Magnetic islands and singular currents at rational surfaces: from MRXMHD to ideal MHD

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Ideal MHD predicts singular currents

$$\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$$
$$\mathbf{j} \times \mathbf{B} = \nabla p$$
$$\nabla \cdot \mathbf{j} = 0\ ,\ \ \nabla \cdot \mathbf{B} = 0$$

$\Longrightarrow$

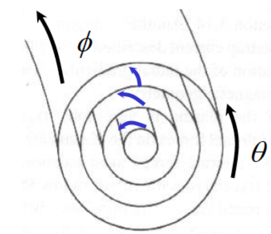
$$\mathbf{j}_\perp = (\mathbf{B} \times \nabla p)/B^2$$
$$\mathbf{B} \cdot \nabla u = -\nabla \cdot \mathbf{j}_\perp$$

Magnetic coordinates

(ψ, θ, ϕ)

ideal MHD $\Longrightarrow$

$$\sqrt{g}\ \mathbf{B} \cdot \nabla \equiv \iota \partial_\theta + \partial_\phi$$



Fourier decomposition

$$u = \sum_{m,n} u_{mn} e^{i(m\theta - n\phi)}$$

$\Longrightarrow$

$$(\iota m - n) u_{mn} = i(\sqrt{g}\ \nabla \cdot \mathbf{j}_\perp)_{mn}$$

Equation type

$$x f(x) = h(x)$$
$$x \equiv \iota m - n,\ h(x) \sim p'$$

$\Longrightarrow$

$$u_{mn}(x) = h(x)/x + \hat{j}_{mn}\delta(x)$$

Pfirsch-Schlutter

Delta current

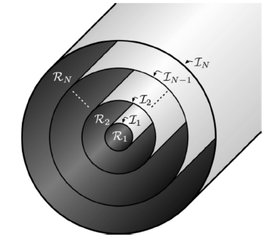
How to calculate 3D MHD equilibrium?

EITHER Insist on nested flux surfaces (ideal MHD)

- VMEC
  - based on energy functional with ideal constraints
  - cannot resolve rational surfaces

OR Relax assumption of flux surfaces

- Resistive MHD
  - initial value calculation, not based on energy minimum
  - islands develop and break rational surfaces
- Relaxed MHD
  - based on energy functional with fewer constraints
  - islands develop at rational surfaces



Multiregion RelaXed MHD

Taylor's theory	MRXMHD	Ideal MHD

Fewer constraints → More constraints

Taylor's theory

$$F = W + \frac{\mu}{2}\Big(\underbrace{\int_V \mathbf{A}\cdot\mathbf{B}\, dV}_{H} - H_0\Big)$$

Topology: $\mathbf{B}\cdot\nabla\psi\big|_{\partial V} = 0$

Given $p, \Delta\psi, H_0$

$$\delta F = 0 \implies \boxed{\nabla\times\mathbf{B} = \mu\mathbf{B}}$$

[Taylor, 1974]

MRXMHD

$$F = \sum_{l=1}^{N}\Big[W_l + \frac{\mu_l}{2}\Big(H_l - H_{l0}\Big)\Big]$$

Topology: $\mathbf{B}\cdot\nabla\psi\big|_{\partial V_l} = 0$

Given $p_l, \Delta\psi_l, \Delta\psi_{p,l}, H_{l0}$

$$\delta F = 0 \implies \boxed{\begin{array}{c}\nabla\times\mathbf{B} = \mu_l\mathbf{B}\\ [[p + B^2/2]] = 0\end{array}}$$

[Dewar, Hole, Hudson, 2006]

Ideal MHD

$$W = \int_V \Big(\frac{p}{\gamma - 1} + \frac{B^2}{2}\Big) dV$$

Topology: $\mathbf{B}\cdot\nabla\psi = 0$

Given $p(\psi), \psi_p(\psi)$

$$\delta W = 0 \implies \boxed{\mathbf{j}\times\mathbf{B} = \nabla p}$$

[Kruskal and Kulsrud, 1958]

Existence of stepped-pressure equilibria?

Existence of Three-Dimensional Toroidal MHD Equilibria with Nonconstant Pressure

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We establish an existence result for the three-dimensional MHD equations

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla p$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\mathbf{B} \cdot n|_{\partial T} = 0$$

with $p \neq$ const in tori T *without symmetry*. More precisely, our theorems insure the existence of sharp boundary solutions for tori whose departure from axisymmetry is sufficiently small; they allow for solutions to be constructed with an arbitrary number of pressure jumps.

