Fractal Pressure Profiles and Equilibria in Cylindrically-Symmetric Ideal MHD

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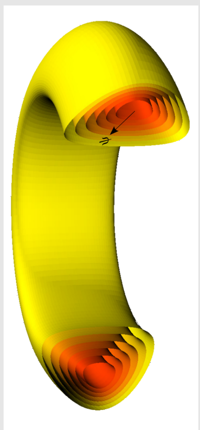
Goals

1. **Ideal MHD** equilibria with nested flux surfaces require fractal pressure
   - Generally perturbed toroidal equilibria suffer from unphysical infinite currents
   - Currents vanish if the pressure is flat on all resonances
2. **Mathematics** for entertaining fractal profiles
   - Non-integrable fields $\implies$ KAM, Diophantine condition
   - Dense sets, nowhere dense sets, and Lebesgue measure
3. Approximate fractality **numerically**
   - Implement a fractal grid
   - Quantify how robust each surface is
4. **Physics** of fractal pressure in a cylinder

1. Nested flux surfaces corrupted by unphysical infinite currents



Force balance: $\nabla p = \boldsymbol{J} \times \boldsymbol{B}$ (1)

$$\boldsymbol{J}_\perp = \frac{\boldsymbol{B} \times \nabla p}{B^2} \qquad \boldsymbol{J}_\parallel = \lambda \boldsymbol{B} \qquad (2)$$

From current conservation:

$$\nabla \cdot \boldsymbol{J}_\perp = -\nabla \cdot (\lambda \boldsymbol{B}) = -\boldsymbol{B} \cdot \nabla \lambda \qquad (3)$$

- Above is a **magnetic differential equation**:

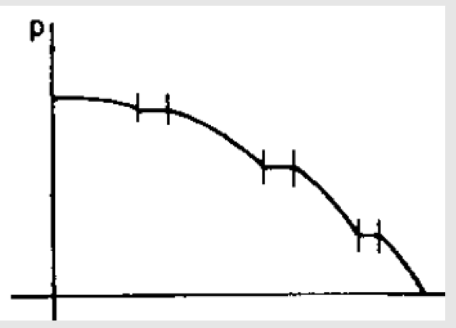
$$(\boldsymbol{B} \cdot \nabla)_{mn} = \partial_\theta \iota + \partial_\phi = \iota m - n \qquad (4)$$

- Decompose $\lambda = \sum_{m,n} \lambda_{mn} \cos(m\theta - n\phi)$

$$\lambda_{mn} = \underbrace{\Delta_{mn} \delta(m\iota - n)}_{\text{1. Delta spike}} - \underbrace{\frac{(\nabla \cdot \boldsymbol{J}_\perp)_{mn}}{m\iota - n}}_{\text{2. } 1/x \text{ singularity}} \qquad (5)$$

1. Finite current through infinitesimal wire: $J = I/a \to J = I\delta(x)$ ☺
2. $\int_0^\epsilon \frac{1}{x} dx$ is logarithmically divergent $\to$ infinite current ☹

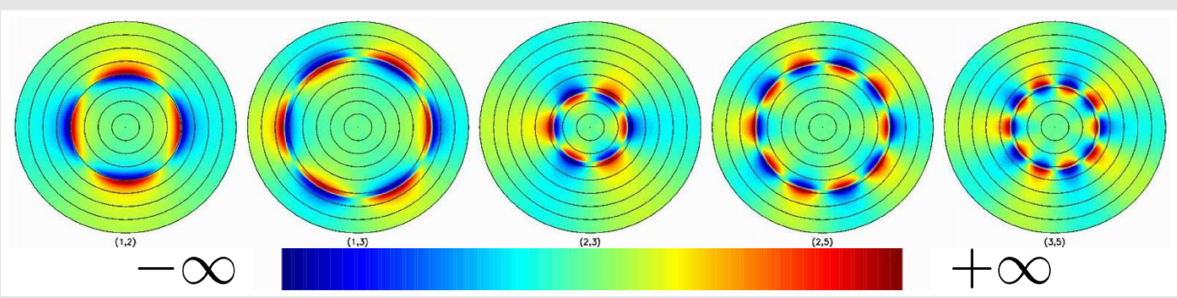
$$\iota = \frac{n}{m}, \quad \nabla p \neq 0 \implies \boldsymbol{I}_\parallel \to \infty \qquad (6)$$



