Computation of singular currents at rational surfaces in non-axisymmetric MHD equilibria

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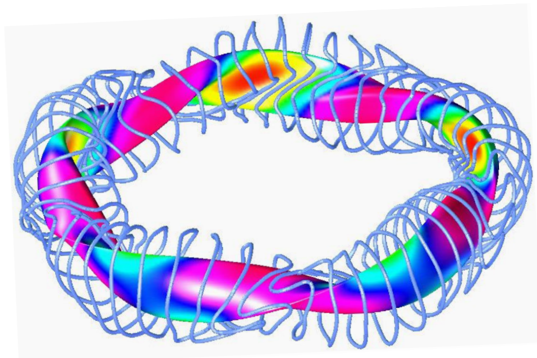
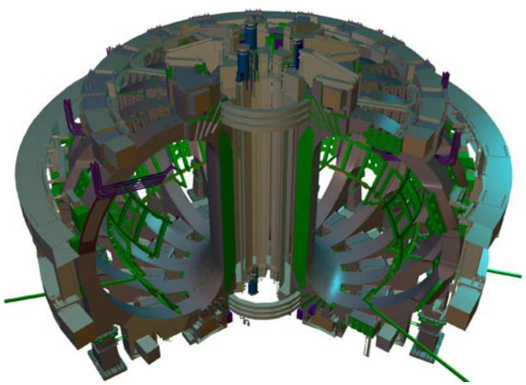
3D MHD brings together tokamaks and stellarators

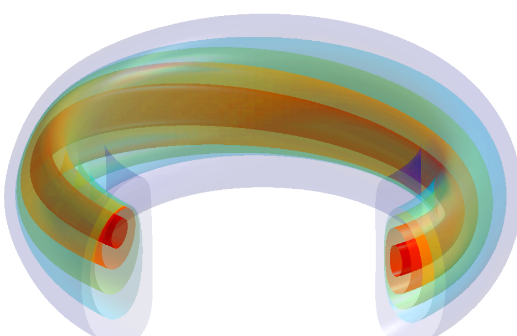


Stellarator three-dimensional topology



Tokamak non-axisymmetric designs

(magnetic ripple, resonant magnetic perturbations,…)



Tokamak MHD helical modes and bifurcations

(saturated internal kink, sawteeth)

Computational 3D MHD is a numerical challenge

- Ideal MHD with continuously nested flux surfaces predicts the existence of singular current densities forming at rational surfaces in 3D equilibria.

- These are critical for:
  - 3D equilibrium (magnetic islands, confinement)
  - 3D macroscopic stability (kink modes, sawteeth)

- Computation of 3D ideal MHD equilibria is a numerical challenge.
  - Magnetic differential equations are densely singular (Newcomb 1959)
  - Non-smooth solutions are ubiquitous to 3D MHD (Grad 1967, Parker 1994)

The story I am about to tell you

① **Origin of singularities in 3D MHD with nested flux surfaces**

② **Questioning the very existence of 3D ideal MHD equilibria**

③ **Ideal MHD as a limit of Multiregion Relaxed MHD**

④ **Numerical computation of singular current densities**

⑤ **Practical application: plasma response to a boundary perturbation**

⑥ **A new class of 3D ideal MHD equilibria with nested surfaces**

Ideal MHD predicts singular current densities

$$\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$$

$$\nabla \cdot \mathbf{j} = 0\ ,\ \ \nabla \cdot \mathbf{B} = 0 \qquad \Longrightarrow \qquad \mathbf{B}\cdot\nabla u = -\nabla\cdot\mathbf{j}_\perp$$

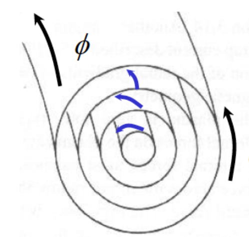
$$\mathbf{j} \times \mathbf{B} = \nabla p \qquad\qquad \mathbf{j}_\perp = (\mathbf{B}\times\nabla p)/B^2$$

magnetic differential equation

nested surfaces

Magnetic coordinates $\Longrightarrow$ $\sqrt{g}\ \mathbf{B}\cdot\nabla \equiv \iota\partial_\theta + \partial_\phi$

$$(\psi, \theta, \phi)$$



Fourier decomposition $\Longrightarrow$ $(\iota m - n)u_{mn} = i(\sqrt{g}\ \nabla\cdot\mathbf{j}_\perp)_{mn}$

$$u = \sum_{m,n} u_{mn} e^{i(m\theta - n\phi)}$$

Equation type $\Longrightarrow$ $\boxed{u_{mn}(x) = h(x)/x + \hat{j}_{mn}\delta(x)}$

$$xf(x) = h(x)$$

$$x \equiv \iota m - n,\ \ h(x) \sim p'$$

Pfirsch-Schlüter current **Dirac δ-current**

Existence of 3D ideal MHD equilibria?

- $\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$ is not the current, but the current density [A/m^2] .

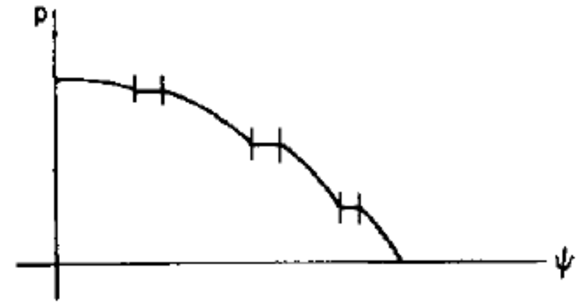
- Singularities are allowed as long as the current $J = \int_\Sigma \mathbf{j} \cdot \mathbf{d}\sigma$ across any surface is finite (weak formulation of the problem).

- Problem: Pfirsch-Schlüter current diverges across certain surfaces.

- Conclusion: pressure gradients cannot be supported at resonant rationals and thus pressure appears pathological.

> The function p is continuous but its derivative is pathological. We have obtained an equilibrium solution without infinite currents, but at the price of a very pathological pressure distribution.