Communications on Pure and Applied Mathematics, Vol. XLIX, 717–764 (1996)

Stepped-Pressure Equilibrium Code (SPEC)

- Implementation of MRXMHD

- Finds minimum of

$$F = \sum_{l=1}^{N} \left[W_l + \frac{\mu_l}{2} \Big(H_l - H_{l0} \Big) \right]$$

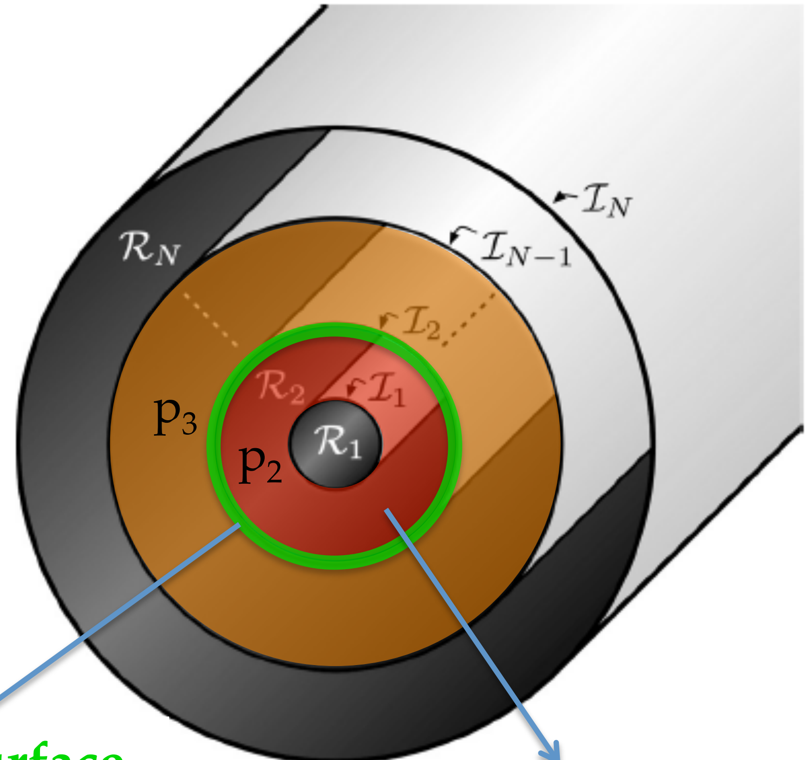
Topology: $\mathbf{B} \cdot \nabla \psi \big|_{\partial V_l} = 0$

