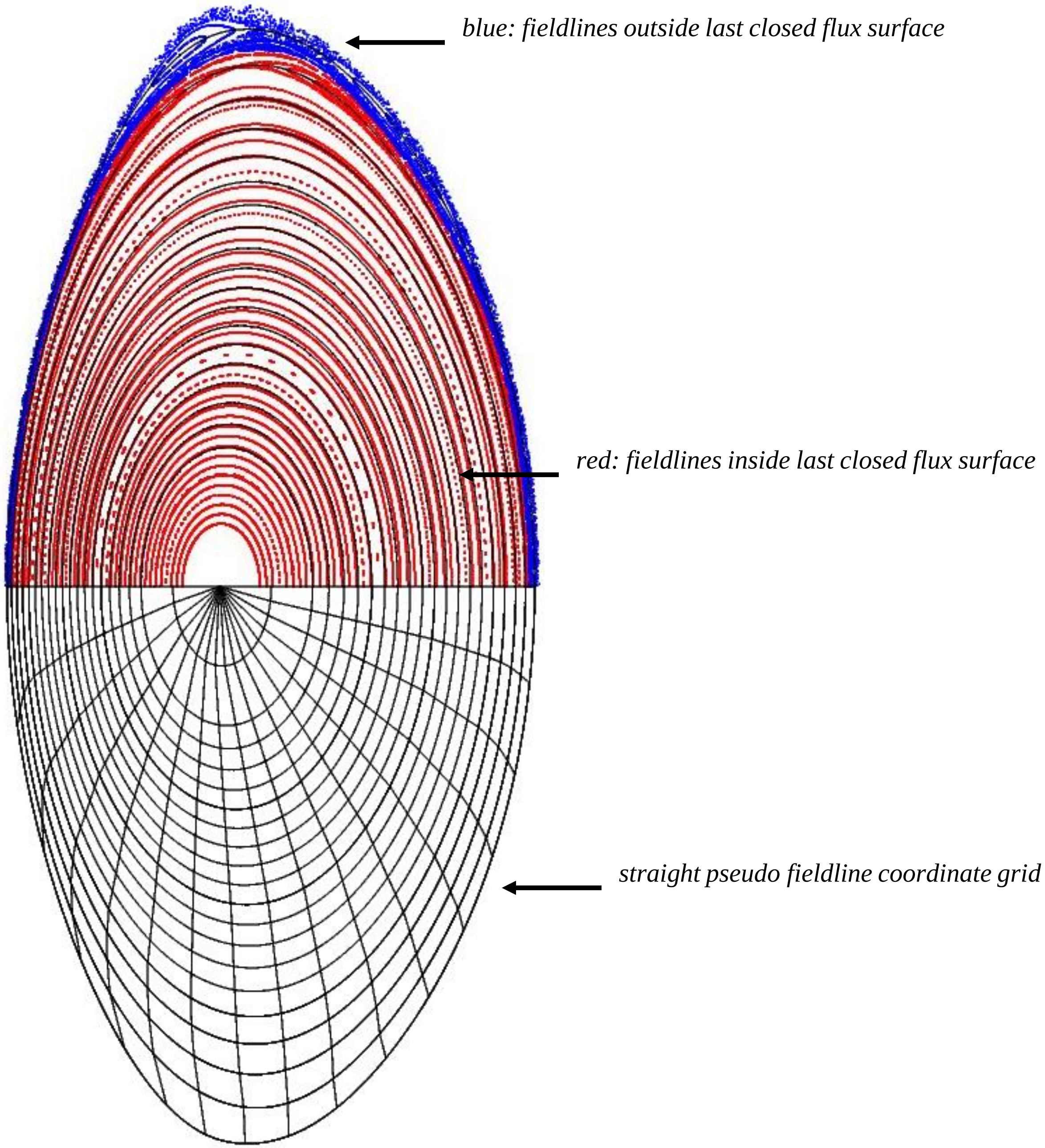
An examination of the chaotic edge of LHD