[H. Grad, 1967]



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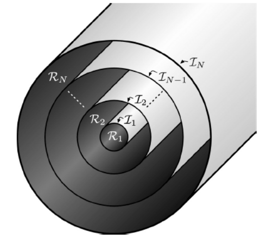
[...] More precisely, our theorems insure the existence of sharp boundary solutions for tori whose departure from axisymmetry is sufficiently small; they allow for solutions to be constructed with an arbitrary number of pressure jumps.

[Bruno and Laurence, 1996]

Multiregion Relaxed MHD



N –> ∞

[Dennis, 2013]

Taylor's theory	MRxMHD	Ideal MHD
Fewer constraints		More constraints
Helicity is conserved globally	**Helicity is conserved discretely**	**Helicity is conserved locally**
$F = W + \frac{\mu}{2}\Big(\underbrace{\int_V \mathbf{A}\cdot\mathbf{B}\, dV}_{H} - H_0\Big)$	$F = \sum_{l=1}^{N}\Big[W_l + \frac{\mu_l}{2}\Big(H_l - H_{l0}\Big)\Big]$	$W = \int_V \Big(\frac{p}{\gamma - 1} + \frac{B^2}{2}\Big) dV$
Topology: $\mathbf{B}\cdot\mathbf{n}\big\|_{\partial V} = 0$	Topology: $\mathbf{B}\cdot\mathbf{n}\big\|_{\partial V_l} = 0$	Topology: $\mathbf{B}\cdot\nabla\psi = 0$
Given $p, \Delta\psi, H_0$	Given $p_l, \Delta\psi_l, \Delta\psi_{p,l}, H_{l0}$	Given $p(\psi), \psi_p(\psi)$
$\delta F = 0 \Longrightarrow \boxed{\nabla\times\mathbf{B} = \mu\mathbf{B}}$	$\delta F = 0 \Longrightarrow \boxed{\begin{array}{c}\nabla\times\mathbf{B} = \mu_l\mathbf{B}\\ [[p + B^2/2]] = 0\end{array}}$	$\delta W = 0 \Longrightarrow \boxed{\mathbf{j}\times\mathbf{B} = \nabla p}$
[Taylor, 1974]	[Dewar, Hole, Hudson, 2006]	[Kruskal, Kulsrud, 1958]

Stepped-**P**ressure **E**quilibrium **C**ode (SPEC)

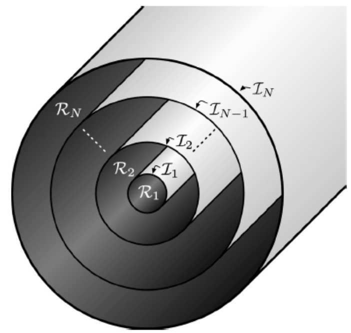
An implementation of MRxMHD

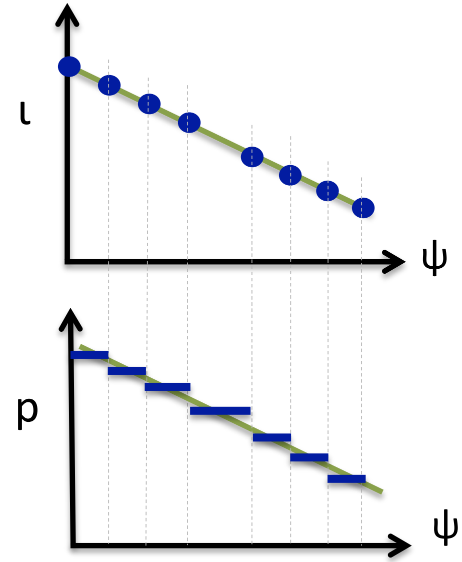
$$\mathcal{R}_l : \quad \nabla \times \mathbf{B} = \mu_l \mathbf{B}$$

$$\mathcal{I}_l : \quad [[p + B^2/2]] = 0$$

Given $p_l, \Delta\psi_l, \iota_l^+, \iota_l^-$

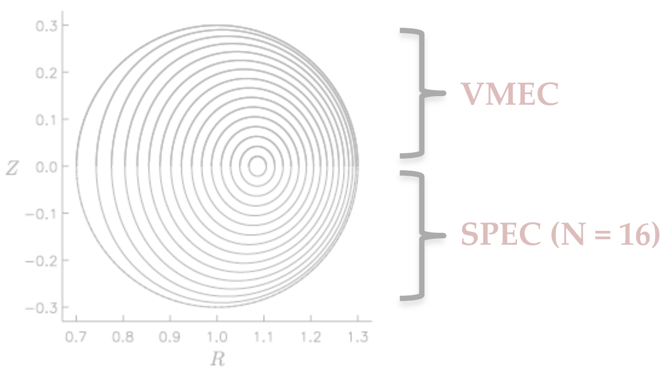
$l = 1, 2, ...N$





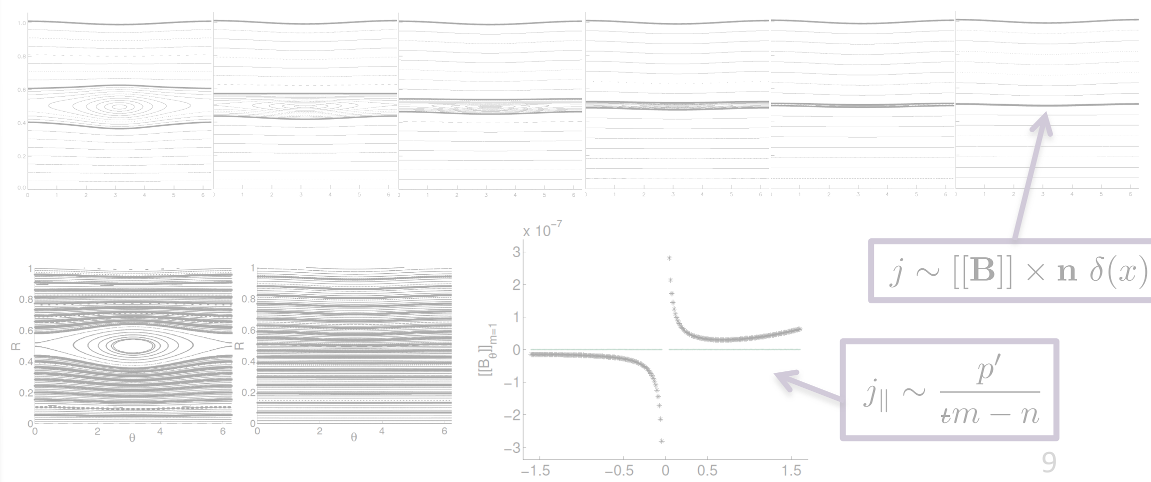
AXISYMMETRIC IDEAL MHD

[Hudson et al, PoP, 2012]



