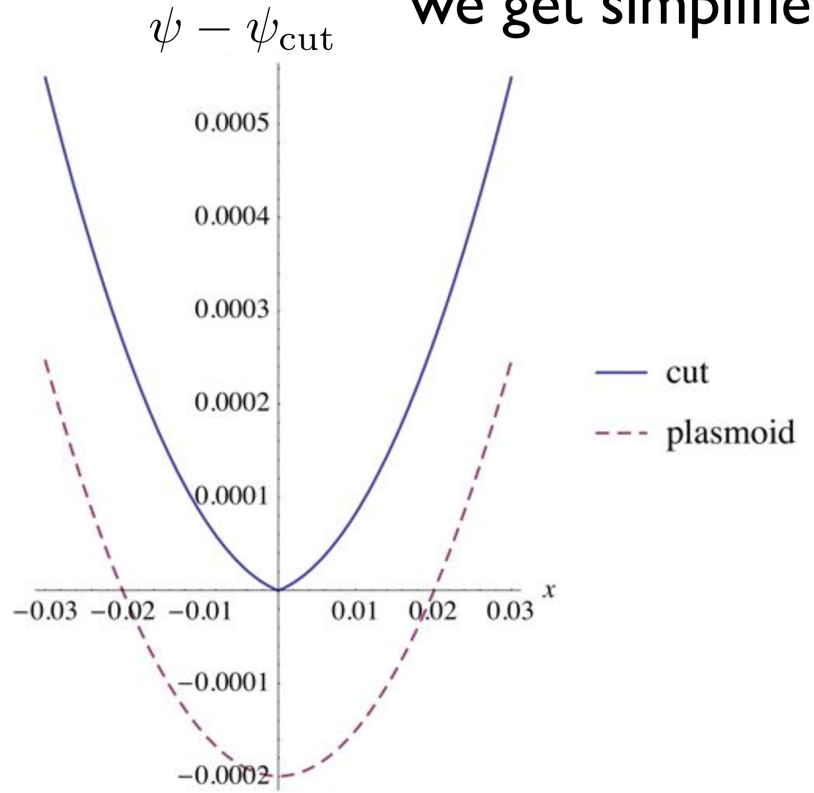
Cuts in *x*,*y* plane correspond to *current sheets*, so jump condition

$$\left[\!\left[p + \frac{B^2}{2} \right]\!\right] = 0$$

$(\text{taking } \mu_0 = 1)$

applies. Using $B^2 = |\nabla\psi|^2 + F^2$ and linear GS assumption: $p(\psi) + \frac{1}{2}F(\psi)^2 = \text{const} - \frac{B_y^a}{a}\psi$