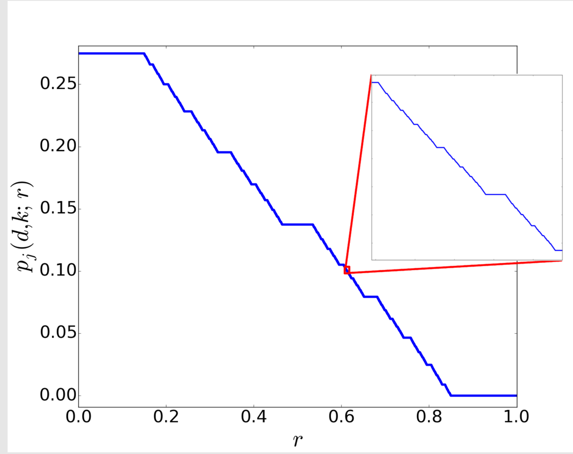
$$\boldsymbol{I}_\parallel \propto \left(\nabla \cdot \frac{(\boldsymbol{B} \times \nabla p)}{B^2} \right)_{mn} \to \pm\infty$$

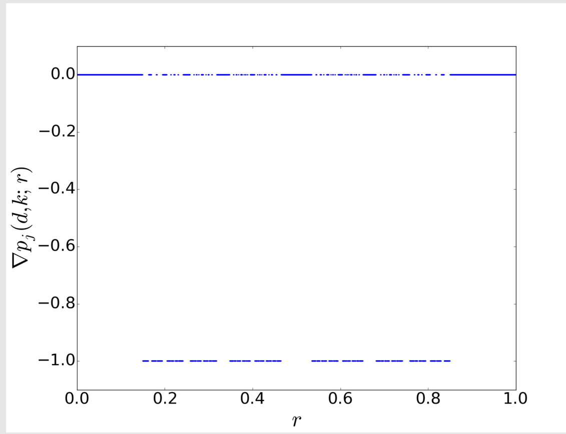
Unless $\nabla p = 0$ when $\iota = \dfrac{n}{m}$

Grad's notion, 1967. [1]



Defining a fractal pressure and understanding its physics





To avoid unphysical currents at resonances:

$\nabla p = 0$ on *excluded rational intervals*,

So prescribe

$$p'(\iota) = \frac{dp}{d\iota} = \begin{cases} 0 & \left|\iota - \frac{n}{m}\right| < \frac{d}{m^k}, \ \forall \frac{n}{m} \in \mathcal{F}_j; \\ -1 & \text{otherwise,} \end{cases}$$

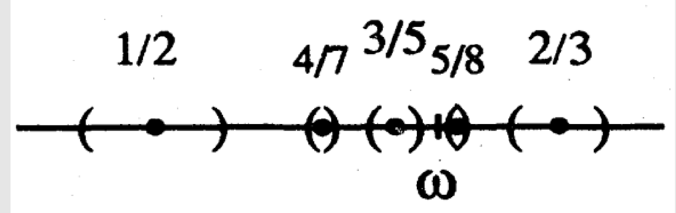
Physics of plasma with fractal p

- $\beta = \int_0^1 dr\ p(r) \implies$ Lebesgue integration
- Prove $p(r) \neq 0, \forall r$?
- Where is $\nabla p = 0$ distributed in ι?
- Staircase $p(r) \implies$ what fields $\boldsymbol{B}, \boldsymbol{J}$?

Numerically: Approximate on a discrete grid?