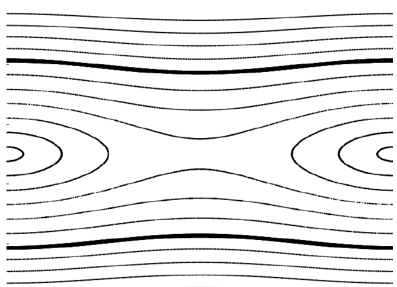
NON-AXISYMMETRIC IDEAL MHD

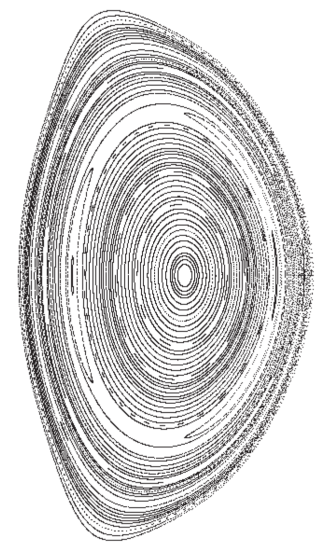
[Loizu et al, PoP, 2015]



Philosophy: build up understanding by steps of increasing complexity

- There are two superimposed singularities in the parallel current
  - Start with constant-pressure plasma to isolate the δ-current
- Geometry introduces most of the complexity
  - Start with slab geometry





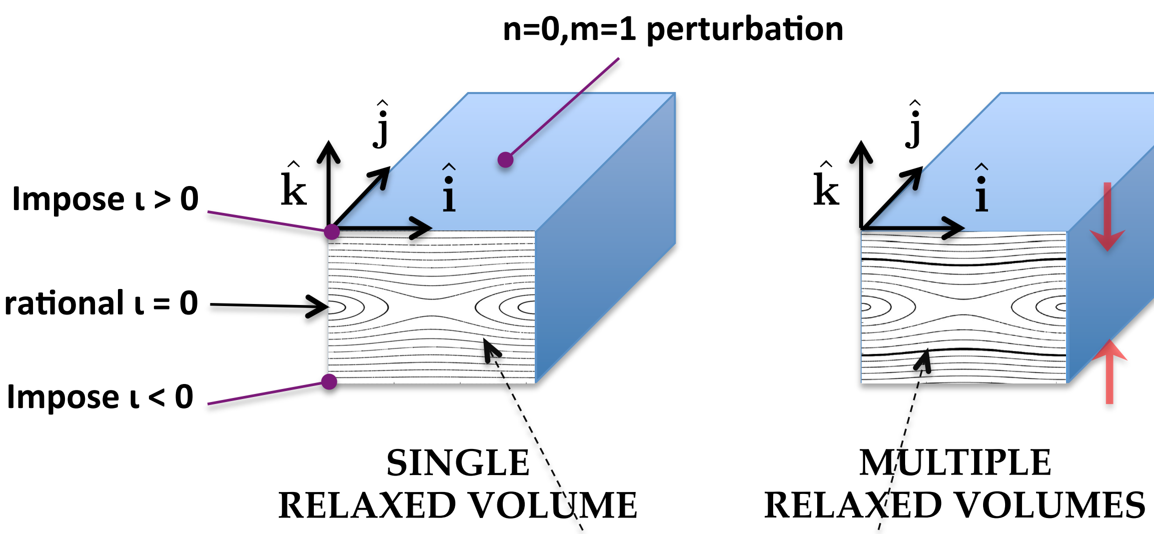
MRxMHD

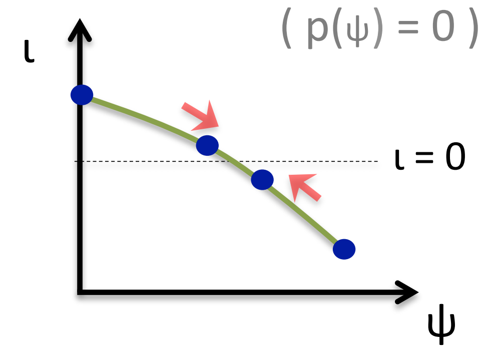
Ideal MHD

Hahm-Kulsrud-Taylor available solution (1985)

Rosenbluth-Dagazian-Rutherford available solution (1973)

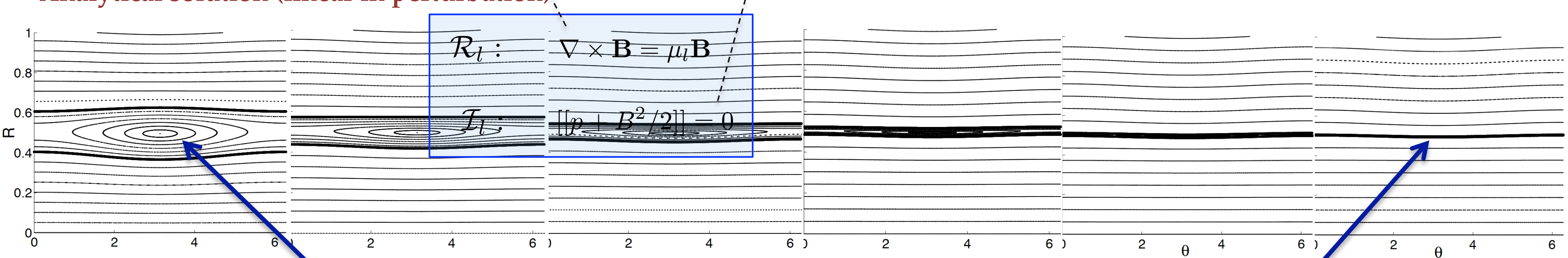
Island shielding produces a δ(x)-current