we get simplified jump condition

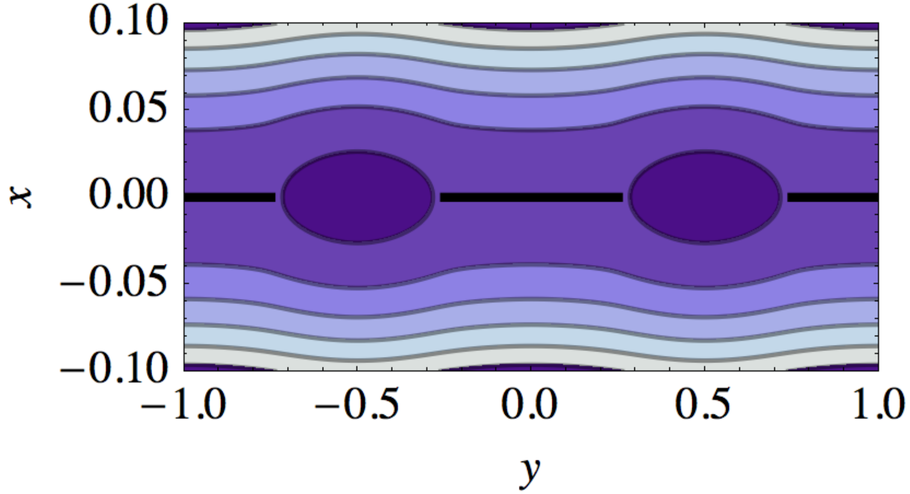


$$\left[\!\left[(\nabla\psi)^2 \right]\!\right] = 0$$

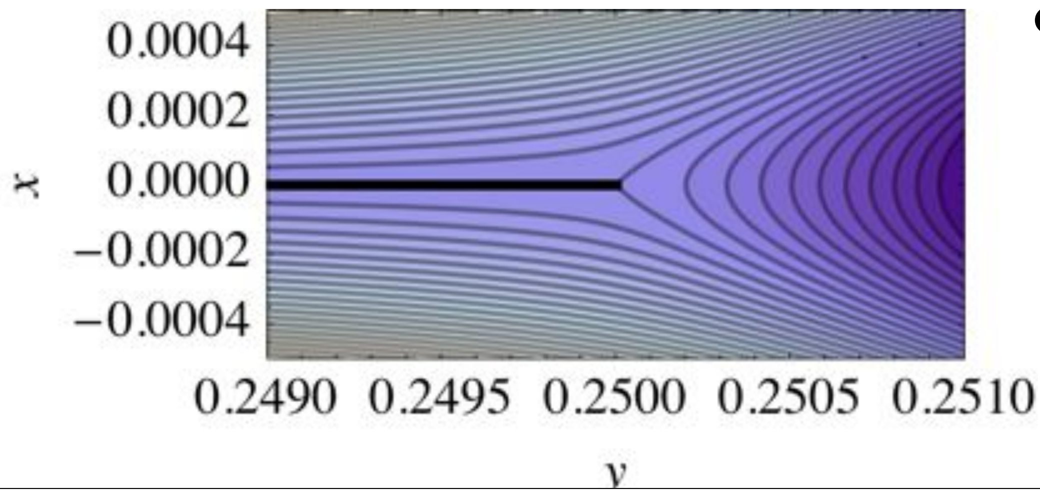
which is automatically satisfied in HKT model if cut is on *x* = 0 line symmetry line

Plasmoid solutions with partial current sheets using conformal mapping

R.L. Dewar, A. Bhattacharjee, R.M. Kulsrud and A.M. Wright
Phys. Plasmas **20**, 082103 (2013):



- partial current sheets between $x = \pm L + n\lambda$
- "plasmoids" between current sheets



- zoomed view of end of Sweet-Parker current sheet

2RxMHD HKT model

HK-style model is natural application of MRxMHD because:

- Linearity of Beltrami equation leads to easily solvable, *linear GS equation (Poisson in small-μ limit.)*
- Symmetry about, and straightness of, current sheet at x = 0: gives most geometrically simple *2-region* geometry

Relaxation scenario:

- Switch-on: *ripple* on upper and lower boundaries slowly increased from zero (plane slab) to final amplitude
- A *shielding current* sheet at x = 0 resonance develops
- Kruskal-Kulsrud damping: evolution through *equilibria*
- Connect equilibrium sequence by *helicity conservation*

Extension of HK shielding solution

Helicity conservation requires *three extensions of HK solution*

Instead of the HK harmonic component ψ_1 we use ansatz

$$\widehat{\psi}(x,y) \equiv \frac{2\alpha\psi_a}{\sinh k_1 a}\left(|\sinh k_1 x| \cos ky + \gamma_{\rm S}\frac{k_1}{\mu}|\sin \mu x|\right) - \overline{\psi}\cos \mu x$$

where:

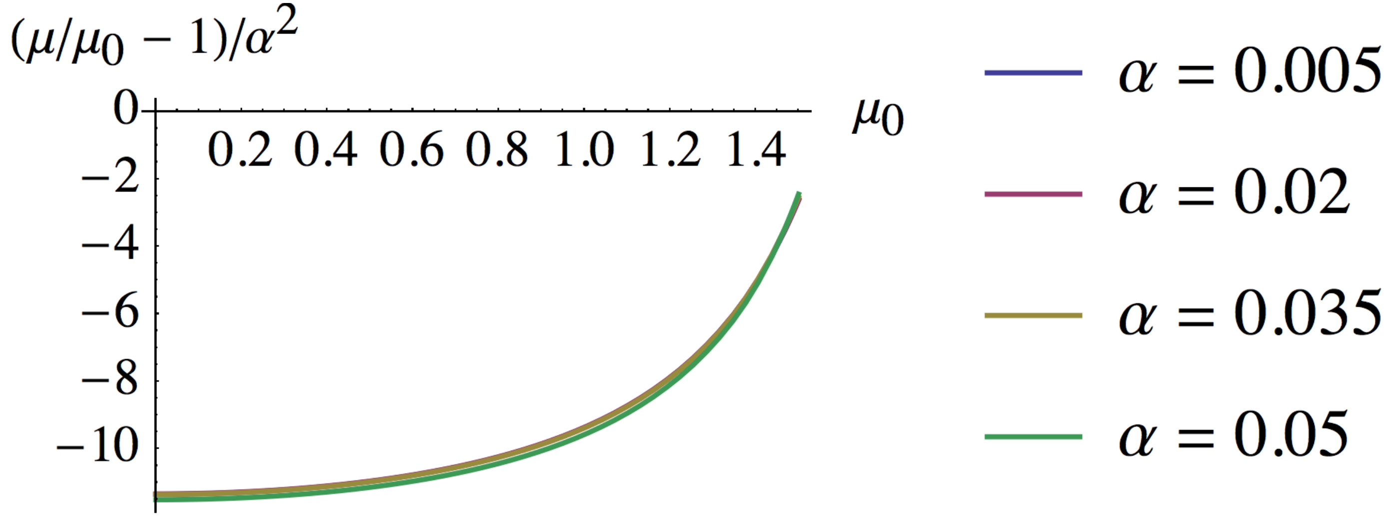
1. $\widehat{\psi}$ is a solution of the *Beltrami equation* $(\nabla^2 + \mu^2)\widehat{\psi} = 0$
   It is only *harmonic* in the *small-μ limit.* Likewise
   $k_1(\mu) \equiv (k^2 - \mu^2)^{1/2} \to k \text{ only as } \mu \to 0$
2. The term in $\gamma_{\rm S}$ was introduced in Dewar *et al.* 2013 to allow control of the *total current* in the sheet
3. The term in $\overline{\psi}$ is required for poloidal flux conservation

