A novel solution for the computation of three-dimensional ideal MHD equilibria with nested surfaces

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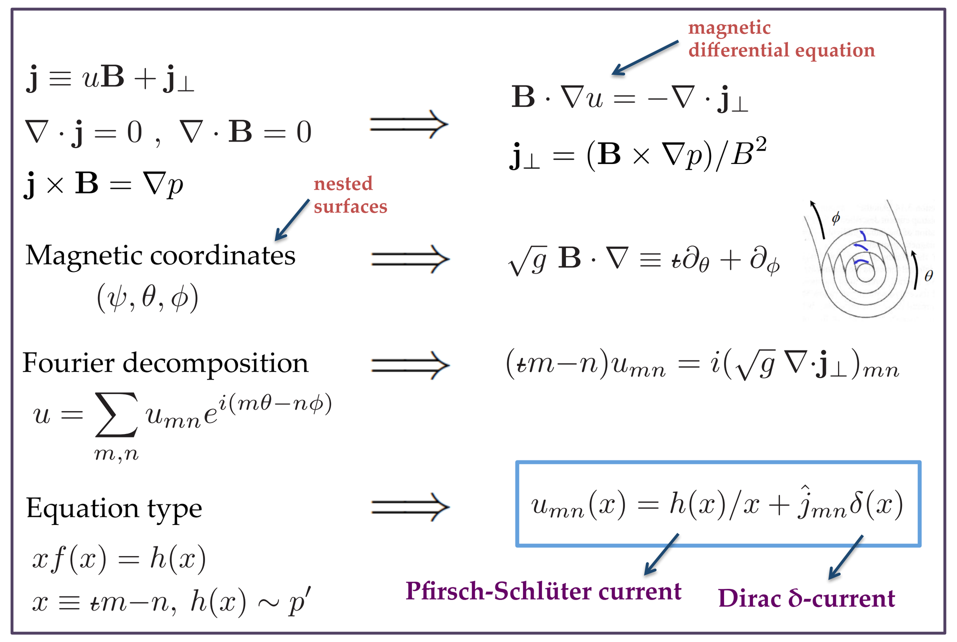
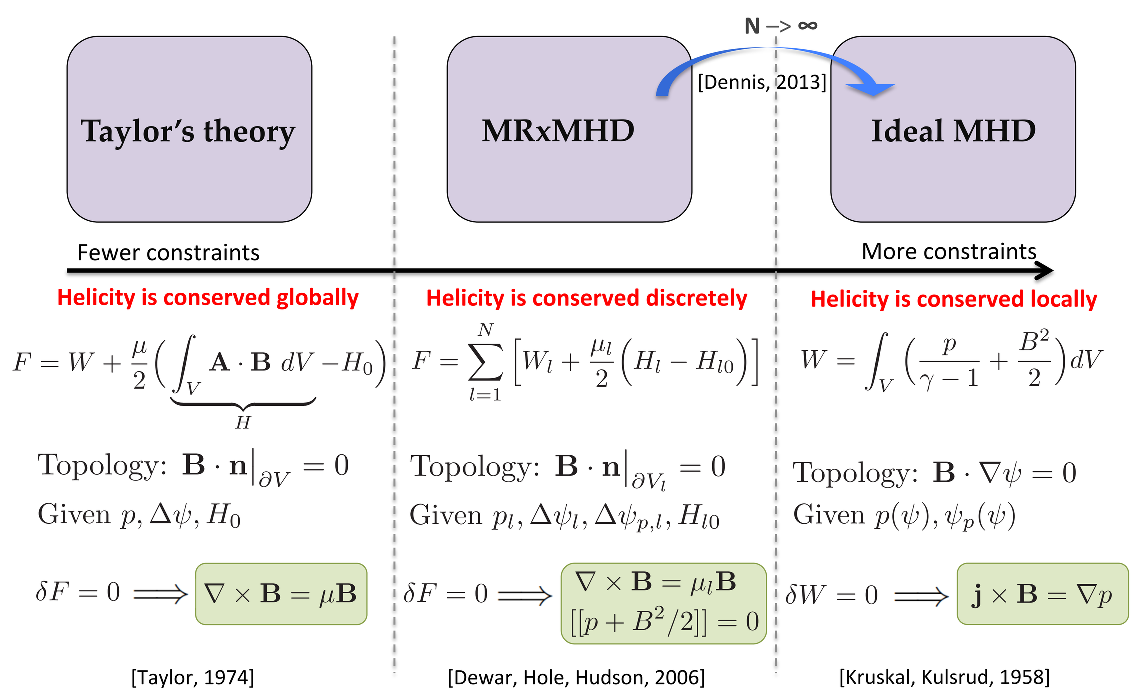
1. Exact computation of current sheets in 3D equilibria