Flatten by Diophantine condition

$$\left|\omega - \frac{n}{m}\right| > \frac{d}{m^k}, \forall n, m \in \mathbb{N}. \qquad \text{(D)}$$



2. Number and measure theory have tools for finding β

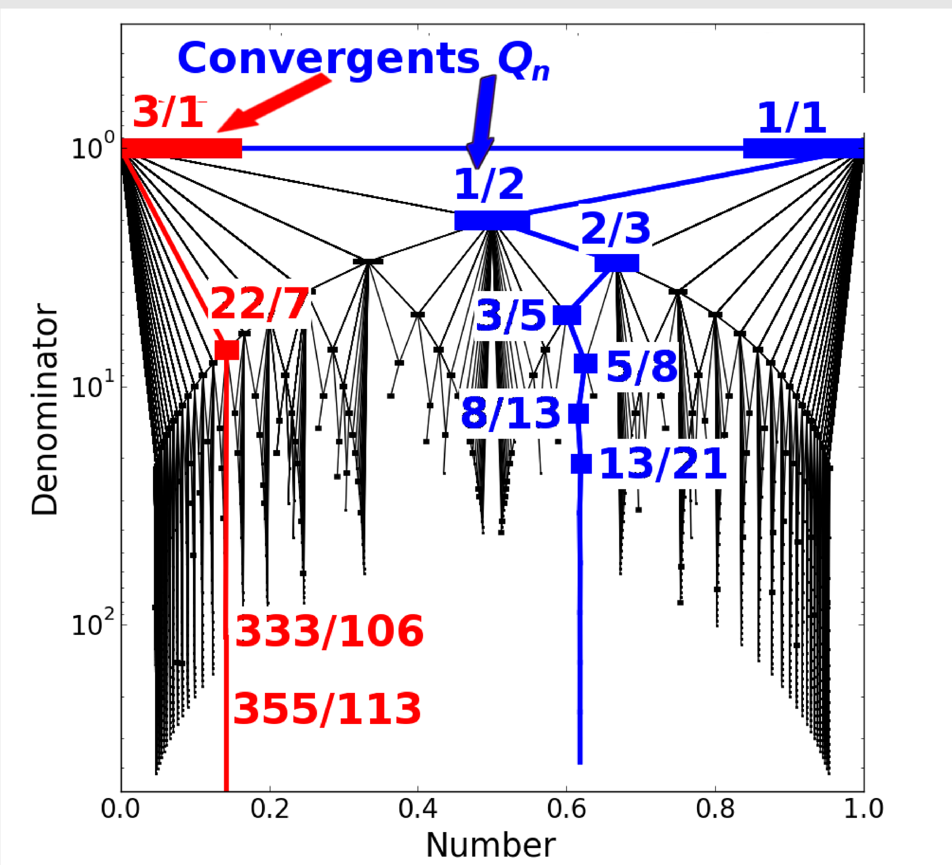
$p'(\iota) \neq 0$ is nowhere dense, yet has finite measure μ

Continued fractions and convergents

$$[a_0; a_1, a_2, \cdots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

Truncate CF to get convergents Q_n:

Value	CF	Farey Tree
π	$[3; 7, 15, 1, 292, \cdots]$	RRRLLLLL...
φ	$[1; \bar{1}]$	RLRLRLRL...



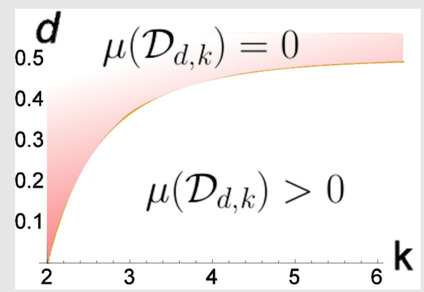
Larger $a_i \to$ closer to rationals

Which irrational numbers are most irrational?