Given $p_l, \Delta\psi_l, \Delta\psi_{p,l}, H_{l0}$

$$\Longleftrightarrow$$

$\mathcal{R}_l \; : \quad \nabla \times \mathbf{B} = \mu_l \mathbf{B}$

$\mathcal{I}_l \; : \quad [[p + B^2/2]] = 0$



KAM surface

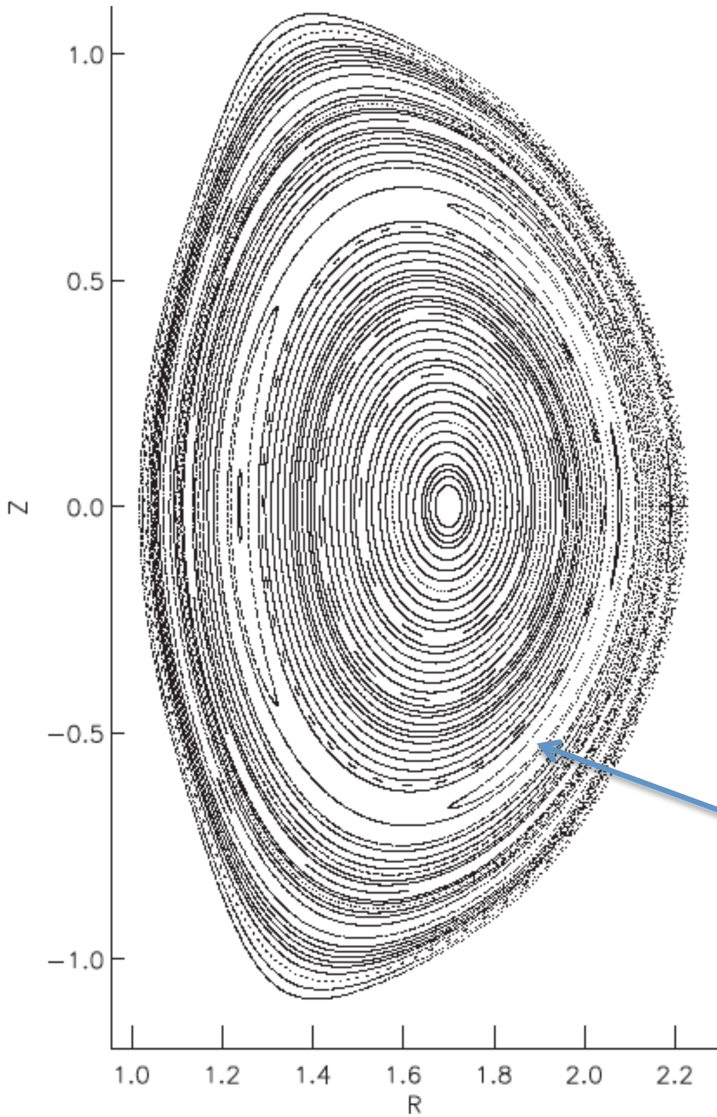
$\Delta\psi, \Delta\psi_p, H_0$

or $\quad \Delta\psi, \Delta\psi_p, \mu$

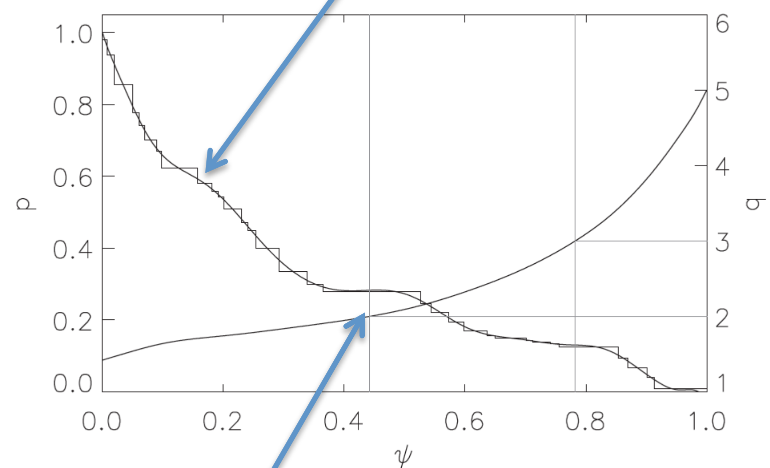
or $\quad \Delta\psi, \iota^-, \iota^+$

[Hudson et al, PoP, 2012]

Example: DIII-D equilibrium with perturbed boundary



KAM interfaces
=
noble irational surfaces



Magnetic island at q=2 rational surface

[Hudson et al, PoP, 2012]

Ideal MHD is a limit of MRXMHD

The infinite interface limit of multiple-region relaxed magnetohydrodynamics

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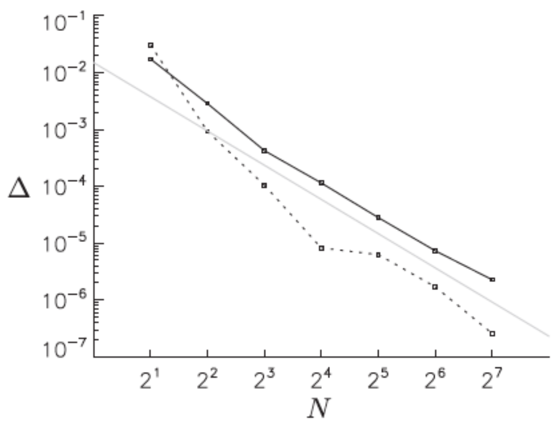
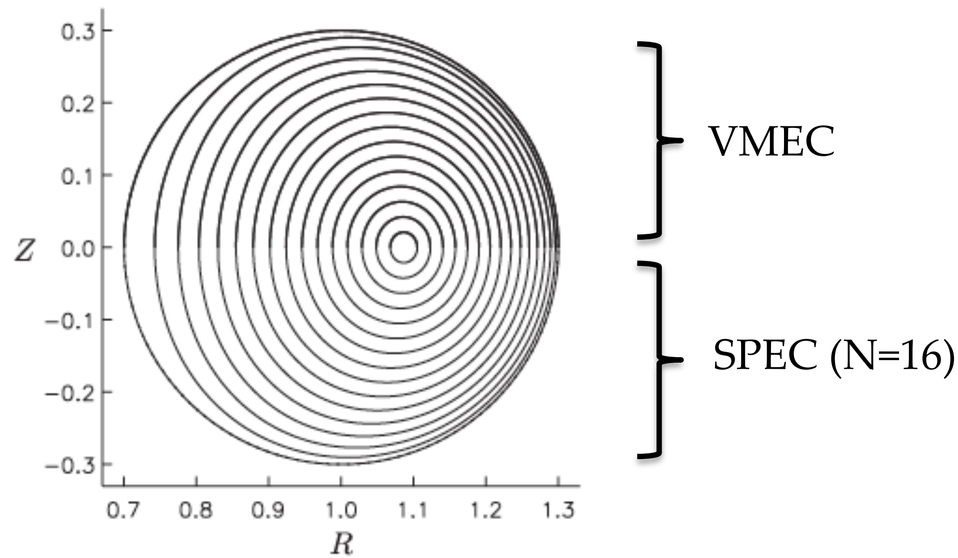
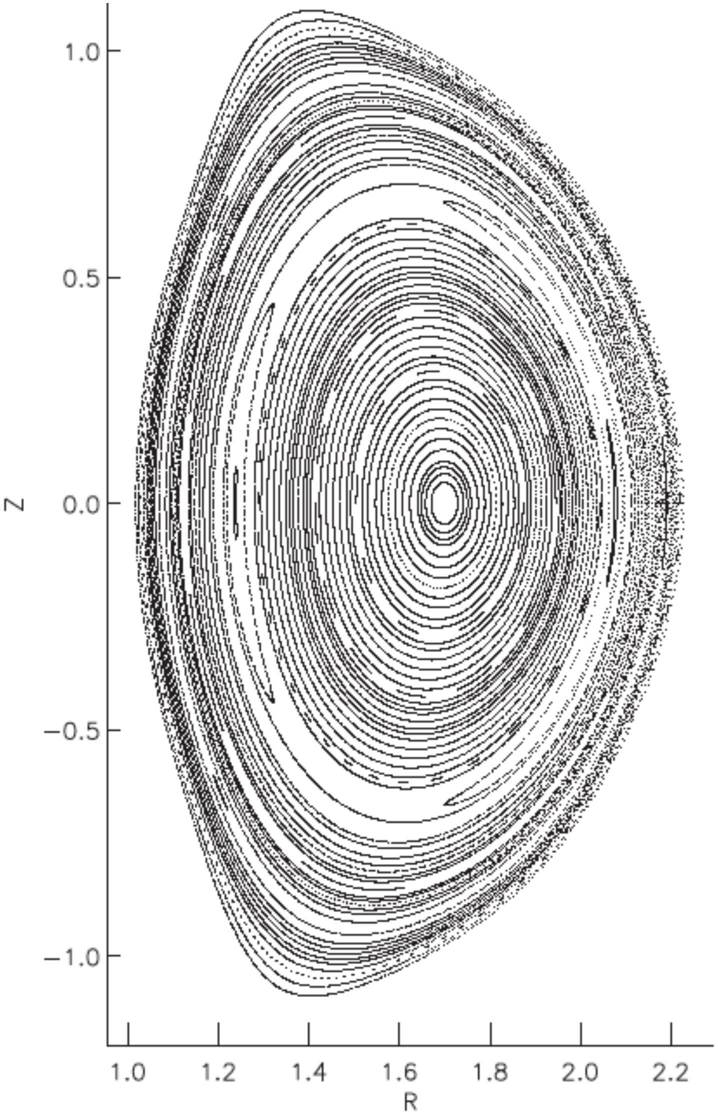
We show the stepped-pressure equilibria that are obtained from a generalization of Taylor relaxation known as multi-region, relaxed magnetohydrodynamics (MRXMHD) are also generalizations of ideal magnetohydrodynamics (ideal MHD). We show this by proving that as the number of plasma regions becomes infinite, MRXMHD reduces to ideal MHD. Numerical convergence studies illustrating this limit are presented. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4795739]