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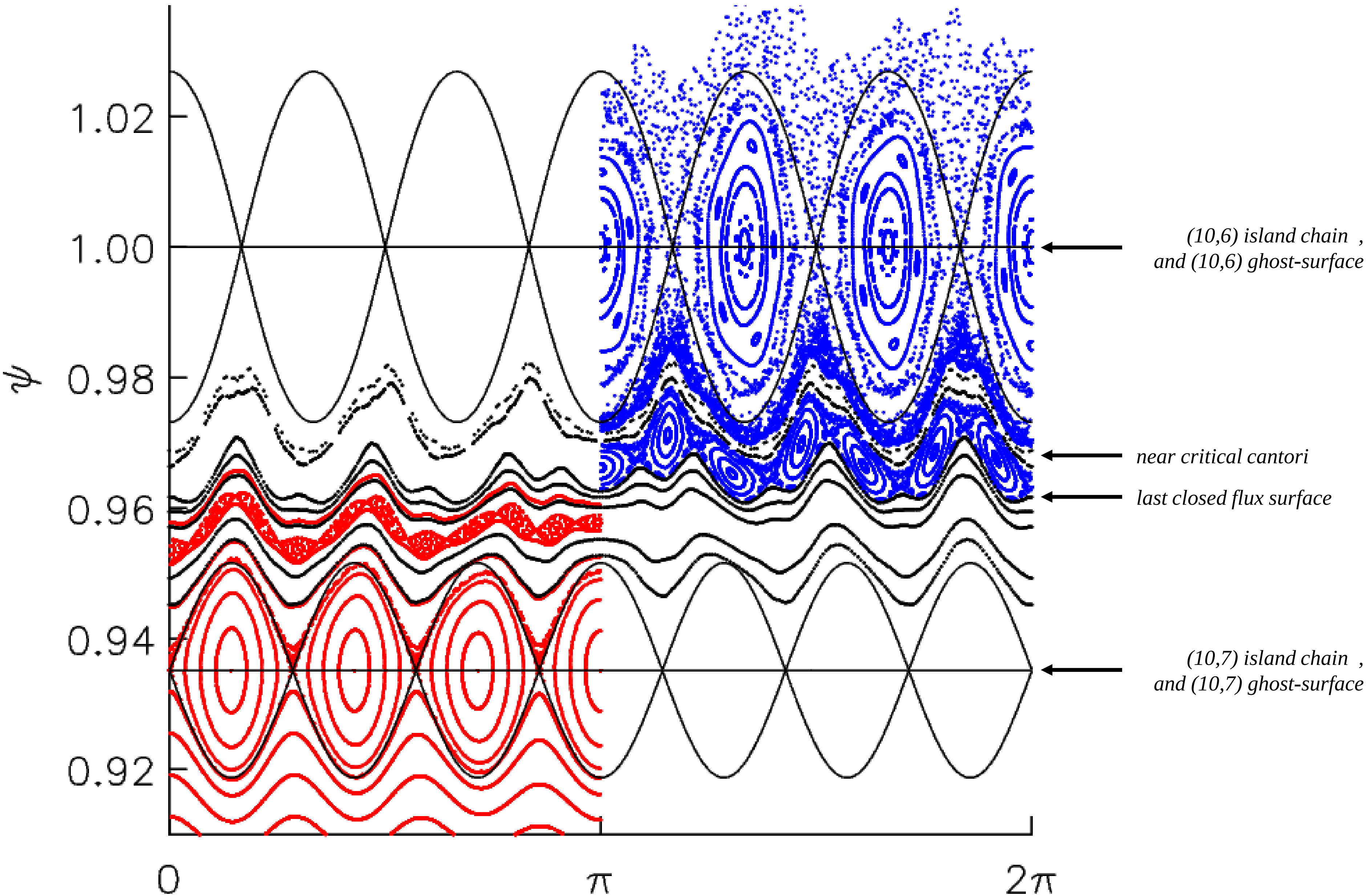
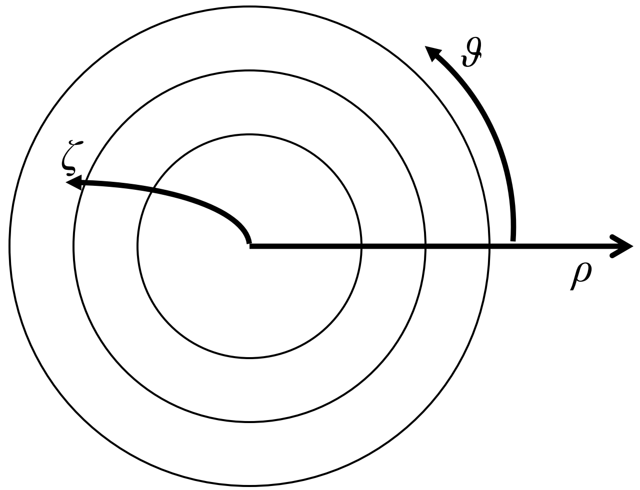


The magnetic field is given in cylindrical coordinates, and arbitrary, toroidal coordinates are introduced.

$$\begin{array}{rclcl} R &=& R(\rho,\theta,\zeta) &=& \sum_{m,n} R_{m,n}(\rho)\cos(m\theta - n\zeta) \\ \phi &=& \zeta && \\ Z &=& Z(\rho,\theta,\zeta) &=& \sum_{m,n} Z_{m,n}(\rho)\sin(m\theta - n\zeta) \end{array}$$



$$\mathbf{B} = B^R\mathbf{e}_R + B^\phi\mathbf{e}_\phi + B^Z\mathbf{e}_Z = B^\rho\mathbf{e}_\rho + B^\theta\mathbf{e}_\theta + B^\zeta\mathbf{e}_\zeta$$

$$\begin{pmatrix} B^R \\ B^\phi \\ B^Z \end{pmatrix} = \begin{pmatrix} R_\rho & R_\theta & R_\zeta \\ \phi_\rho & \phi_\theta & \phi_\zeta \\ Z_\rho & Z_\theta & Z_\zeta \end{pmatrix} \begin{pmatrix} B^\rho \\ B^\theta \\ B^\zeta \end{pmatrix}$$

Begin with circular cross section coordinates, centered on the magnetic axis.

In practice, we will have a discrete set of toroidal surfaces that will be used as "coordinate surfaces".

The Fourier harmonics, $R_{m,n}$ & $Z_{m,n}$, of a discrete set of toroidal surfaces are interpolated using piecewise cubic polynomials.