Diophantine set contains $\nabla p \neq 0$ surfaces

$$\mathcal{D}_{d,k} = \left\{ \alpha : \text{(D)}, \quad \forall n, m \in \mathbb{N} \right\}$$

$$\mathcal{D} = \left\{ \alpha : \alpha \in \mathcal{D}_{d,k}, d > 0, k \geq 2 \right\}$$



Brjuno numbers: growth of convergent denominators

$$\mathcal{B} = \left\{ \alpha : \sum_{n=0}^{\infty} \frac{\ln Q_{n+1}}{Q_n} < \infty \right\}$$

$$\mathcal{B}_j = \left\{ \alpha : \sum_{n=0}^{\infty} \frac{\ln Q_{n+j}}{Q_n} < \infty \right\}$$

$$\mathcal{B}_{j+1} \subsetneq \mathcal{B}_j \subsetneq \cdots \subsetneq \mathcal{B}_2 \subsetneq \mathcal{B}_1 = \mathcal{B}$$

Proven [2]:

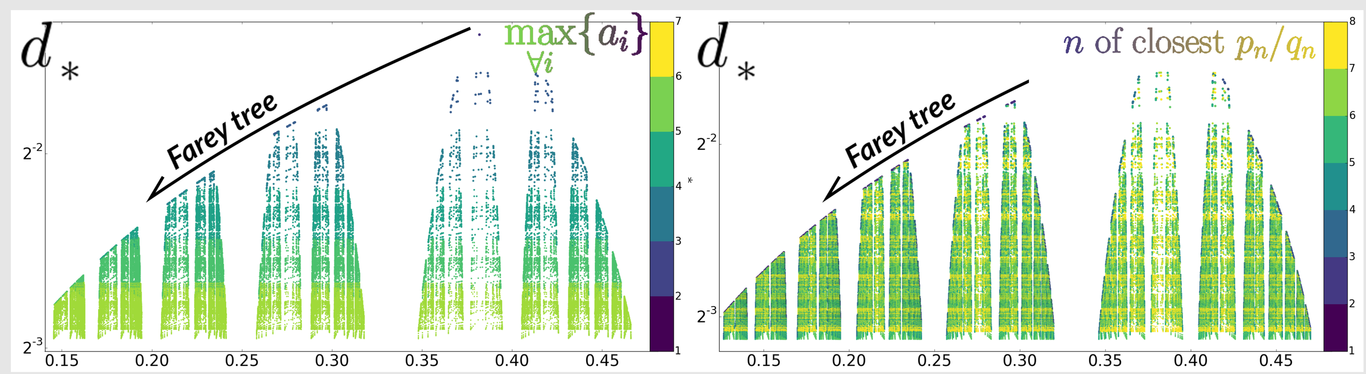
$\mathcal{D} \subset \mathcal{B}_\infty$

Assertion:

$\mathcal{D} = \mathcal{B}_\infty$

Particular $d = d_*$ exactly overlaps irrationals

$$d_* = \min_{n,m} \left| \alpha m^k - n m^{k-1} \right| \qquad \text{More } \textbf{robust} \equiv \text{higher } d_*$$



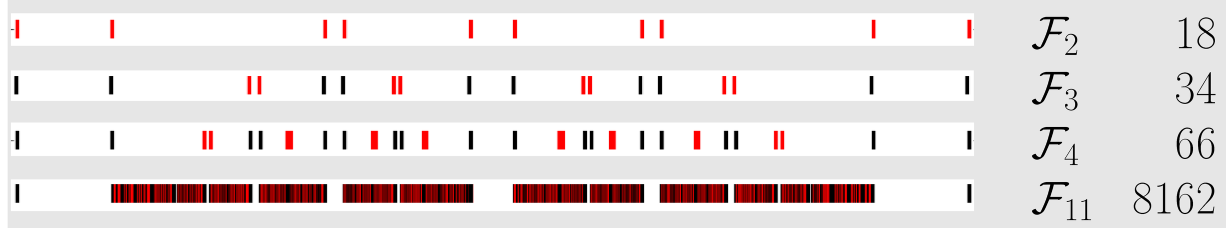
3. Incorporating fine-scale structure requires a fractal grid

- Most numerical methods smooth over fractality

Fractal objects exhibit **self-similarity** at all scales.

