Penetration and amplification of resonant perturbations in 3D ideal-MHD equilibria

S. R. Hudson[1], J. Loizu[2,1], S. Lazerson[1], A. Bhattacharjee[1] & P. Helander[2]

1) Princeton Plasma Physics Laboratory, Princeton NJ, 08543, USA
2) Max-Planck-Institut fűr Plasmaphysik, 17491, Greifswald, Germany

2016 IAEA Fusion Energy Conference, Kyoto, Hudson.S-TH-C-1.1

1. Macroscopic force-balance, $\nabla p = \mathbf{j} \times \mathbf{B}$, is fundamental for magnetic confinement.
   defines the equilibrium state, linear stability, neoclassical transport, experimental design & reconstruction

2. Conventional solutions in arbitrary (three-dimensional) geometry with rational rotational-transform flux-surfaces are non-analytic and non-physical.

3. We compute physically acceptable, mathematically self-consistent solutions to $\nabla p = \mathbf{j} \times \mathbf{B}$ with nested flux-surfaces in 3D geometry.

4. Our new class of solutions have sheet-currents and discontinuous rotational-transform.

5. In our solutions, RMPs penetrate into the core, and are amplified by pressure.

Question : Why is $\nabla p = \mathbf{j} \times \mathbf{B}$ important?

Answer : Simplest, fastest and most-widely used approximation to the equilibrium state

1. Consider e.g. *dynamical,* resistive, single-fluid equations with flow.

$$
\begin{aligned}
\frac{\partial p}{\partial t} &= (\gamma - 1)\left(Q + \nabla \cdot (\kappa_{\parallel} \nabla_{\parallel} p + \kappa_{\perp} \nabla_{\perp} p) + \eta\, j^2 - \mathbf{\Pi} : \nabla \mathbf{v}\right) - (\mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v}) \\
\frac{\partial n}{\partial t} &= -\nabla \cdot (n\mathbf{v}) \\
\frac{\partial \mathbf{v}}{\partial t} &= \frac{1}{n m_i}\left(\mathbf{j} \times \mathbf{B} - \nabla p - \nabla \cdot \mathbf{\Pi}\right) - \mathbf{v} \cdot \nabla \mathbf{v} \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\eta \nabla \times \mathbf{B} - \mathbf{v} \times \mathbf{B})
\end{aligned}
$$

2. Numerically too complicated, too slow for stellarator design optimization.

Question : Why is $\nabla p = \mathbf{j} \times \mathbf{B}$ important?
Answer : Simplest, fastest and most-widely used approximation to the equilibrium state

1. Consider resistive, single-fluid equations with flow.

$$0 = (\gamma - 1)\left(Q + \nabla \cdot (\kappa_{\|}\nabla_{\|}p + \kappa_{\perp}\nabla_{\perp}p) + \eta j^2 - \mathbf{\Pi} : \nabla \mathbf{v}\right) - (\mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v})$$

$$0 = -\nabla \cdot (n\mathbf{v})$$

$$0 = \frac{1}{n m_i}(\mathbf{j} \times \mathbf{B} - \nabla p - \nabla \cdot \mathbf{\Pi}) - \mathbf{v} \cdot \nabla \mathbf{v}$$

$$0 = \nabla \times (\eta \nabla \times \mathbf{B} - \mathbf{v} \times \mathbf{B})$$

2. Numerically too complicated, too slow for stellarator design optimization.

    (i) let $\partial_t = 0$,

Question : Why is $\nabla p = \mathbf{j} \times \mathbf{B}$ important?
Answer : Simplest, fastest and most-widely used approximation to the equilibrium state

1. Consider single-fluid equations with flow.

$$
\begin{aligned}
0 &= (\gamma - 1)\left(Q + \nabla \cdot (\kappa_{\|} \nabla_{\|} p \qquad\qquad) \qquad\qquad\qquad \right) - (\mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v}) \\
0 &= -\nabla \cdot (n\mathbf{v}) \\
0 &= \frac{1}{n m_i} (\mathbf{j} \times \mathbf{B} - \nabla p \qquad\qquad) - \mathbf{v} \cdot \nabla \mathbf{v} \\
0 &= \nabla \times (\qquad\qquad - \mathbf{v} \times \mathbf{B})
\end{aligned}
$$

2. Numerically too complicated, too slow for stellarator design optimization.

(i) let $\partial_t = 0$, (ii) ignore non-ideal terms, $\eta, \kappa_{\perp}, \mathbf{\Pi}$,