If the surfaces are smooth and well separated, this "simple-minded" interpolation works.

A regularization factor is introduced, e.g. $R_{m,n}(\rho) = \rho^{m/2}\bar{X}_{m,n}(\rho) + R_{m,n}(0)$ to ensure that the interpolated surfaces do not overlap near the coordinate origin=magnetic axis.

A magnetic vector potential, in a suitable gauge, is quickly determined by radial integration.

$$\mathbf{A} = A_\theta(\rho,\theta,\zeta)\nabla\theta + A_\zeta(\rho,\theta,\zeta)\nabla\zeta$$

$$\begin{array}{rcrcl}
\sqrt{g}B^\rho & = & \partial_\theta A_\zeta & - & \partial_\zeta A_\theta \\
\sqrt{g}B^\theta & = & & - & \partial_\rho A_\zeta \\
\sqrt{g}B^\zeta & = & \partial_\rho A_\theta & &
\end{array}$$

$$\begin{array}{rccc}
\partial_\rho A_{\theta,m,n} & = & & (\sqrt{g}B^\zeta)_{m,n} \\
\partial_\rho A_{\zeta,m,n} & = & - & (\sqrt{g}B^\theta)_{m,n}
\end{array}$$

hereafter, we will use the commonly used notation

$$\mathbf{A} \equiv \psi\nabla\theta - \chi\nabla\zeta$$

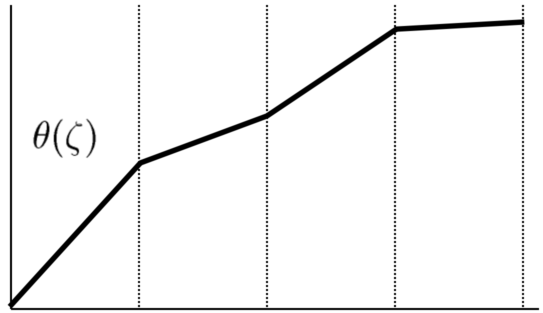
ψ is the toroidal flux, and χ is called the magnetic field-line Hamiltonian

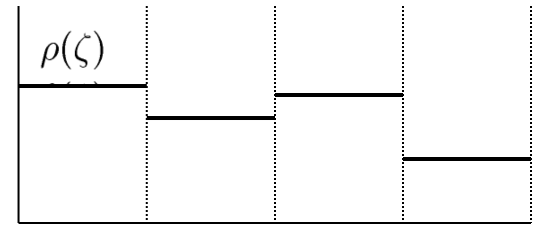
The magnetic field-line **action** is the **line integral of the vector potential**

$$S \equiv \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}$$

$\mathcal{C}$ is an arbitrary "trial" curve

piecewise-constant, piecewise-linear





For $\zeta \in (\zeta_{i-1}, \zeta_i)$

$$\begin{aligned} \rho(\zeta) &= \rho_i \\ \theta(\zeta) &= \theta_{i-1} + \dot{\theta} \ (\zeta - \zeta_{i-1}) \end{aligned}$$

where $\dot{\theta} \equiv (\theta_i - \theta_{i-1})/\Delta\zeta$ is constant,

$$S \equiv \sum_{i=1}^{N} \int_{\zeta_{i-1}}^{\zeta_i} d\zeta \ \mathbf{A} \cdot d\mathbf{l} \equiv \sum_{i=1}^{N} \sum_{m,n} \left[\psi_{mn}(\rho_i) \ \dot{\theta} - \chi_{mn}(\rho_i) \right] \int_{\zeta_{i-1}}^{\zeta_i} d\zeta \cos(m\theta - n\zeta)$$

the piecewise-linear approximation allows
the cosine integral to be evaluated analytically,
i.e. method is FAST

$$\int_{\zeta_{i-1}}^{\zeta_i} d\zeta \ \cos(m\theta - n\zeta) = \frac{\sin(m\theta_i - n\zeta_i) - \sin(m\theta_{i-1} - n\zeta_{i-1})}{m\dot{\theta} - n}$$

To find extremizing curves, use Newton method to set $\partial_\rho S=0$, $\partial_\vartheta S=0$

$\dfrac{\partial S}{\partial \rho_i} = 0$ reduces to $\dfrac{\partial S_i}{\partial \rho_i} = 0$, which can be solved locally, $\rho_i = \rho_i(\theta_{i-1}, \theta_i)$

$$\frac{\partial S}{\partial \theta_i} = \partial_2 S_i(\theta_{i-1}, \theta_i) + \partial_1 S_{i+1}(\theta_i, \theta_{i+1})$$