IDEA: shield the island by reducing flux and transform in the resonant volume

Analytical solution (linear in perturbation)



$\iota^{\pm} = \pm X^{\alpha}$

$\Delta\Psi = X^{\beta}$

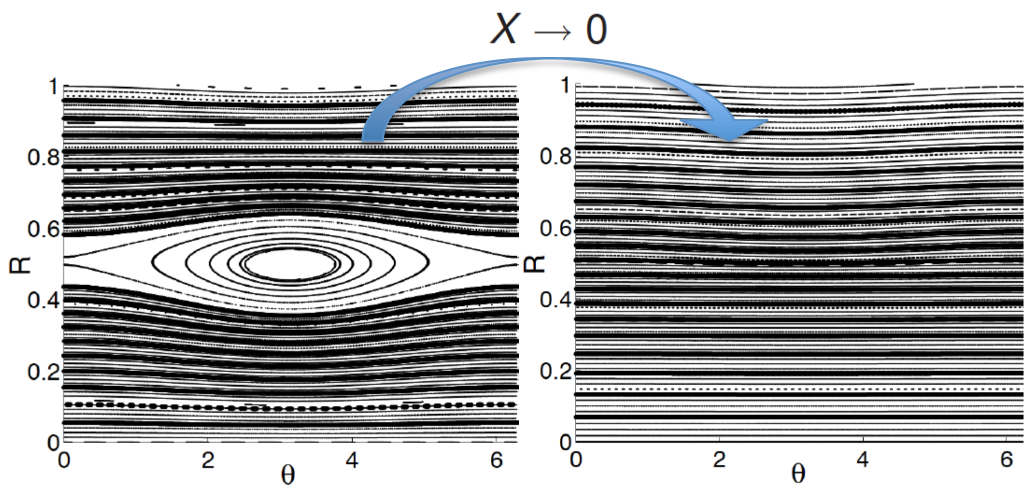
$X \to 0$

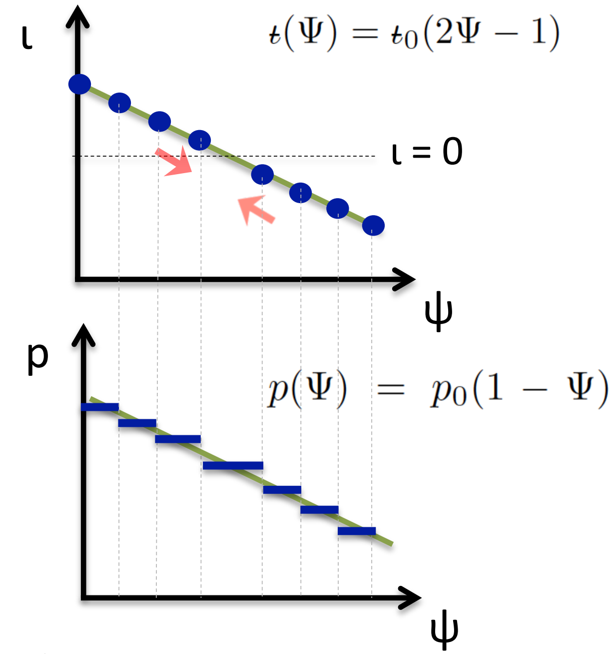
$\beta > \alpha > 0$

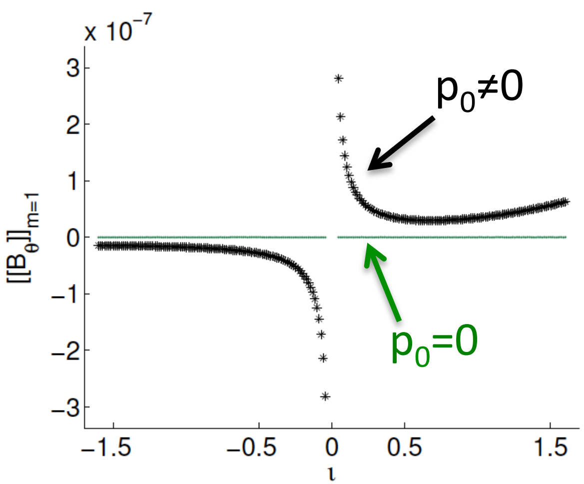
$[[B_\theta]]_{n=0,m=1} \neq 0$

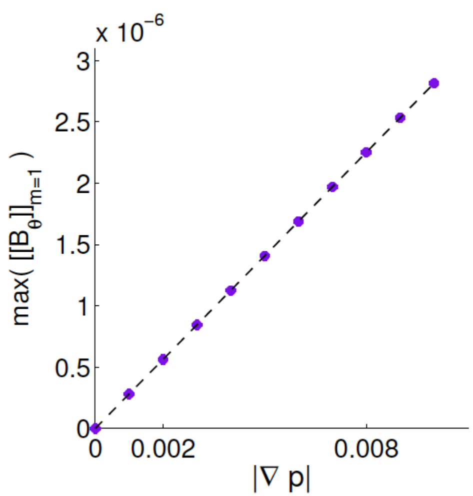
retrieve HKT solution

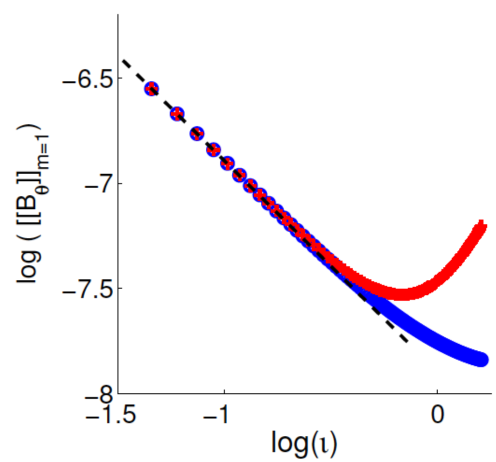
With pressure, island shielding produces a 1/x current