Question : Why is $\nabla p = \mathbf{j} \times \mathbf{B}$ important?
Answer : Simplest, fastest and most-widely used approximation to the equilibrium state

1. Consider single-fluid equations

$$0 = (\gamma - 1)\left(Q + \nabla \cdot (\kappa_{\|} \nabla_{\|} p \qquad)\qquad\qquad \right)$$

$$0 = \frac{1}{nm_i}\left(\mathbf{j} \times \mathbf{B} - \nabla p \qquad \right)$$

2. Numerically too complicated, too slow for stellarator design optimization.

(i) let $\partial_t = 0$, (ii) ignore non-ideal terms, $\eta, \kappa_\perp, \mathbf{\Pi}$, (iii) ignore velocity, $\color{blue}{\mathbf{v}}$,

Question : Why is $\nabla p = \mathbf{j} \times \mathbf{B}$ important?

Answer : Simplest, fastest and most-widely used approximation to the equilibrium state

1. Consider single-fluid equations

$p(\psi)$ is given (i.e. equilibrium model, not transport model)

$$0 = \frac{1}{nm_i} (\mathbf{j} \times \mathbf{B} - \nabla p \qquad)$$

2. Numerically too complicated, too slow for stellarator design optimization.

(i) let $\partial_t = 0$, (ii) ignore non-ideal terms, $\eta, \kappa_\perp, \mathbf{\Pi}$, (iii) ignore velocity, $\mathbf{v}$,

(iv) *prescribe* p, to obtain $\nabla p = \mathbf{j} \times \mathbf{B}$, . . . calculation is still too slow . . .

Question : Why is $\nabla p = \mathbf{j} \times \mathbf{B}$ important?
Answer : Simplest, fastest and most-widely used approximation to the equilibrium state

1. Consider single-fluid equations

$p(\psi)$ is given (i.e. equilibrium model, not transport model)

$$0 = \frac{1}{nm_i}\left(\mathbf{j} \times \mathbf{B} - \nabla p \qquad \right) \qquad \text{with } \mathbf{B} \cdot \nabla \psi = 0$$

2. Numerically too complicated, too slow for stellarator design optimization.

   (i) let $\partial_t = 0$, (ii) ignore non-ideal terms, $\eta, \kappa_\perp, \mathbf{\Pi}$, (iii) ignore velocity, $\mathbf{v}$,

   (iv) *prescribe* p, to obtain $\nabla p = \mathbf{j} \times \mathbf{B}$, . . . calculation is still too slow . . .

   (v) restrict attention to integrable magnetic fields.

3. The most widely used equilibrium code for stellarators, VMEC, and the IPEC code for perturbed tokamaks $\boxed{\text{solve } \nabla p = \mathbf{j} \times \mathbf{B} \text{ with nested flux surfaces}}$.

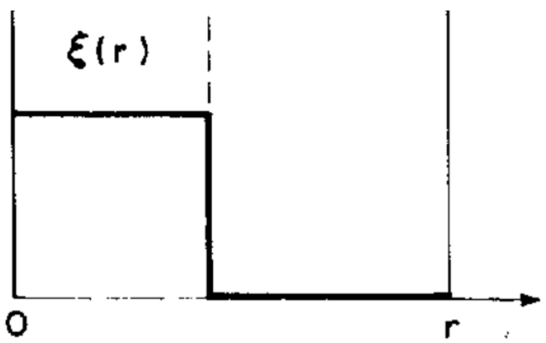
4. Foundation of equilibrium, linear stability, and neo-classical transport.

Problem: ideal-MHD equilibria with rational flux-surfaces have a non-analytic dependence on the 3D boundary.

1. **Breakdown of perturbation theory:**

   Following [Rosenbluth, Dagazian & Rutherford, Phys. Fluids **16**, 1894 (1973)]

   " .. the standard perturbation theory approach .. is not applicable here due to the singular nature of the lowest order step function solution for $\boldsymbol{\xi}$"



2. The mathematics:

   i. Equilibrium: $\nabla p - \mathbf{j} \times \mathbf{B} = 0$

   ii. Displacement: $\boldsymbol{\xi} = \epsilon\, \boldsymbol{\xi}_1 + \epsilon^2\, \boldsymbol{\xi}_2 + ..$