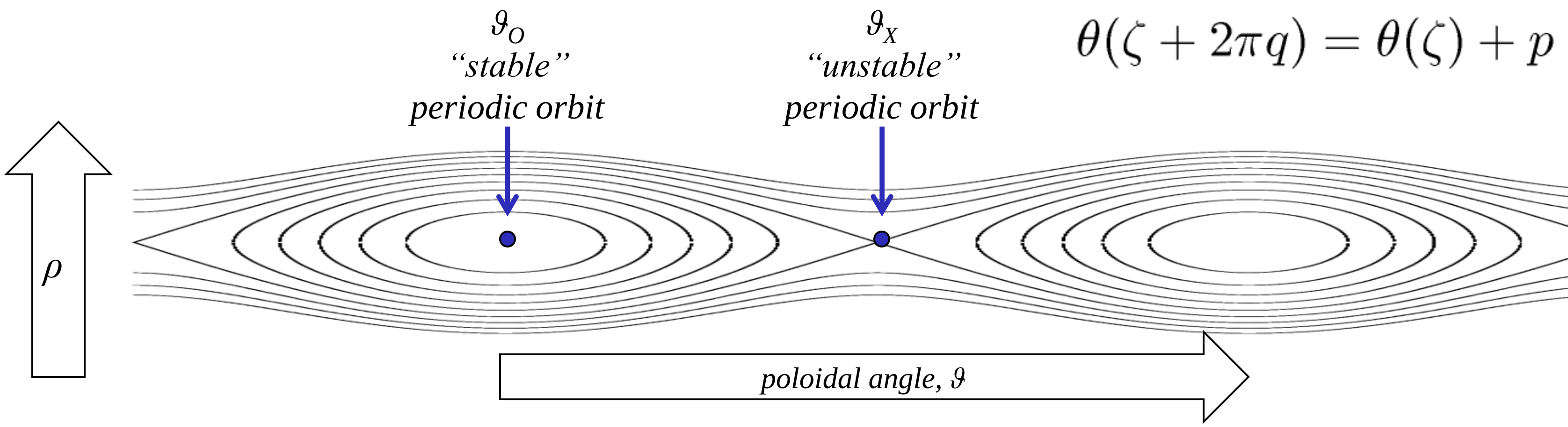
tridiagonal Hessian, inverted in O(N) operations, ***i.e. method is FAST***

Not required to follow magnetic field lines, and does not depend on coordinate transformation.

The trial-curve is constrained to be periodic, and a family of periodic curves is constructed.

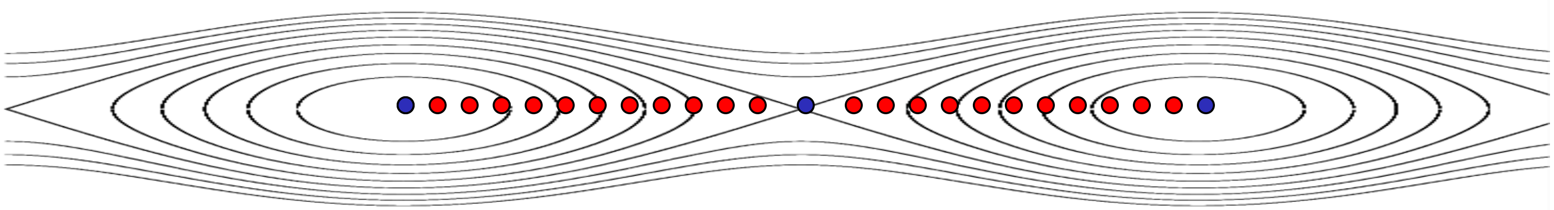
Usually, there are only the “stable” periodic field-line and the “unstable” periodic field line,

$$\theta(\zeta + 2\pi q) = \theta(\zeta) + p\,2\pi/N$$



However, we can “artificially” constrain the poloidal angle, i.e. *ϑ(0)=given constant*,
and search for extremizing periodic curve of the *constrained* action-integral

$$S \equiv \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l} \;-\; \nu \;[\;\theta(0) - \theta_0\;]$$



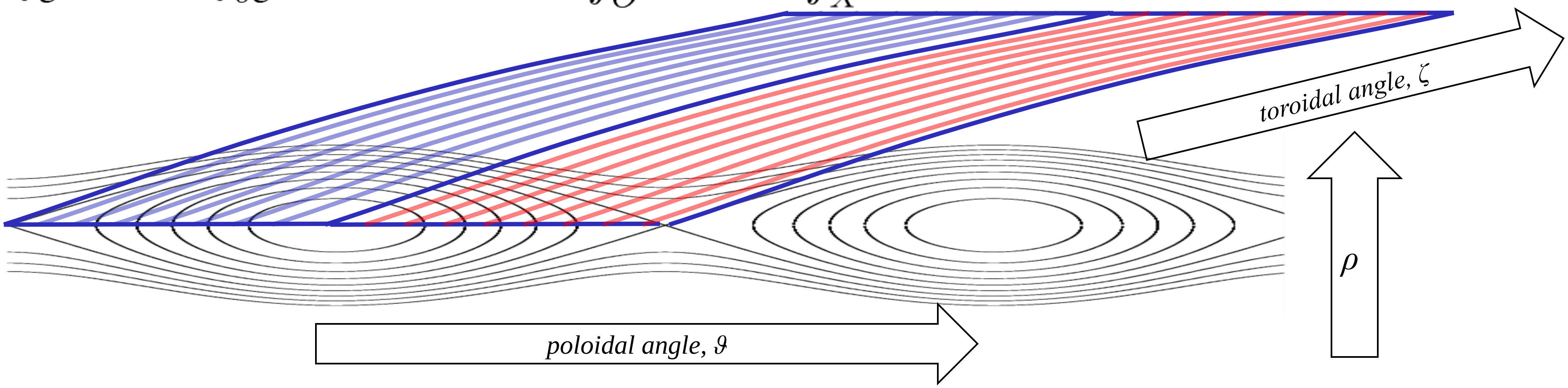
A rational, quadratic-flux minimizing surface
is a family of periodic, extremal curves of the constrained action integral, and
is closely to related to the rational ghost-surface,
which is defined by an action-gradient flow between the minimax periodic orbit and the minimizing orbit.