μ is a *variable*

- In plane slab, *before* ripple is turned on, the *unperturbed* equilibrium flux function is

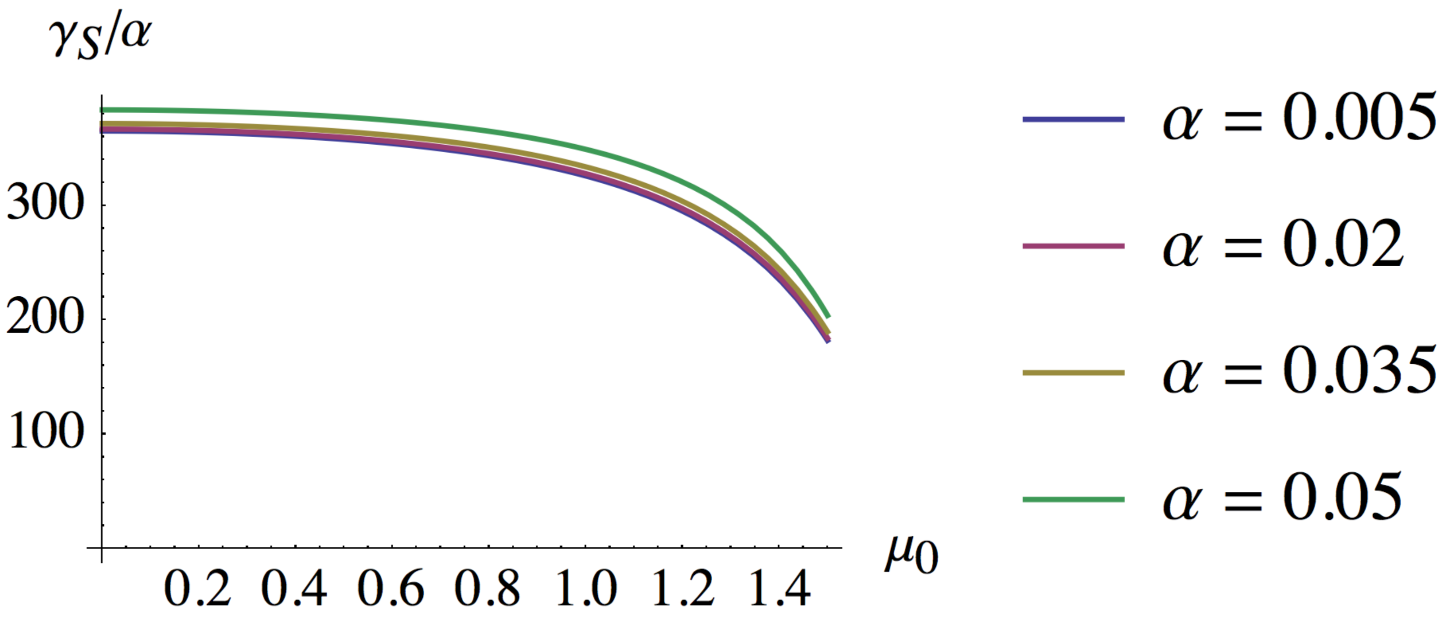
$$\psi_0(x) \equiv \frac{B_0}{\mu_0}(1 - \cos \mu_0 x)$$

- As amplitude parameter α is increased from 0, μ must *change* to preserve helicity and fluxes:



Current sheet has a strong d.c. component

- HK implicitly assumed the total current in the sheet was zero, but MRxMHD switch-on shows there is a *nonzero* total current $J = \dfrac{2\alpha\psi_a k_1 \lambda}{\sinh k_1 a}\gamma_S$ proportional to γ_S :



Conclusions

- Multi-region generalization of Taylor relaxation has been extended to a self-consistent dynamics through Hamilton's Principle of Stationary Action.
- A rippled slab model has been used to illustrate the formation of a resonant current sheet as boundary ripple is switched on
- In future a topologically more complex "plasmoid" scenario for the MRxMHD interfaces, combined with helicity transport through reconnection, will be explored to see if it resolves the KAM equilibrium existence paradox and connects continuously to island solution