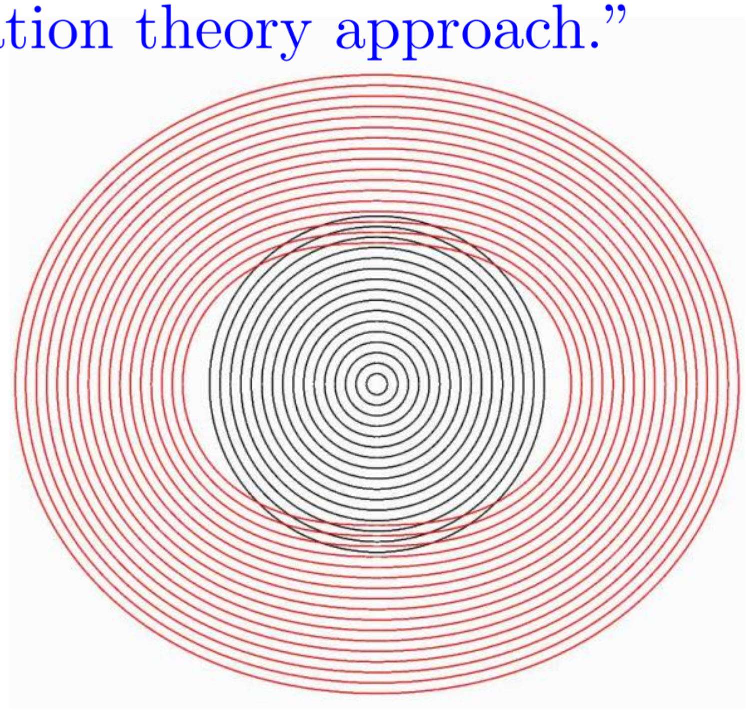
   iii. Ideal Variations: $\delta\mathbf{B} \equiv \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$, $\delta p \equiv (\gamma - 1)\boldsymbol{\xi} \cdot \nabla p - \gamma \nabla \cdot (p\boldsymbol{\xi})$

   iv. Linearized Equation: $\mathcal{L}_0[\boldsymbol{\xi}_1] = \nabla \delta p - \delta \mathbf{j} \times \mathbf{B} - \mathbf{j} \times \delta\mathbf{B} = 0$

3. "$\mathcal{L}_0$ is a singular operator .. we must abandon the perturbation theory approach."

   i. The linear displacement is discontinuous.

   $\Longrightarrow$ ii. The linearly perturbed flux surfaces overlap!



4. The singularity in $\mathcal{L}_0 \equiv \nabla_{\mathbf{x}} \mathbf{F}$, where $\mathbf{F} \equiv \nabla p - \mathbf{j} \times \mathbf{B}$, also affects:

   i. iterative solvers, e.g. $\mathbf{x}_{i+1} \equiv \mathbf{x}_i - \nabla_{\mathbf{x}} \mathbf{F}^{-1} \cdot \mathbf{F}[\mathbf{x}_i]$,

   ii. and descent algorithms, e.g. $\dfrac{\partial \mathbf{x}}{\partial \tau} = -\nabla_{\mathbf{x}} \mathbf{F}$.

Problem: solutions to force balance with nested surfaces have non-integrable singularities in the current-density.

1. $\nabla p = \mathbf{j} \times \mathbf{B}$ yields $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$. Note: $\mathbf{j}$ is current-*density*, current through $\mathcal{S}$ is $\int_{\mathcal{S}} \mathbf{j} \cdot ds$.

2. Write $\mathbf{j} = \sigma \mathbf{B} + \mathbf{j}_\perp$, then $\nabla \cdot \mathbf{j} = 0$ yields $\mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$.

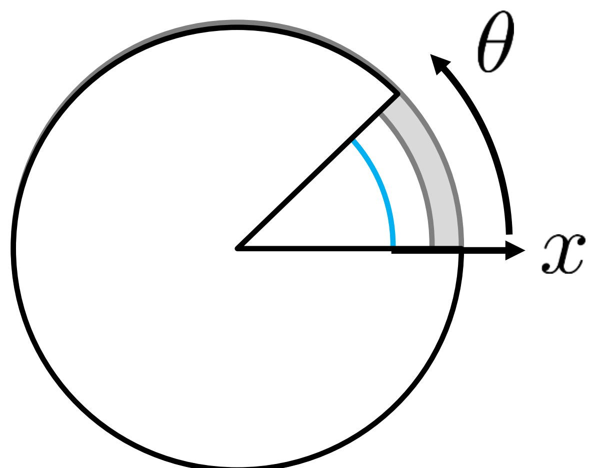
3. Nested flux surfaces allows $\mathbf{B} = \nabla\psi \times \nabla\theta + \iota\!\!\!-(\psi)\, \nabla\zeta \times \nabla\psi$, $\sigma \equiv \sum_{m,n} \sigma_{m,n}(\psi) \exp[i(m\theta - n\zeta)]$;