$$\varphi_2 \equiv \frac{1}{2} \int_{\Gamma} w |B_n|^2 \, \mathrm{d}S,$$

The "upward" flux = "downward" flux across a toroidal surface passing through an island chain can be computed.

$$\int_{\partial\mathcal{V}} \mathbf{B}\cdot d\mathbf{S} \equiv \int_{\mathcal{V}} \nabla\cdot\mathbf{B} = 0$$ *the total flux across any closed surface of a divergence free field is zero.*

$$\int_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{s} \equiv \int_{\partial\mathcal{S}} \mathbf{A}\cdot d\mathbf{l} \qquad \Psi_{p/q} \equiv \int_{O} \mathbf{A}\cdot d\mathbf{l} - \int_{X} \mathbf{A}\cdot d\mathbf{l}$$



consider a sequence of rationals, *p/q*, that approach an irrational,

$$\text{If } \Psi_{p/q} \to 0 \text{ as } p/q \to ɨ, \text{ then KAM surface exists}$$

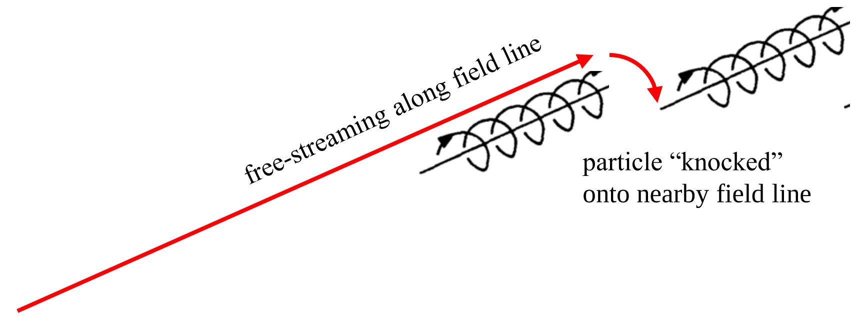
If $\Psi_{p/q} \to \Delta$, where $\Delta \neq 0$, then the KAM surface is "broken", and $\Psi_{p/q}$ is the upward-flux across the cantorus

The diagnostics include:

1. Greene's residue criterion: the existence of an irrational surface can be determined by calculating the stability of nearby periodic orbits.
2. Chirikov island overlap: flux surfaces are destroyed when magnetic islands overlap.
3. Cantori: can present effective, partial barriers to fieldline transport, and cantori can be approximated by high-order periodic orbits.

The construction of chaotic coordinates simplifies anisotropic diffusion

$$\frac{\partial T}{\partial t} = \nabla \cdot (\kappa_\parallel \nabla_\parallel T + \kappa_\perp \nabla_\perp T) + Q,$$



In chaotic coordinates, the temperature becomes a surface function, *T=T(s)*, where *s* labels invariant (flux) surfaces or almost-invariant surfaces.

If *T=T(s)*, the anisotropic diffusion equation can be solved analytically, $\dfrac{dT}{ds} = \dfrac{c}{\kappa_\parallel \varphi + \kappa_\perp G},$

where c is a constant, and

$\varphi = \int \int d\theta d\phi \sqrt{g} B_n^2,$ is related to the quadratic-flux across an invariant or almost-invariant surface,

$G = \int \int d\theta d\phi \sqrt{g} g^{ss},$ is a geometric coefficient.

An expression for the temperature gradient in chaotic fields
S.R. Hudson, Physics of Plasmas, 16:010701, 2009
Temperature contours and ghost-surfaces for chaotic magnetic fields
S.R.Hudson and J.Breslau
Physical Review Letters, 100:095001, 2008

When the upward-flux is sufficiently small,
so that the parallel diffusion across an almost-invariant surface is comparable to the perpendicular diffusion,
the plasma cannot distinguish between a perfect invariant surface and an almost invariant surface

List of publications, *http://w3.pppl.gov/~shudson/*

Generalized action-angle coordinates defined on island chains
R.L.Dewar, S.R.Hudson and A.M.Gibson
Plasma Physics and Controlled Fusion, 55:014004, 2013

Unified theory of Ghost and Quadratic-Flux-Minimizing Surfaces
Robert L.Dewar, Stuart R.Hudson and Ashley M.Gibson
Journal of Plasma and Fusion Research SERIES, 9:487, 2010

Are ghost surfaces quadratic-flux-minimizing?
S.R.Hudson and R.L.Dewar
Physics Letters A, 373(48):4409, 2009