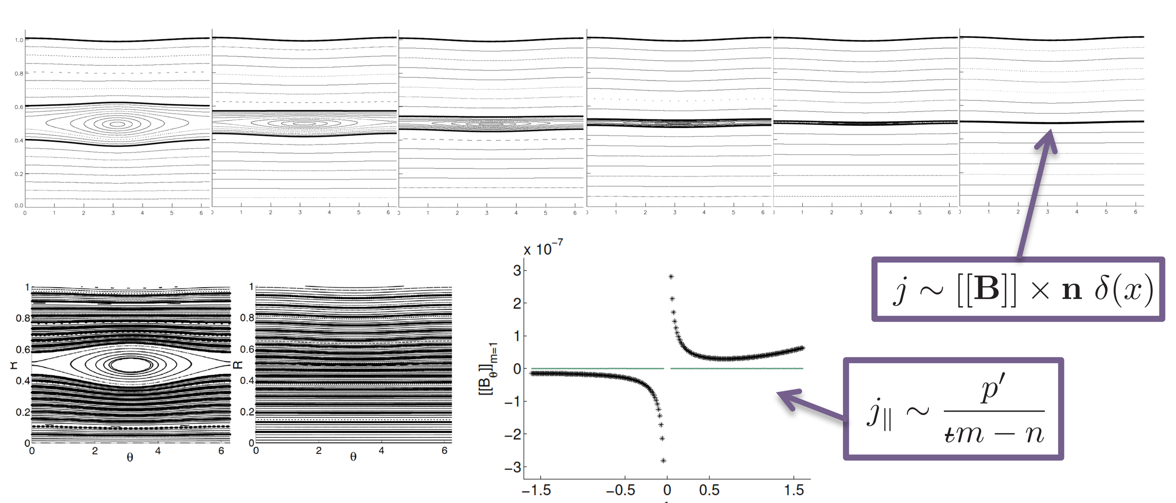
- 3D MHD equilibria with nested surfaces exhibit **singular currents** at resonant surfaces [1]:
  - A Pfirsch-Schlüter $1/x$**-current**, which arises as a result of finite pressure gradient.
  - A $\delta(x)$**-current** which is necessary to prevent the formation of islands.



- Singular current densities, or **current sheets**, are **critical for** :
  - The **ideal and resistive stability** of 3D MHD equilibria [2-4].
  - The computation of the ideal response to **RMP perturbations** in tokamaks [5].
- **QUESTION:** How to numerically compute singular current densities in ideal MHD equilibria?
- **ANSWER:** **M**ultiregion **Relax**ed **MHD** [6] allows for discontinuities and converges to ideal MHD.



- The **SPEC code** [7] is a numerical implementation of MRxMHD.
- **First numerical proof** of the mutual existence of singular current densities [8].



- **CONCLUSION:** well-defined solutions require locally-infinite shear at the resonant surface.