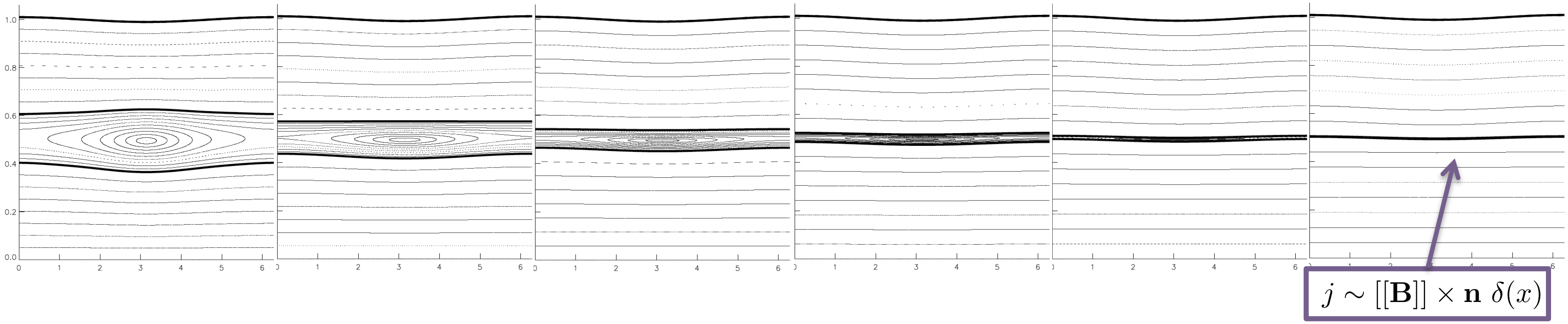


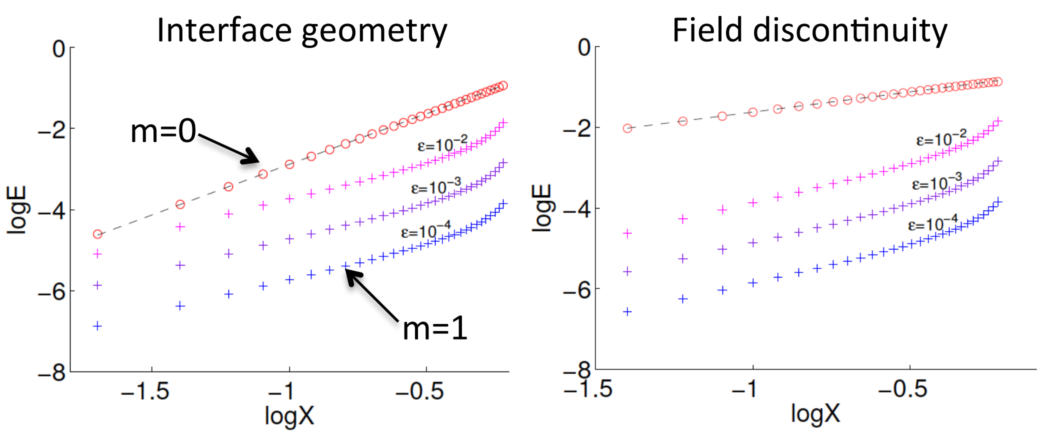


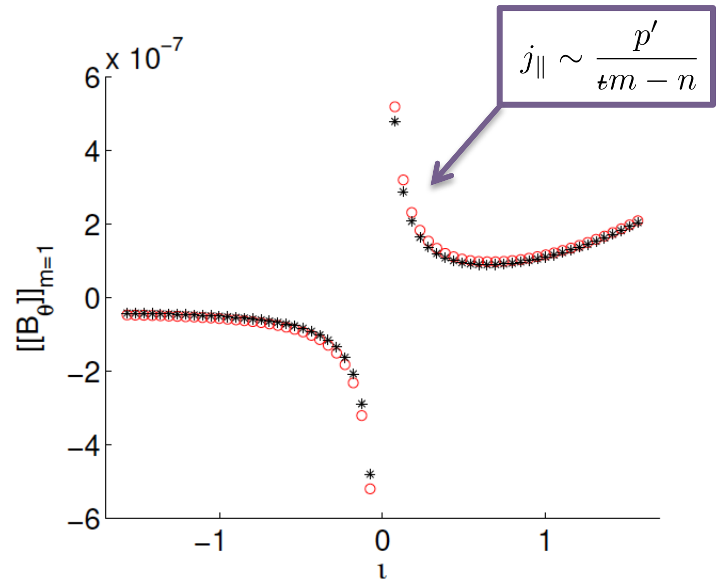




SPEC reproduces the analytical results





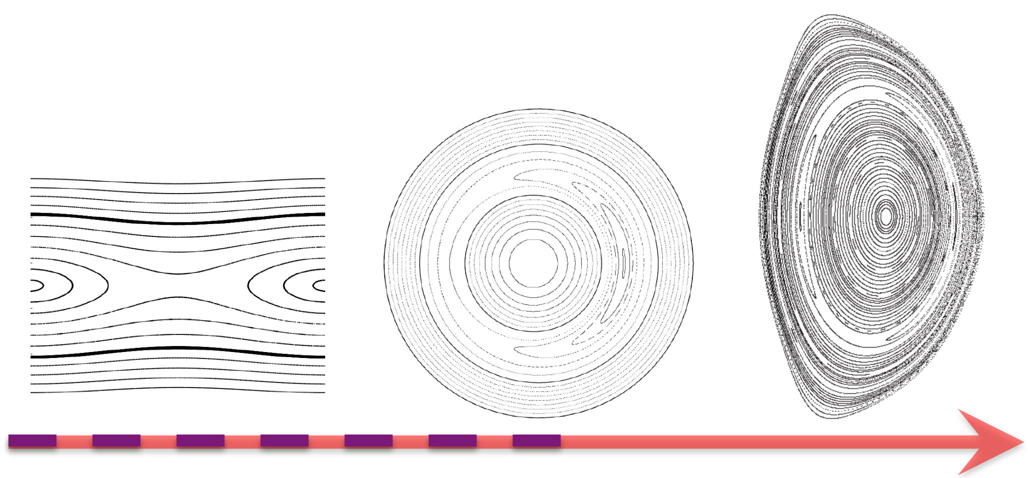


The agreement between SPEC and the linear analytical prediction improves with decreasing perturbation