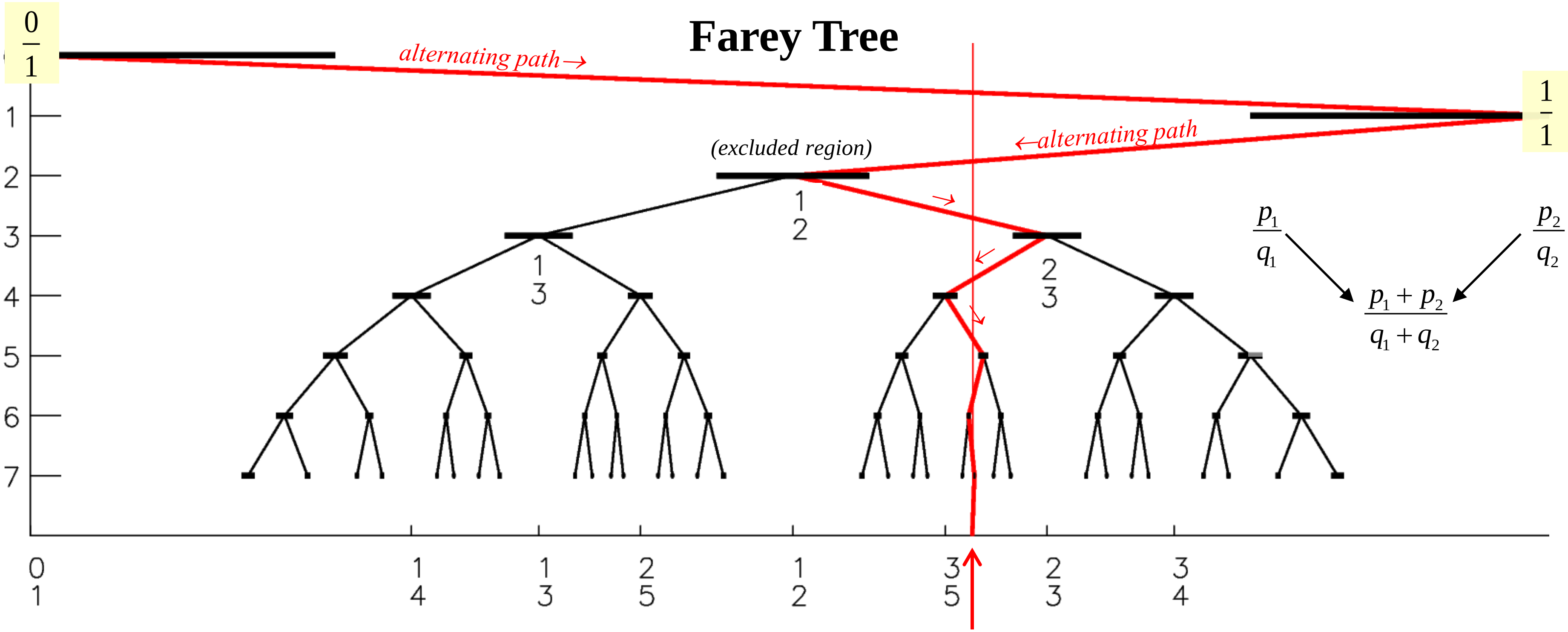
An expression for the temperature gradient in chaotic fields
S.R.Hudson
Physics of Plasmas, 16:010701, 2009

Temperature contours and ghost-surfaces for chaotic magnetic fields
S.R.Hudson and J.Breslau
Physical Review Letters, 100:095001, 2008

Calculation of cantori for Hamiltonian flows
S.R.Hudson
Physical Review E, 74:056203, 2006

Almost invariant manifolds for divergence free fields
R.L.Dewar, S.R.Hudson and P.Price
Physics Letters A, 194(1-2):49, 1994

The fractal structure of chaos is related to the structure of numbers



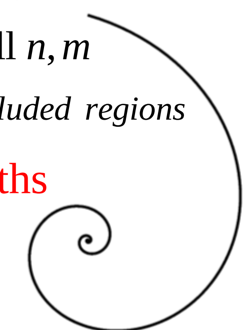
islands & chaos emerge at every rational → about each rational n/m, introduce excluded region, width r/m^k

KAM Theorem → flux surface can survive if $|\omega - n/m| > r/m^k$, for all n, m

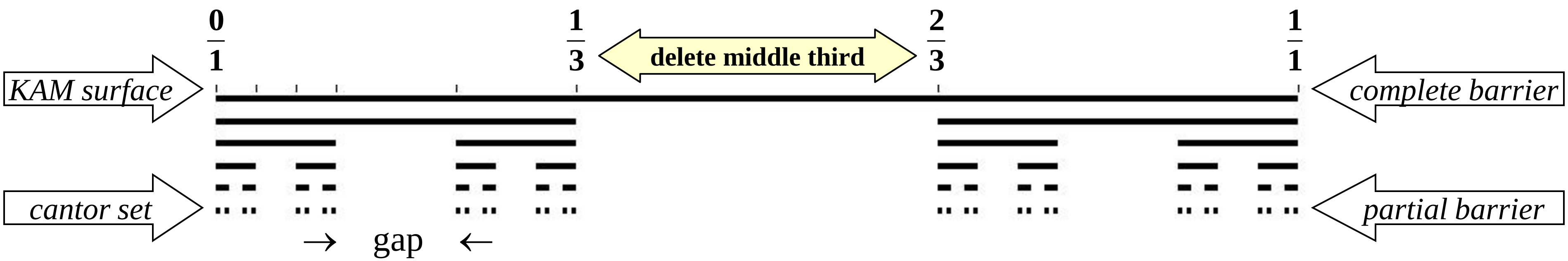
(Kolmogorov, Arnold, Moser) *we say that* ω *is* "*strongly - irrational*" *if* ω *avoids all excluded regions*