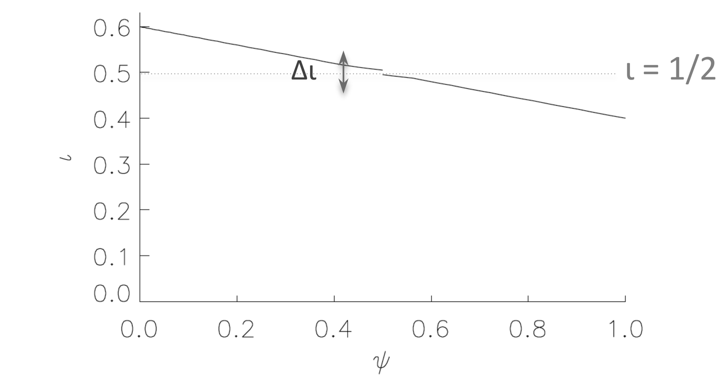
2. Existence of 3D MHD equilibria with nested surfaces

- A **physically valid equilibrium** must have finite integrated current densities:
  - δ-current densities are always integrable.
  - $1/x$ pressure-driven current densities give **divergent currents**.
- **Historical conclusion**: 3D equilibria have either **fractal** [9] or **stepped** [10] pressure profiles.
- **QUESTION:** are there 3D MHD equilibria with nested surfaces and smooth pressure profiles?
- **ANSWER:** we present a **new class of 3D MHD equilibria** [11] with
  (1) nested surfaces, arbitrary 3D geometry, and arbitrary smooth pressure,
  (2) agreement between linear and nonlinear equilibrium calculations in the appropriate limit.
- **Idea:** consider 3D MHD equilibria with discontinuous transform across resonant surfaces.

- Screw pinch axisymmetric equilibrium with $p(r) \equiv 0$ and

$$\iota(r) = \iota_0 - \iota_1\,(r/a)^2\ \pm\Delta\iota$$

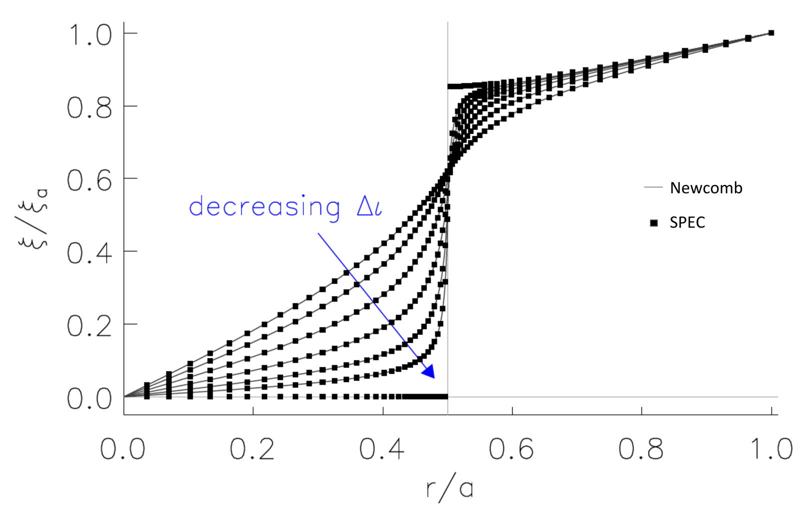
where $\Delta\iota$ results from a "DC" current sheet.



- Linear response to boundary perturbation $\xi_a\cos(m\theta + kz)$:

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0 \quad \text{(Newcomb's equation)}$$

where $f(r)$ and $g(r)$ depend on the equilibrium.