$$\Delta \equiv \int_0^{2\pi} d\theta |\mathbf{x}_{\mathrm{VMEC}}(\theta) - \mathbf{x}_{\mathrm{SPEC}}(\theta)|$$



Singular currents at rational surfaces should arise in MRXMHD with sufficiently high N



- MRXMHD allows **B** discontinuities and therefore delta currents

- Ideal MHD delta currents represent the shielding of magnetic islands

$$\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$$

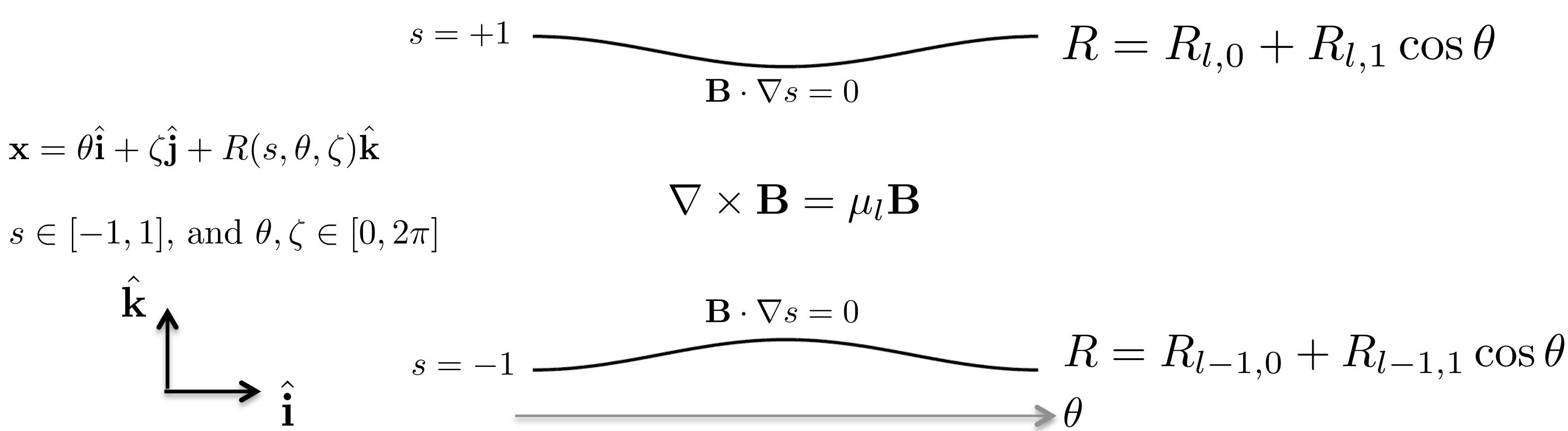
$$u_{mn}(x) = h(x)/x + \hat{j}_{mn}\delta(x)$$