Greene's residue criterion → the most robust flux surfaces are associated with alternating paths

→ Fibonacci ratios $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \ldots$



For non-integrable fields, field line transport is restricted by KAM surfaces and cantori

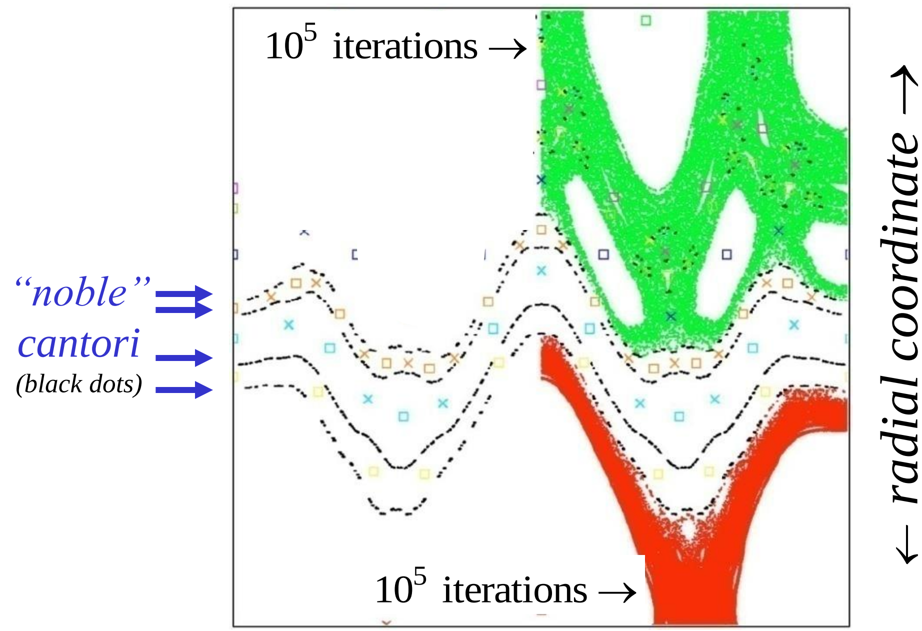


→ *KAM surfaces are closed, toroidal surfaces that* ***stop*** *radial field line transport*

→ *Cantori have "holes" or "gaps";*
but ***cantori can severely "slow down"*** *radial field line transport*

→ *Example: all flux surfaces destroyed by chaos,*
but even after ***100 000 transits*** *around torus*
the field lines ***cannot get past cantori***

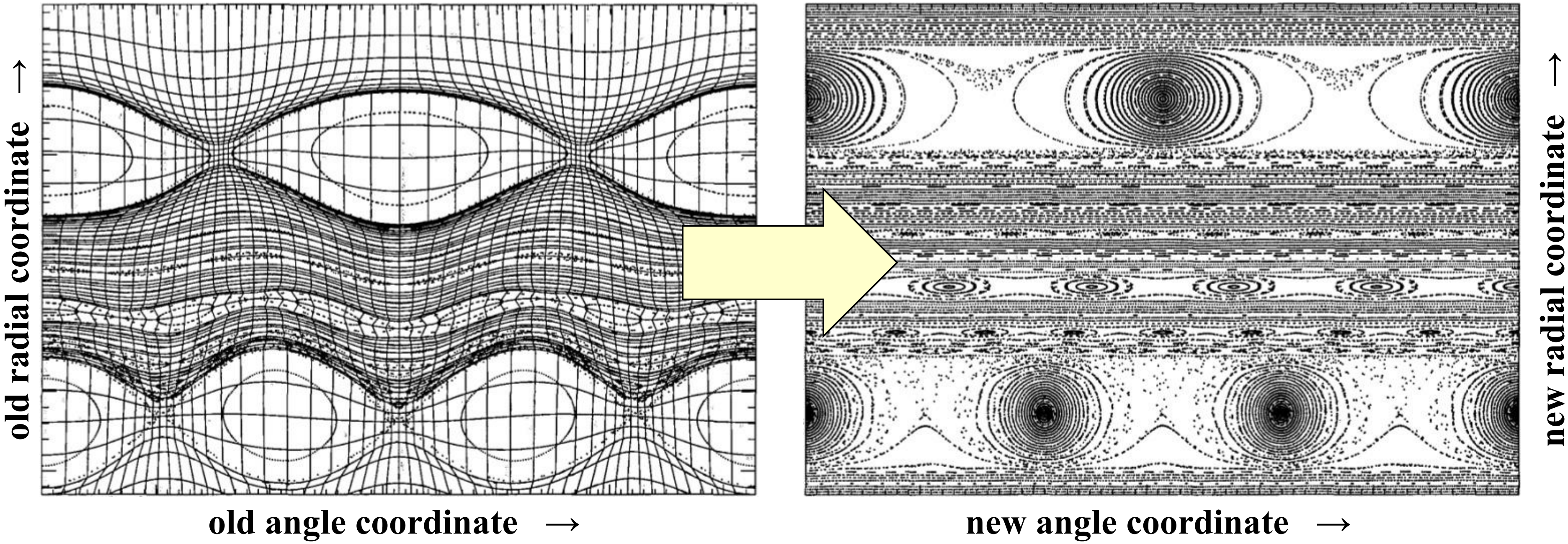
Calculation of cantori for Hamiltonian flows
S.R. Hudson, Physical Review E 74:056203, 2006