- Radial displacement, $\xi(r)$, gives overlap of surfaces unless

$$\left|\frac{d\xi}{dr}\right| < 1 \quad (\textit{sine qua non} \text{ condition})$$

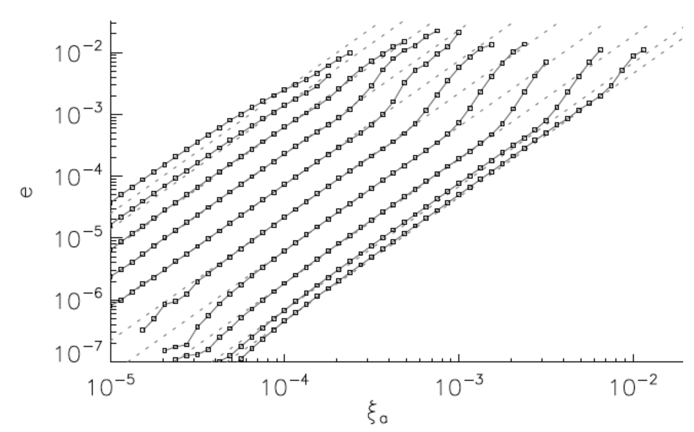
- Analytical expression for $|d\xi/dr|_{r=r_s}$

$$\xi_s' = 2\iota_s'\frac{\xi_s}{\Delta\iota}$$

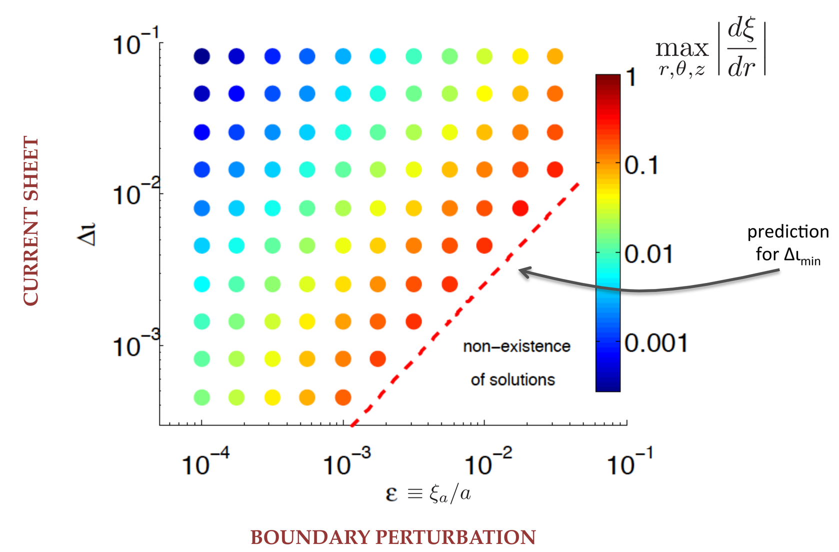
provides a minimum current sheet

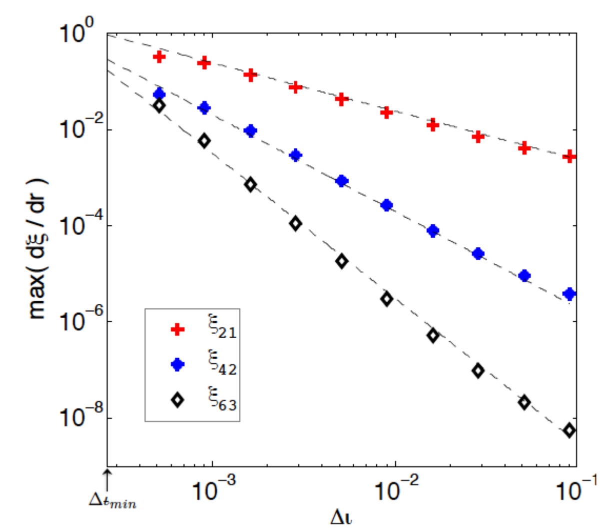
$$\Delta\iota > \Delta\iota_{min} = 2\iota_s'\xi_s \approx \iota_s'\xi_a$$

as the *sine qua non* condition for the existence of equilibria.



- Confirmed by linear and nonlinear simulations [11].