Approach 1: interface at rational surface

Approach 2: squeeze island with irationals

A simple analytical model can be built

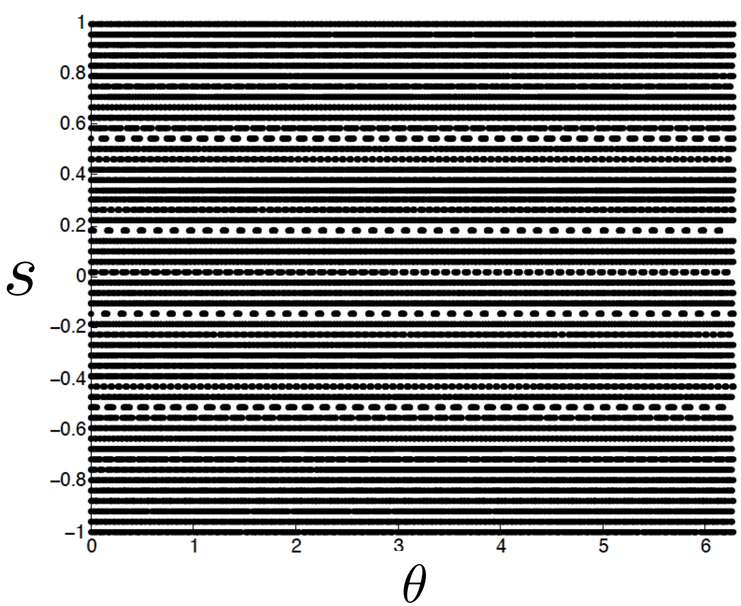
- Non-axisymmetric, shaped, toroidal geometry is too complicated
- Simplify to reveal fundamental physics and build understanding
  - Geometry: cartesian, topologically toroidal plane
  - Boundary: m=1, n=0, small perturbation (resonance n/m=0)
  - Number of volumes: N=1 (Taylor), N=2 , N=3
  - Constant pressure: [[B^2]]=0 at interfaces



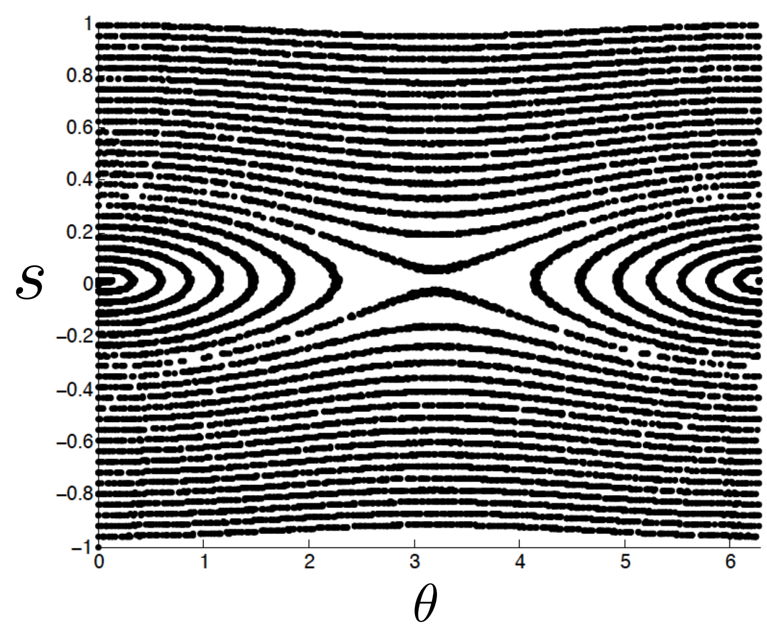
Toroidally symmetric solution for small perturbation

- Toroidal symmetry is assumed: $\partial_\zeta = 0$
- Solution of the form: $\mathbf{B} = (B_s, B_\theta, B_\zeta) = \mathbf{B}(s, \theta; \mu, B_0, \hat{B}_0, \Delta_l, \epsilon_{l-1}, \epsilon_l)$
- Can relate $(\mu, B_0, \hat{B}_0) \leftrightarrow (\Delta\Psi_l, \iota_l^+, \iota_l^-)$ then e.g. choose $\iota_l^\pm = \pm 1.618033...$
- Poincaré section for the magnetic field trajectories

UNPERTURBED (ε = 0)