Chaotic coordinates "straighten out" chaos

Poincaré plot of chaotic field
(in **action-angle** coordinates of **unperturbed** field)

Poincaré plot of chaotic field
in chaotic coordinates



phase-space is partitioned into (1) regular ("irrational") regions with "good flux surfaces", temperature gradients
and (2) irregular (" rational") regions with islands and chaos, flat profiles

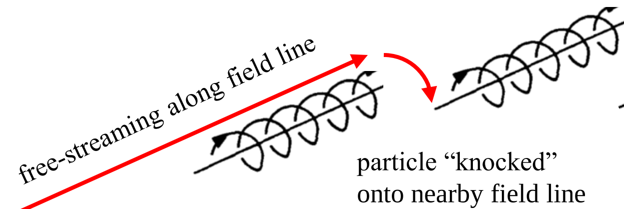
Generalized magnetic coordinates for toroidal magnetic fields
S.R. Hudson, Doctoral Thesis, The Australian National University, 1996

Chaotic coordinates simplify anisotropic transport

The temperature is constant on ghost surfaces, *T=T(s)*

1. Transport *along* the magnetic field is *unrestricted*

→ consider parallel random walk, with **long** steps≈*collisional mean free path*



2. Transport *across* the magnetic field is *very small*

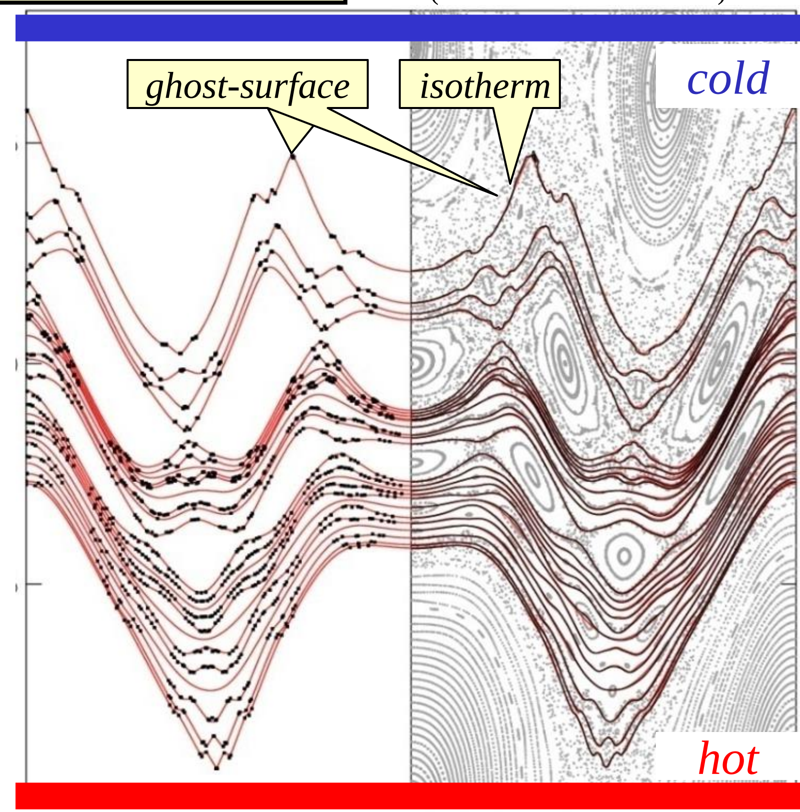
→consider perpendicular random walk with **short** steps≈*Larmor radius*

3. Anisotropic diffusion balance

$$\kappa_{\parallel} \nabla_{\parallel}^2 T + \kappa_{\perp} \nabla_{\perp}^2 T = 0 , \quad \kappa_{\parallel} \gg \kappa_{\perp}, \quad \kappa_{\perp} / \kappa_{\parallel} \sim 10^{-10}$$

4. Compare solution of numerical calculation to ghost-surfaces

$2^{12} \times 2^{12}$ = *4096 ×4096* grid points (to resolve small structures)



5. The temperature adapts to KAM surfaces,cantori,

and ghost-surfaces!

i.e. T=T(s), where s=const. is a ghost-surface

from T=T(s,θ,ϕ) to T=T(s) is a fantastic simplification, allows analytic solution

$$\frac{dT}{ds} \propto \frac{1}{\kappa_{\parallel} \varphi_2 + \kappa_{\perp} G}$$

Temperature contours and ghost-surfaces for chaotic magnetic fields
S.R. Hudson et al., Physical Review Letters, 100:095001, 2008
Invited talk 22^{nd} IAEA Fusion Energy Conference, 2008
Invited talk 17th International Stellarator, Heliotron Workshop, 2009

An expression for the temperature gradient in chaotic fields
S.R. Hudson, Physics of Plasmas, 16:100701, 2009