- **CONCLUSION**: 3D ideal equilibria exist as long as $\Delta\iota > \Delta\iota_{min}$, and may be computed with arbitrary smooth pressure.

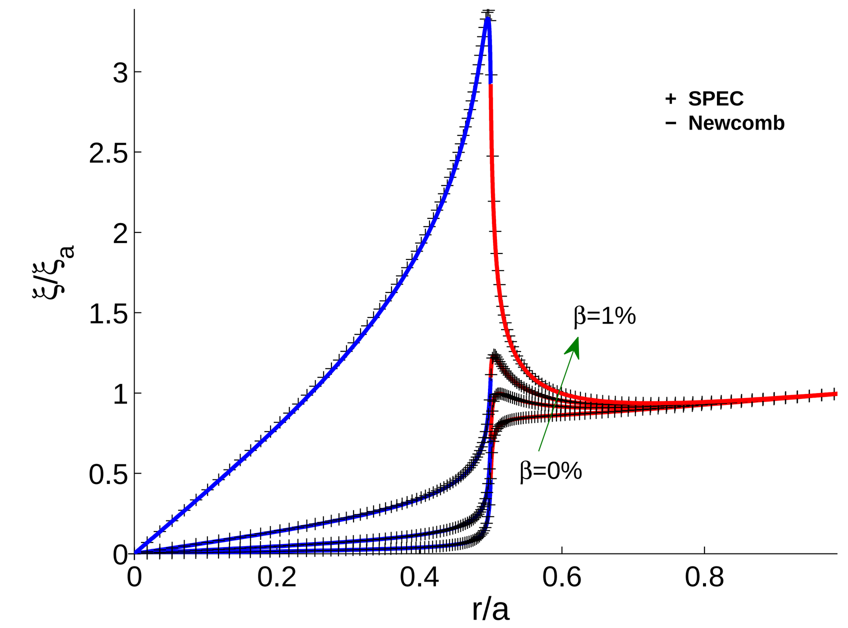
3. RMP penetration and amplification with β in an ideal fluid

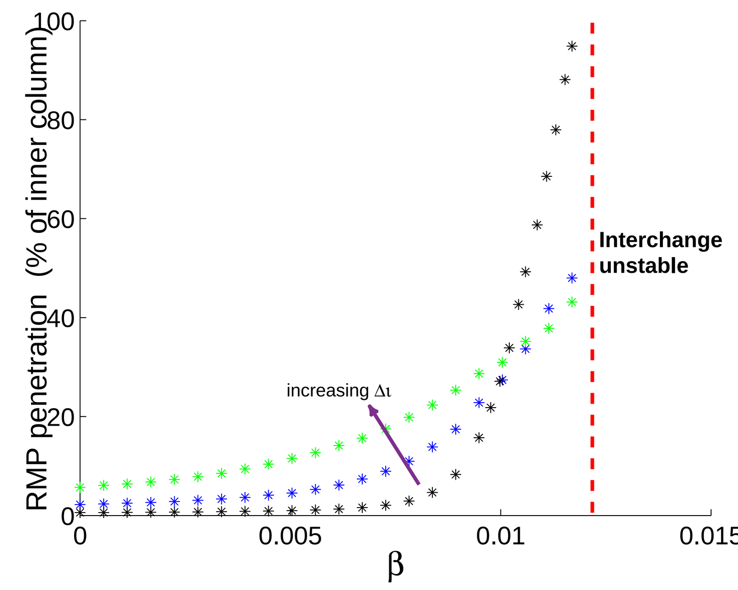
- Screw pinch axisymmetric equilibrium with finite pressure,