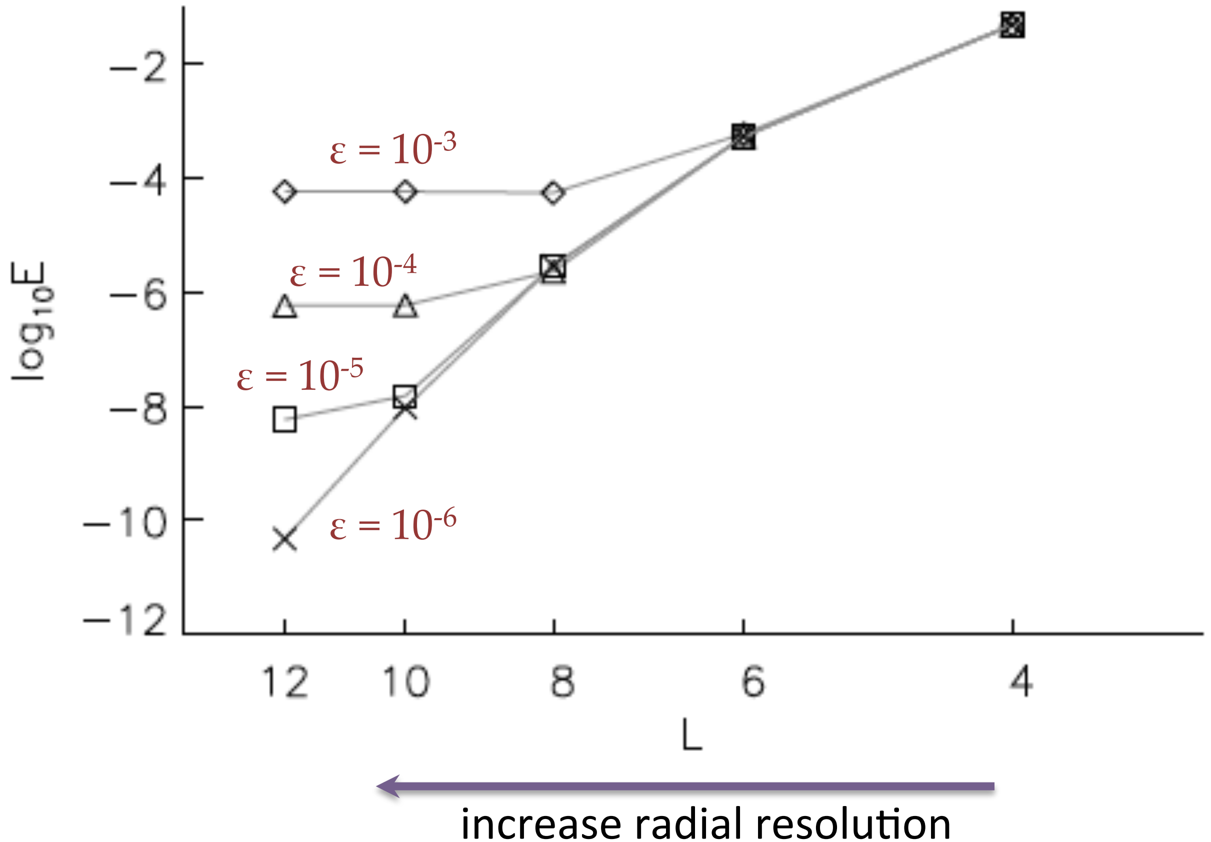


PERTURBED ($\varepsilon = \pm 2\text{x}10^{-2}$)

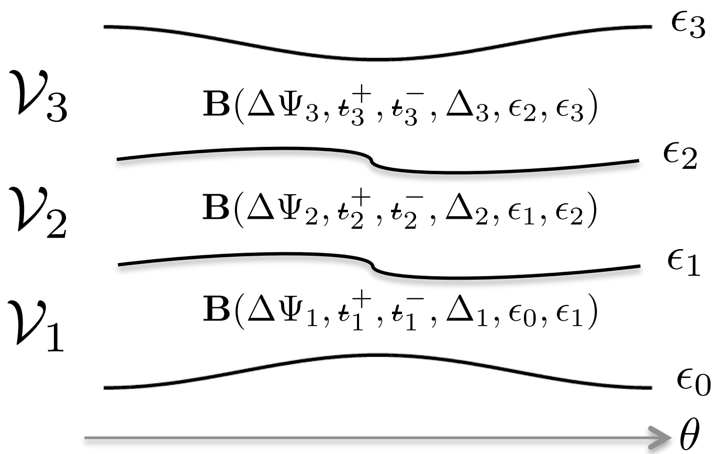


SPEC converges to the analytical solution

- Compare analytical solution with SPEC (for small perturbations)