We have a guideline for more complex geometries

- Ideal MHD predicts two types of singular currents at rational surfaces in 3D
  - We have provided **the first numerical proof** of their mutual existence
  - We have developed an **analytical linear slab model** that
    (1) describes the formation of islands around resonant rational surfaces
    (2) retrieves the ideal MHD limit in which magnetic islands are shielded
    (3) computes the subsequent formation of δ-currents and 1/x-currents
  - Results provide a **guideline** for the computation of 3D ideal MHD equilibria

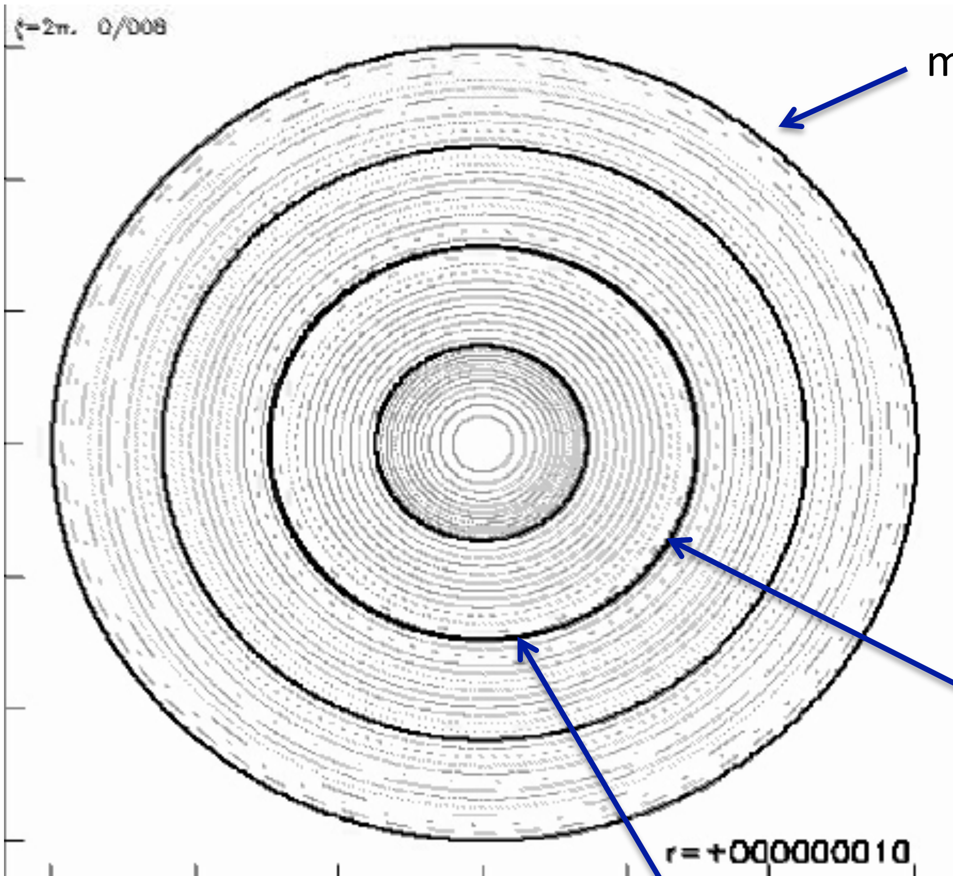
More details in [Loizu et al, PoP 2015]



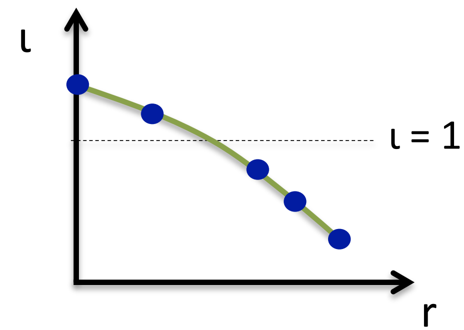
Application to cylindrical geometry

m=1, n=1 perturbation: $\xi_{11}/a = 10^{-2}$

Z



R



$\iota^{\pm} = 1 \pm X^{\alpha}$

$\Delta\Psi = X^{\beta}$

$X \rightarrow 0$

$\beta > \alpha > 0$

$[[B_{\theta}]]_{n=1,m=1} \neq 0$

Shielding obtained directly with discontinuous ι

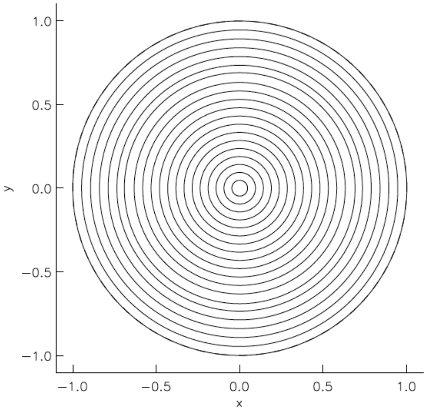
- It is not very practical to create a sequence to shield an island.

- Notice that the sequence { $\iota^{\pm} = 1 \pm X^{\alpha}$, $\Delta\Psi = X^{\beta}$ } implies "infinte shear":

$$\frac{\Delta\iota}{\Delta\Psi} \sim X^{\alpha-\beta} \rightarrow \infty$$

- Suggests direct method: place an ideal interface with $\iota = 1 \pm \Delta\iota/2$.