4. Resonant harmonic of the parallel current-density is

$$\boxed{\sigma_{m,n} = \underbrace{\frac{(\sqrt{g}\, \nabla \cdot \mathbf{j}_\perp)_{m,n}}{m\iota\!\!\!- - n}}_{\text{Pfirsch-Schlüter}} + \Delta_{m,n} \underbrace{\delta_{m,n}(x)}_{\delta\text{-function}}}\ , \text{ where } x \equiv \iota\!\!\!- - n/m.$$

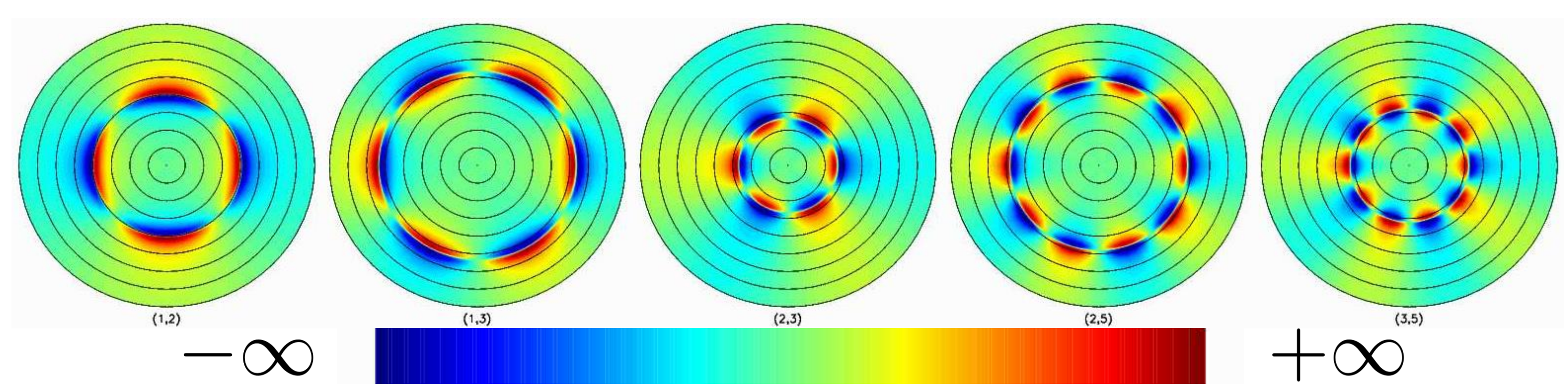
$$\left\{ \begin{array}{c} \textbf{thin wire} \\ \text{finite conductivity} \\ \mathbf{j} = \mathbf{I}/a \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{zero wire} \\ \textbf{infinite conductivity} \\ \mathbf{j} = \mathbf{I}\,\delta(\mathbf{x}) \end{array} \right\}$$

So-called "sheet" or "surface" currents are physically acceptable; total current is ***FINITE***.