Multiple relaxed volumes can be coupled

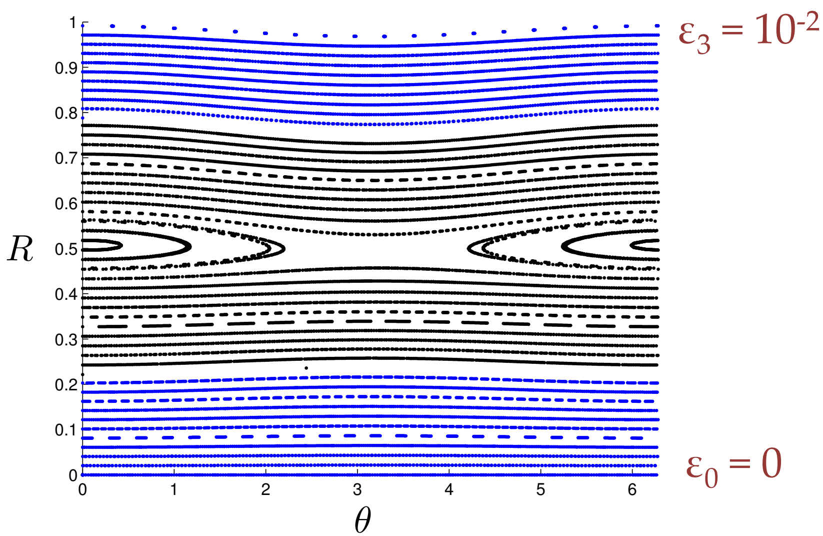


Additional geometric freedom is constrained by

$$[[B^2]] = [[B^2]]_{m=0} + [[B^2]]_{m=1} \cos\theta = 0$$

at the internal interfaces

provides Δ_1, Δ_2 provides ϵ_1, ϵ_2

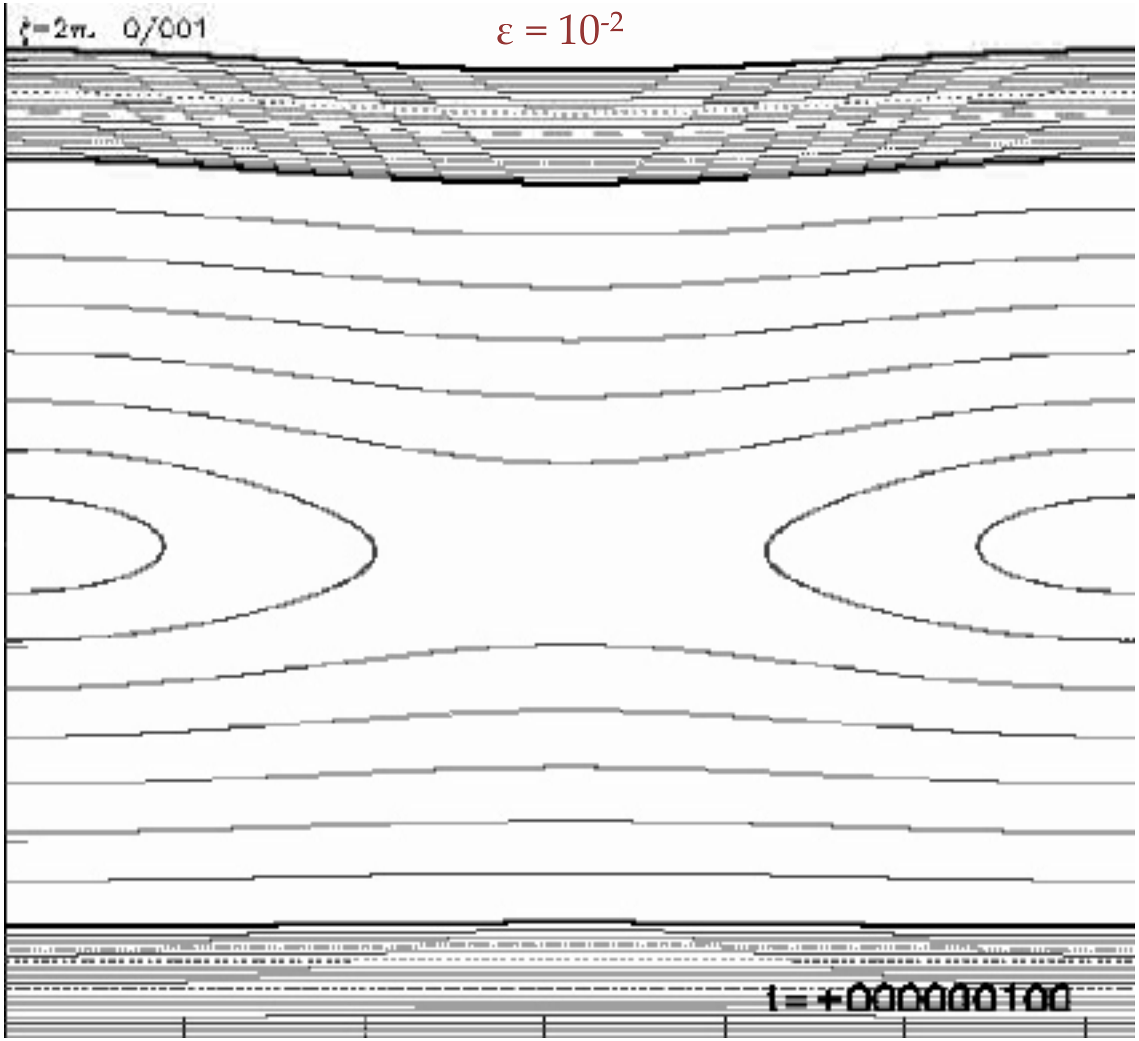


Can solve analytically: $\mathcal{M} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \mathcal{D} \begin{bmatrix} \epsilon_0 \\ \epsilon_3 \end{bmatrix}$