- SPEC produces perfectly converged equilibria !



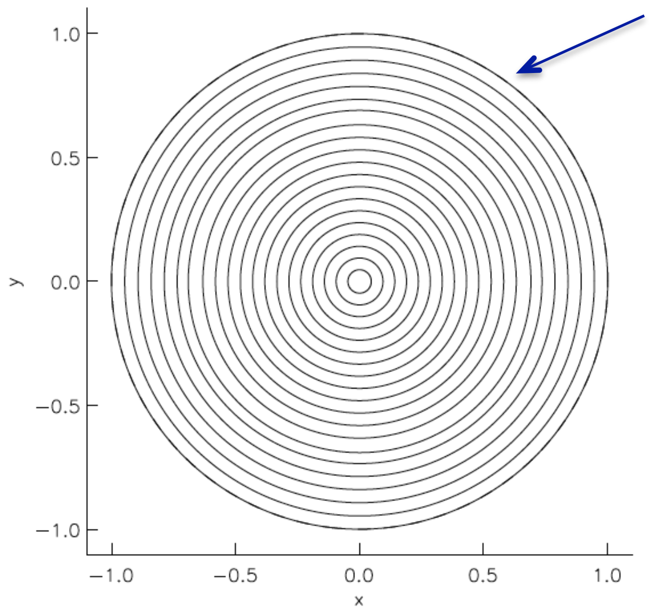
- **Questions to answer:**

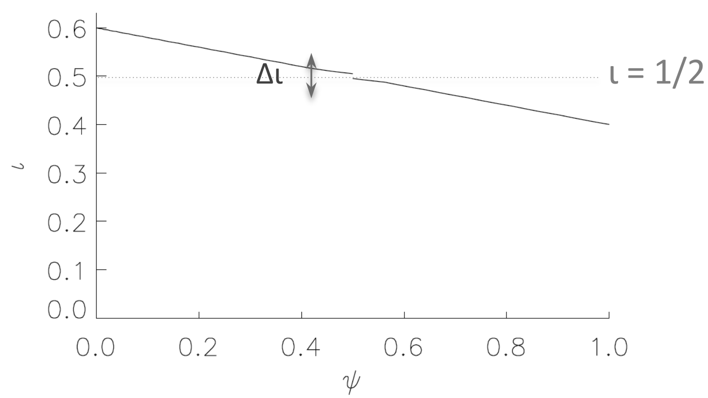
  (1) physical meaning of Δι ?

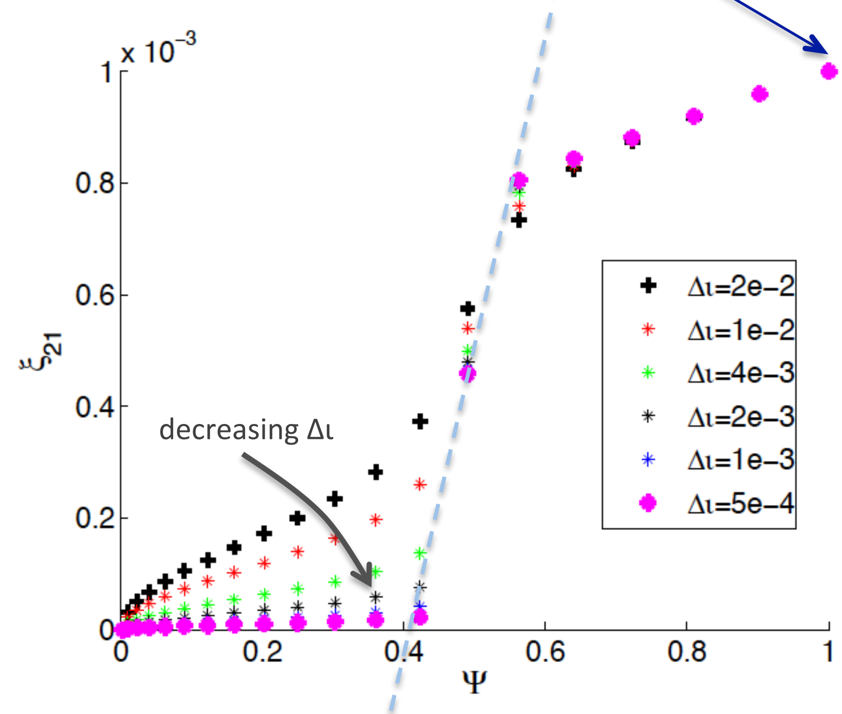
  (2) how to choose Δι ?

Plasma ideal response to a boundary perturbation

m=2, n=1 perturbation: $\xi_{21}/a = 10^{-3}$







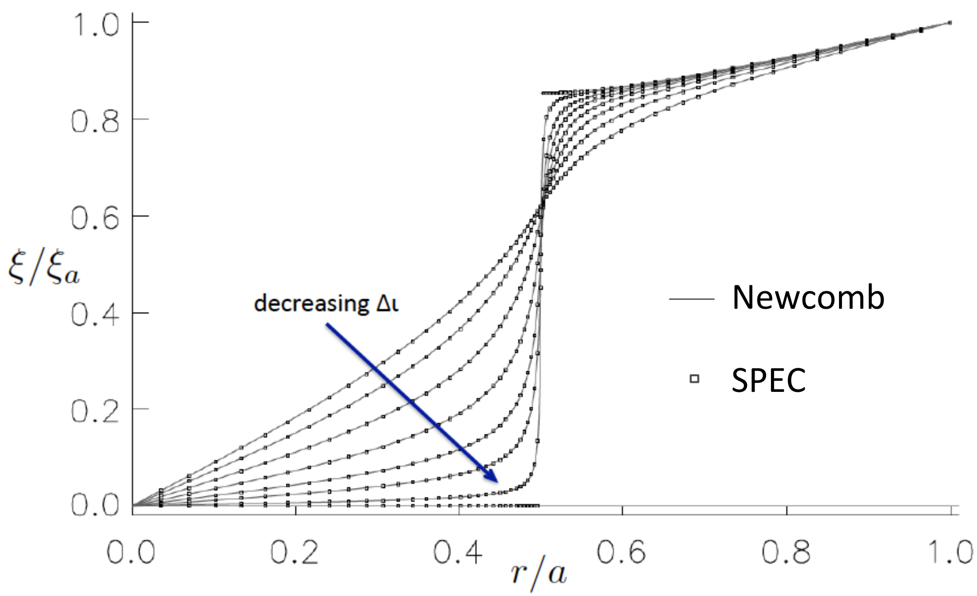
Necessary condition $\left|\dfrac{d\xi}{dr}\right| < 1$ for nested surfaces