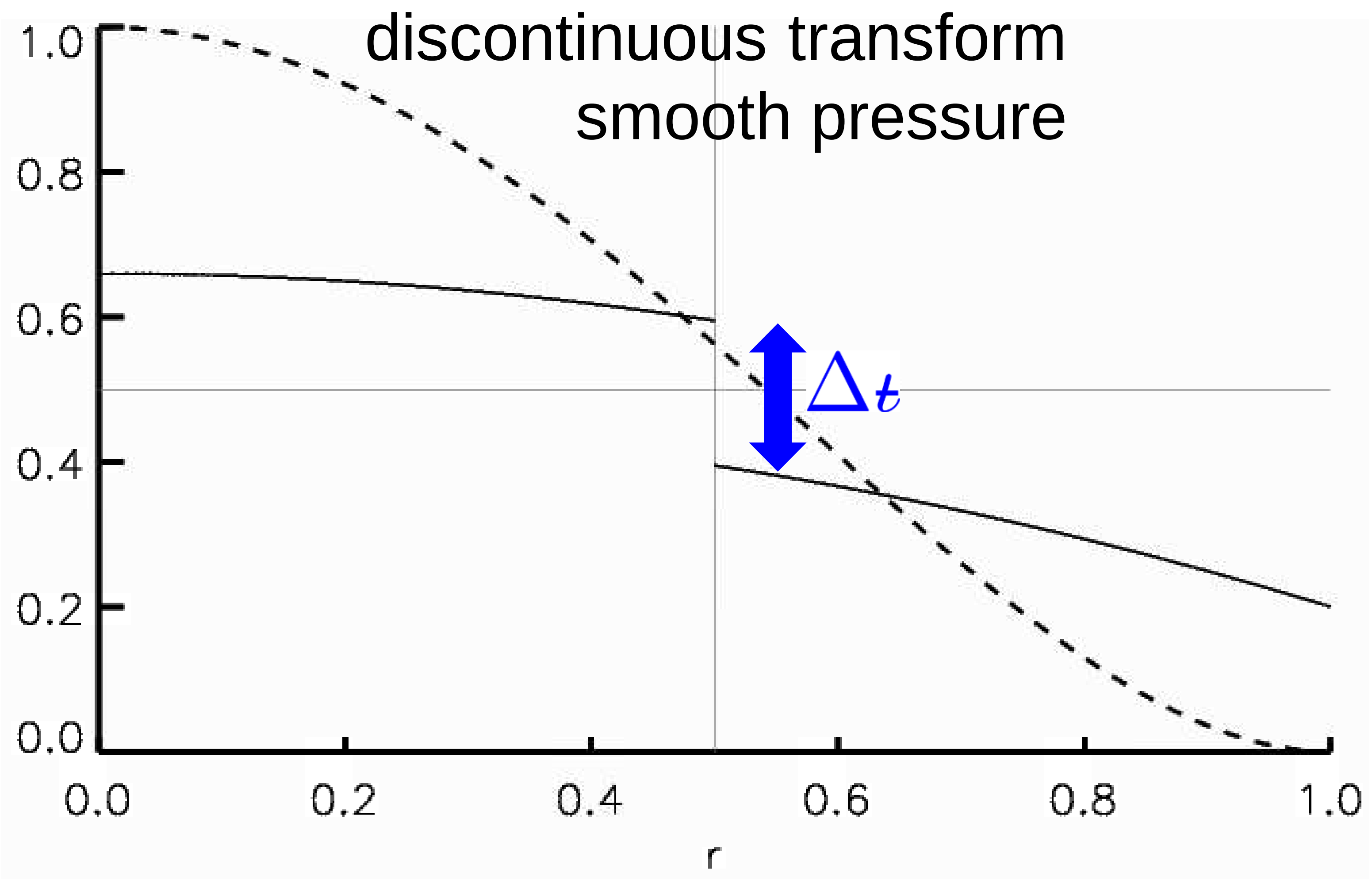


$$\begin{aligned} \int_{\mathcal{S}} \mathbf{j}_\parallel \cdot d\mathbf{s} &= g_{m,n}\, p' \, \frac{2}{m} \int_\epsilon^\delta dx \frac{1}{x} \\ &= g_{m,n}\, p' \, \frac{2}{m} (\ln\delta - \ln\epsilon) \\ &\rightarrow \infty \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

A dense collection of alternating, ***INFINITE*** currents is NOT physical.



Solution: introduce new class of solutions to $\nabla p = \mathbf{j} \times \mathbf{B}$, with sheet currents and discontinuous transform