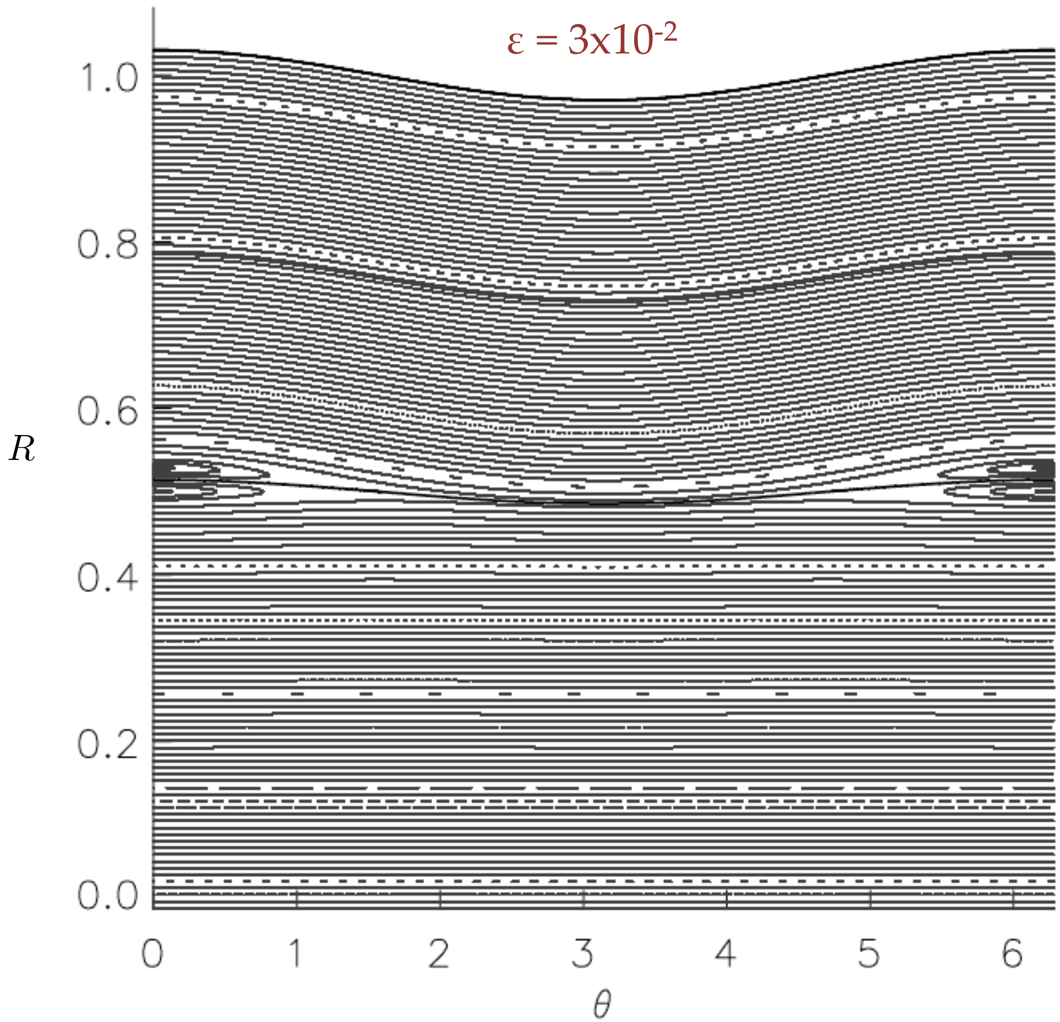
However , $\lim_{\Delta\Psi_2 \to 0,\ \iota_2^\pm \to 0} \left[\det \mathcal{M} \right] = 0$

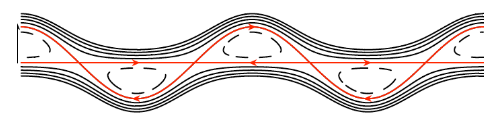
Solution becomes non-unique!

Example: island squeezing with SPEC



Example: interface at rational surface with SPEC





[Boozer and Pomphrey, PoP, 2010]

Suggested that delta current at the rational surface is not enough to completely shield the island

Questions we need to answer

- We are developing a fundamental understanding of how MRXMHD can handle island shielding at rational surfaces

- Both theory and numerics suggest that magnetic island shielding may not be unique (delta current not unique).

- Is non-uniqueness possible in MRXMHD?

- Does magnetic island shielding require additional constraints?

- How does this connect to the Hahm-Kulsrud-Taylor model?

- How does this connect to the plasmoid solutions of Dewar et al?

- How does this connect to the Boozer-Pumphrey residual islands?