Revisiting solutions to Newcomb equation

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0$$

$$f = B_z^2(\iota - \iota_s)^2 \frac{r^3}{R^2 + r^2\iota_s^2}$$

$$g = B_z^2(\iota - \iota_s)^2 r \frac{k^2r^2 + m^2 - 1}{R^2 + r^2\iota_s^2} + B_z^2(\iota_s^2 - \iota^2)2\iota_s^2 \frac{r^3}{(R^2 + r^2\iota_s^2)^2}$$



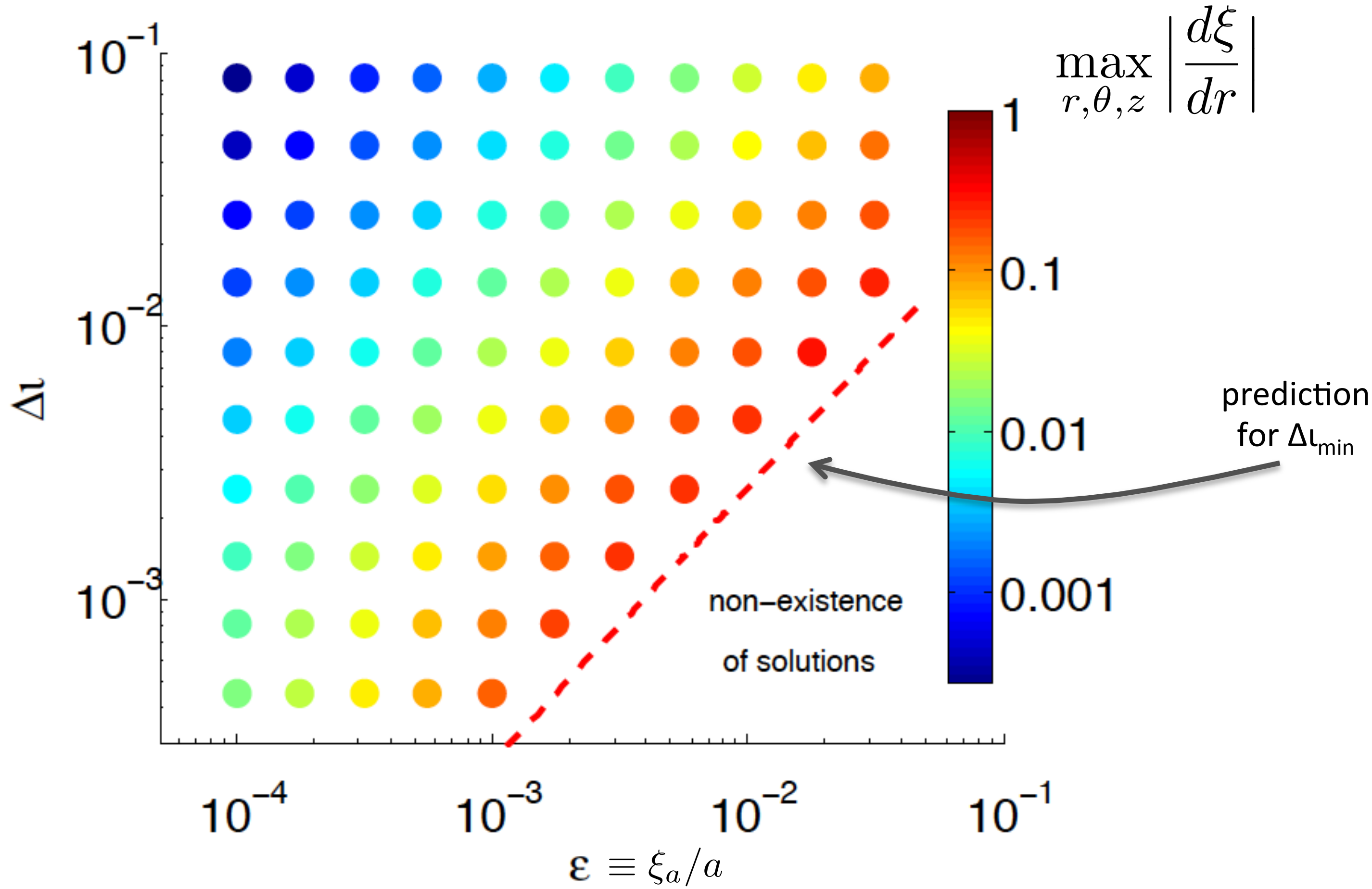
Analytical prediction

$$|\xi_s'| = 2\iota_s' \frac{\xi_s}{\Delta\iota}$$

Minimum “DC” current sheet

$$\Delta\iota_{min} \propto \xi_a/a \equiv \epsilon$$

Existence space of nonlinear equilibria



CURRENT SHEET

BOUNDARY PERTURBATION

A new class of 3D MHD equilibria with nested surfaces

Conjecture

3D MHD equilibria with nested surfaces
and discontinuous transform across rationals are well defined.

Corollaries

- DC current sheets can manifest at resonant surfaces. *
- Perturbation theory is now well-posed (allowing linear/nonlinear benchmark).
- Technically speaking, there are no resonant rational surfaces.
- The possibility of continuous and smooth pressure profiles is rescued.

* present in [Rosenbluth, Dagazian and Rutherford 1973] & [Boozer and Pomphrey 2010]