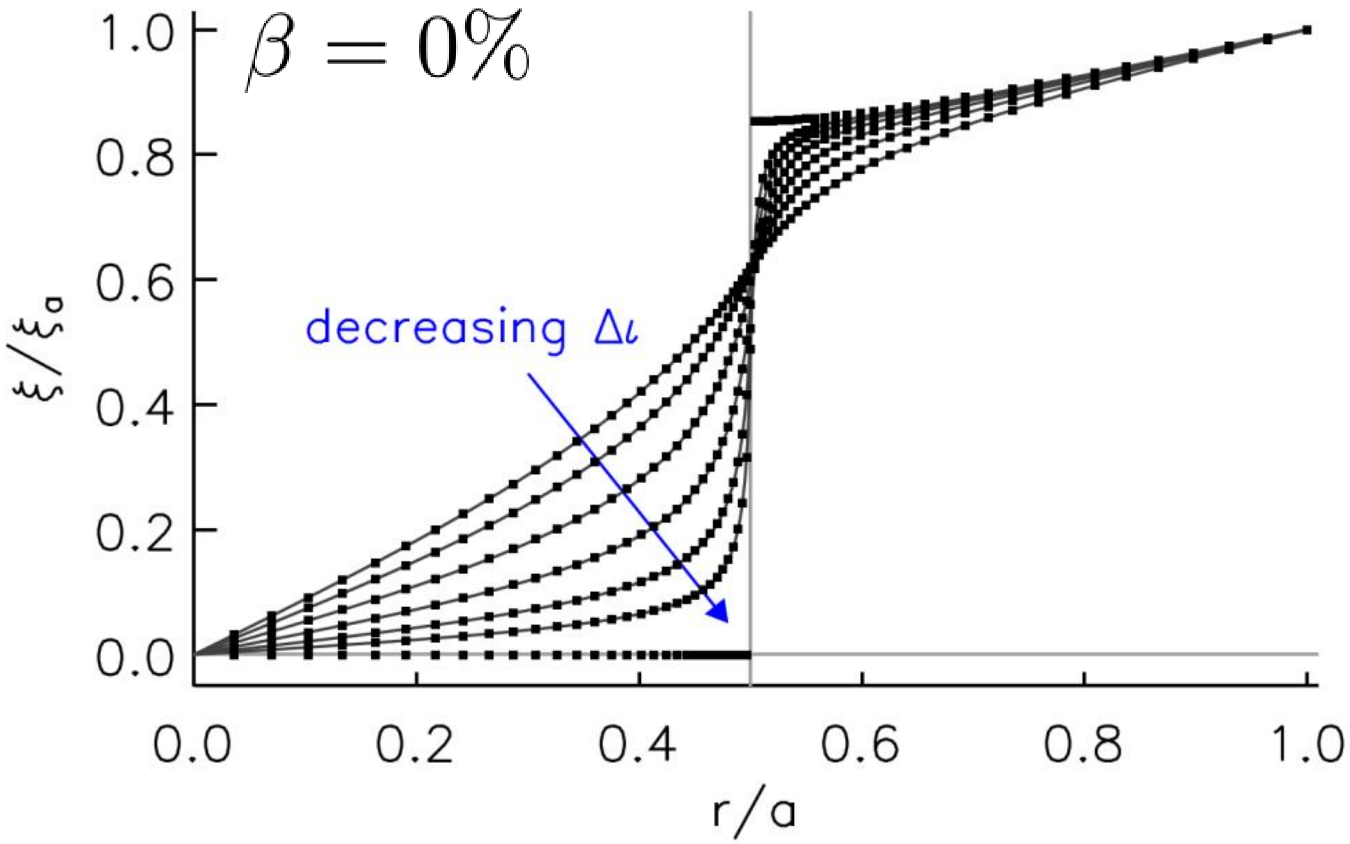
NO RESONANT SURFACE

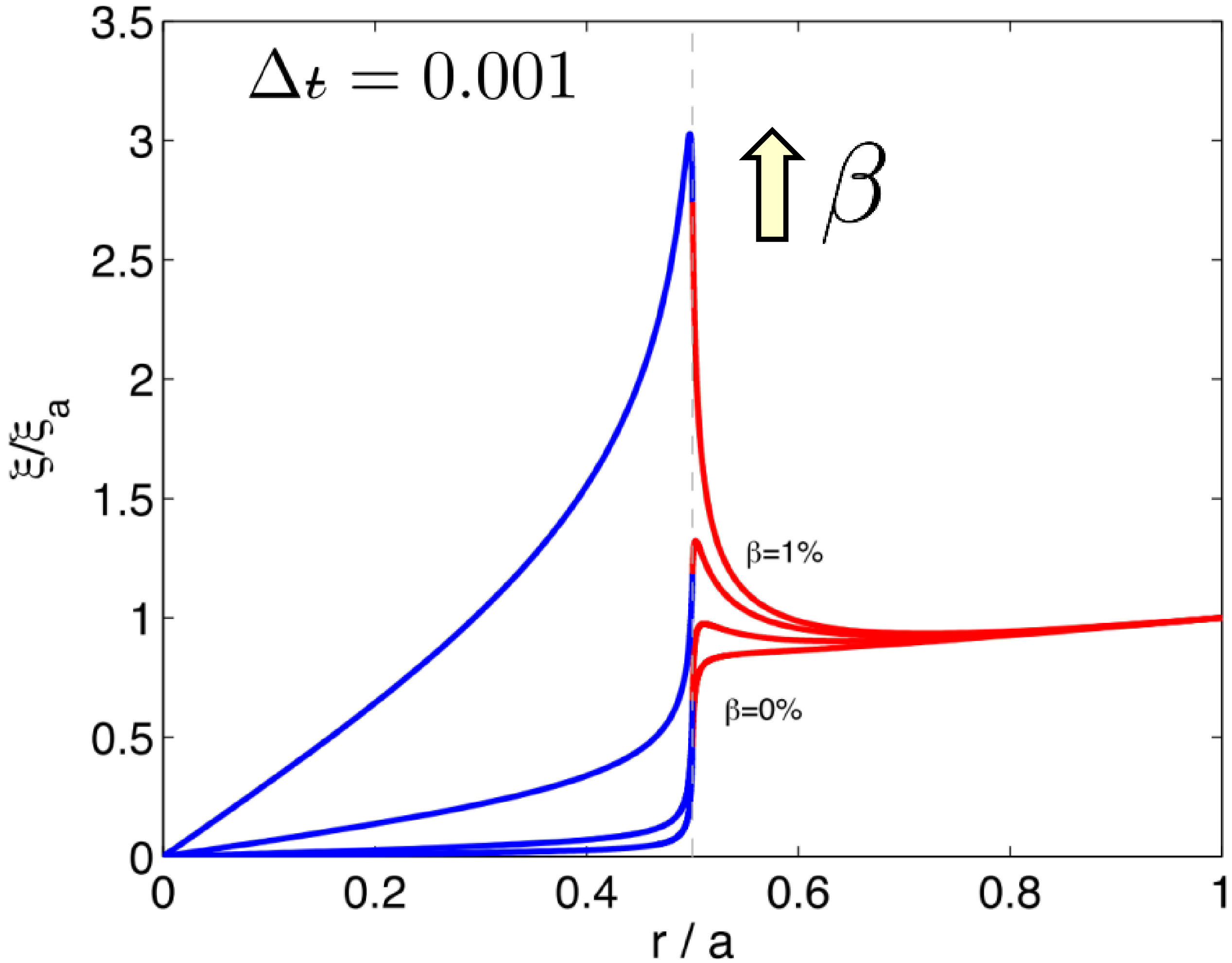
$$\mathcal{L}_0[\boldsymbol{\xi}] \equiv \nabla \delta p - \delta \mathbf{j} \times \mathbf{B} - \mathbf{j} \times \delta \mathbf{B} = 0 \textbf{ IS } \textbf{\textit{NOT}} \textbf{ SINGULAR IF NO RESONANCES}$$