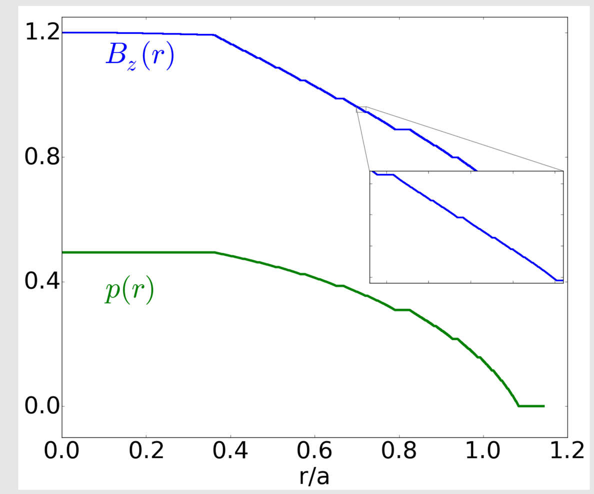
- Solution: Build fine structure into grid, one Farey level at a time

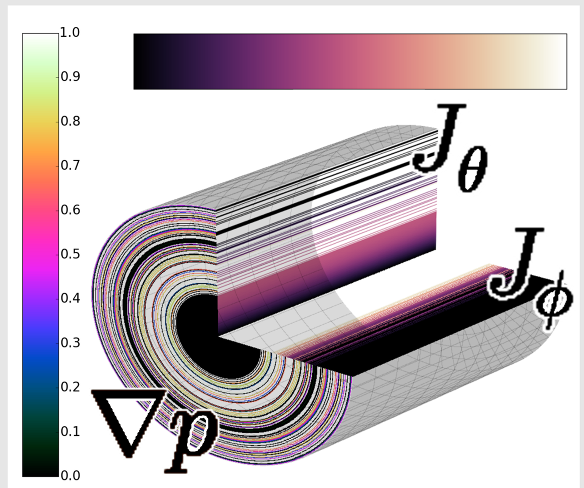
$$x_0 = [0, 1] \to x_1 = \left[0, \quad \frac{d}{1^k}, \quad 1 - \frac{d}{1^k}, \quad 1\right]$$

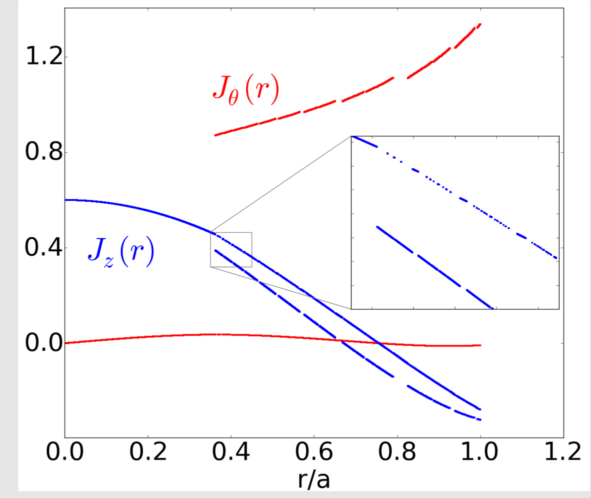


4. MHD equilibria computed in a cylinder

- $\boldsymbol{B}$ parallels $p(r)$
  - Continuous everywhere
  - Not smooth on irrationals
- $\boldsymbol{J}$ parallels $p'(r)$
  - Discontinuities on nowhere-dense subset
  - **Always finite!**







Conclusions and upcoming work

- Grad's "pathological" solution is a valid equilibrium state
- Number, measure theory establish existence on a nowhere-dense subset
- Farey grid spacing converges rigorously to exact solution
- Fractal pressure is compatible with non-smooth $\boldsymbol{B}(\boldsymbol{r})$ and discontinuous (but finite!) $\boldsymbol{J}(\boldsymbol{r})$

Questions:

- What sets of (d, k) are typical for plasma discharges?
- How are the most robust irrationals related?

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References

[1] H. Grad. Toroidal containment of a plasma. *Phys. Fluids*, 10 (1):137, 1967.
[2] E. F. Lee. *The Structure and Geometry of the Brjuno Numbers*. PhD thesis, Boston University, 1998.