$$\iota(r) = \iota_0 - \iota_1\,(r/a)^2\ \pm\Delta\iota$$

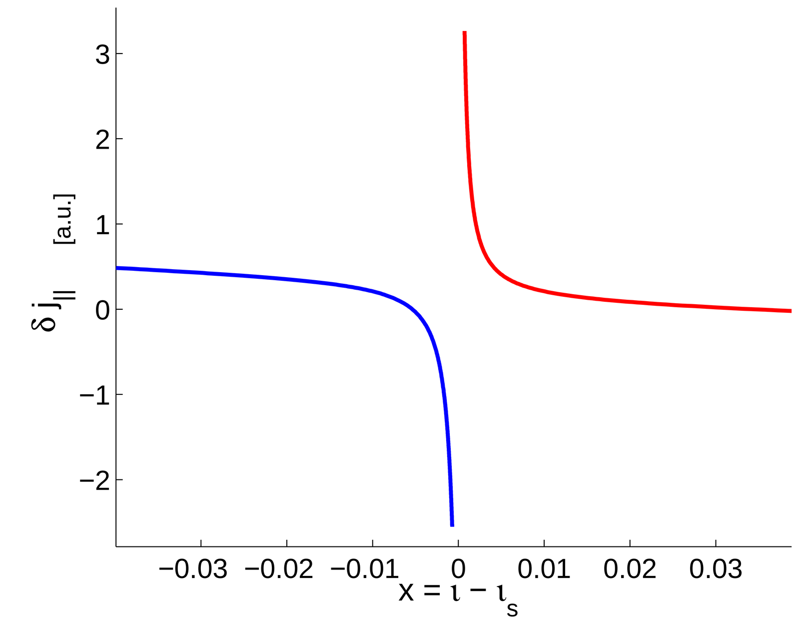
$$p(r) = p_0\left(1 - 2(r/a)^2 + (r/a)^4\right)$$

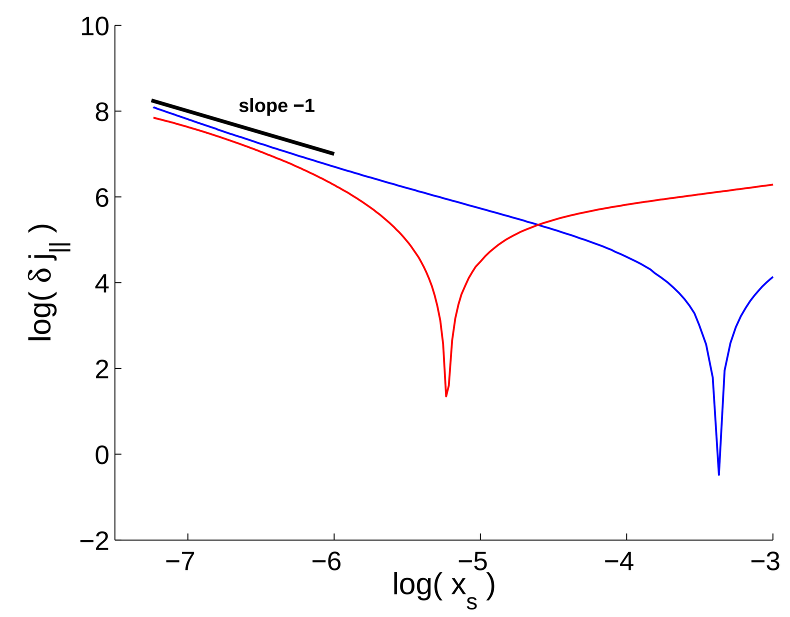
- The solution to Newcomb's equation, $\xi(r)$, gives the response to a resonant magnetic perturbation (RMP).
- Very large penetration and amplification of the RMP, even for low values of $\beta\ \sim 1\%$ (interchange stable).
- SPEC nonlinear simulations reproduce the same results.





- Pressure-driven **parallel current** shows $\sim 1/x$ behaviour but is **bounded** by $\sim 1/\Delta\iota$.





- **CONCLUSION**: RMP significantly penetrates all the way into the centre of a tokamak, even within ideal MHD.
- **Note**: The nonlinear code VMEC [12] shows qualitatively similar behaviour for the displacement solution.

4. References

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