In cylindrical geometry, reduces to Newcomb's equation,

$$\frac{d}{dr}\left(f\,\frac{d\xi}{dr}\right) - g\,\xi = 0$$

Results: perturbation penetrates past "resonant" surface, perturbation amplified by pressure.





RMP penetrates into the core.

[Loizu, Hudson *et al*., Phys. Plasmas **22**, 090704 (2015)]

RMP amplified by pressure near ideal stability limit

[Loizu, Hudson *et al*., Phys. Plasmas **23**, 055703 (2016)]

1. Even in ideal-MHD, there are no resonant "shielding" currents at the resonant rational surface.

2. The perturbation penetrates, even in cylindrically symmetric equilibria (where there is no poloidal or toroidal "coupling" to lowest order).

3. Similar penetration and amplification of RMPs are expected for tokamak equilibria, with consequences for transport.

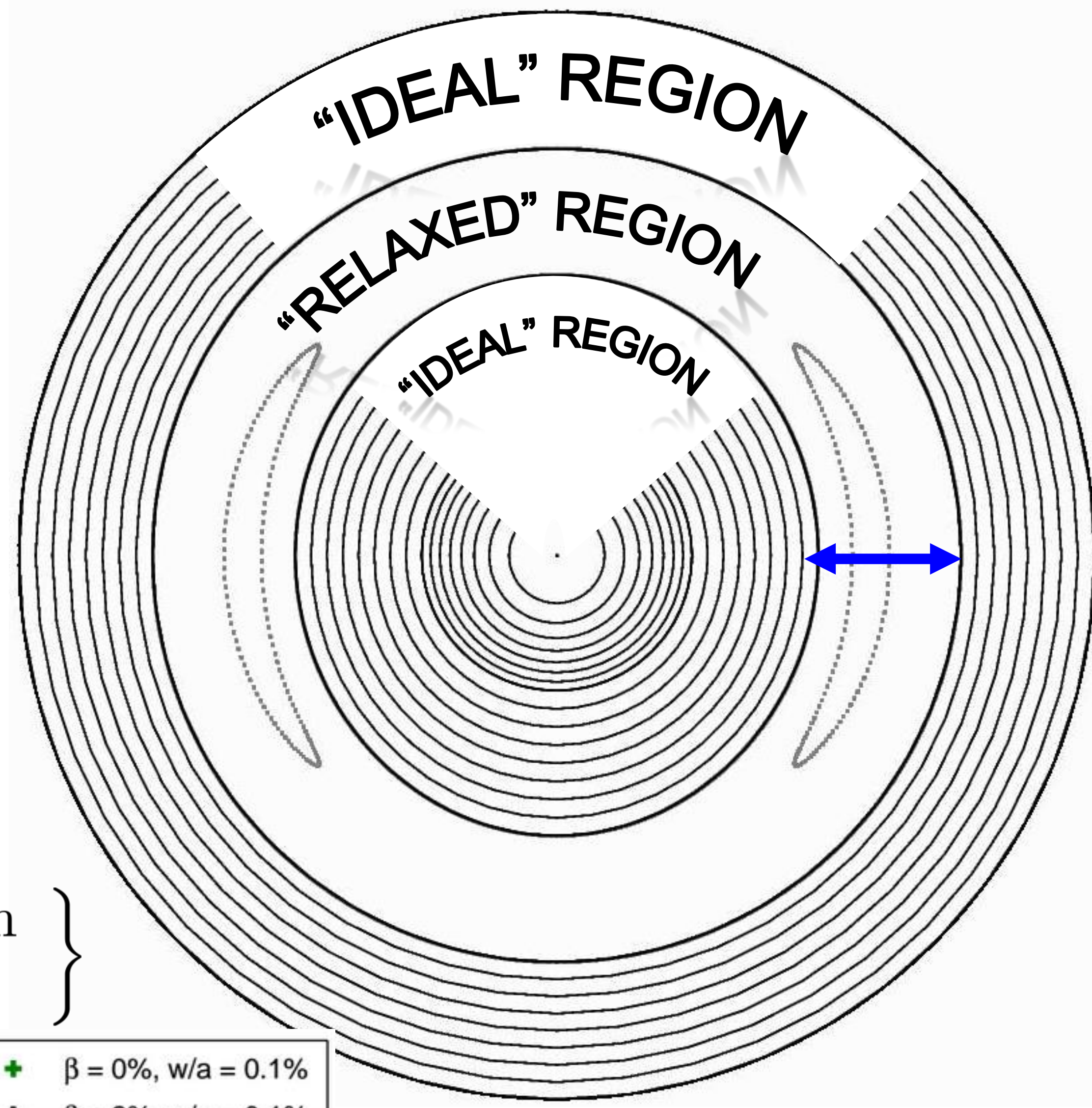
New solutions match smoothly to "tearing" solution: amplification and penetration of the RMP is still present.

1. Now, include a "relaxed" region,

   i. $\Delta\psi_t \equiv$ toroidal flux in relaxed region

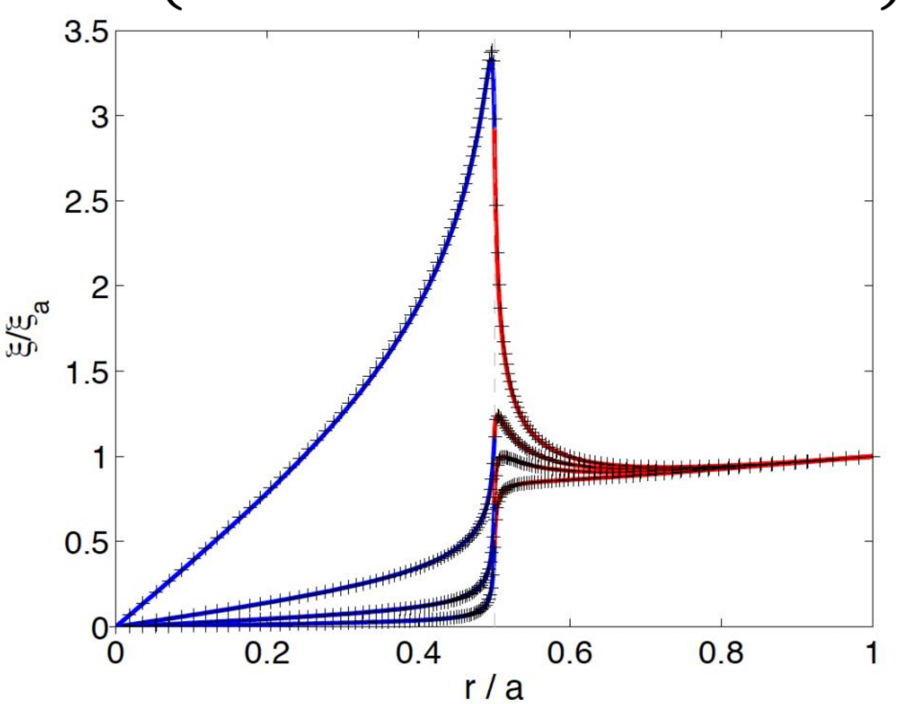
   ii. $\Delta\iota\!\!\!-\equiv$ jump in transform across relaxed region

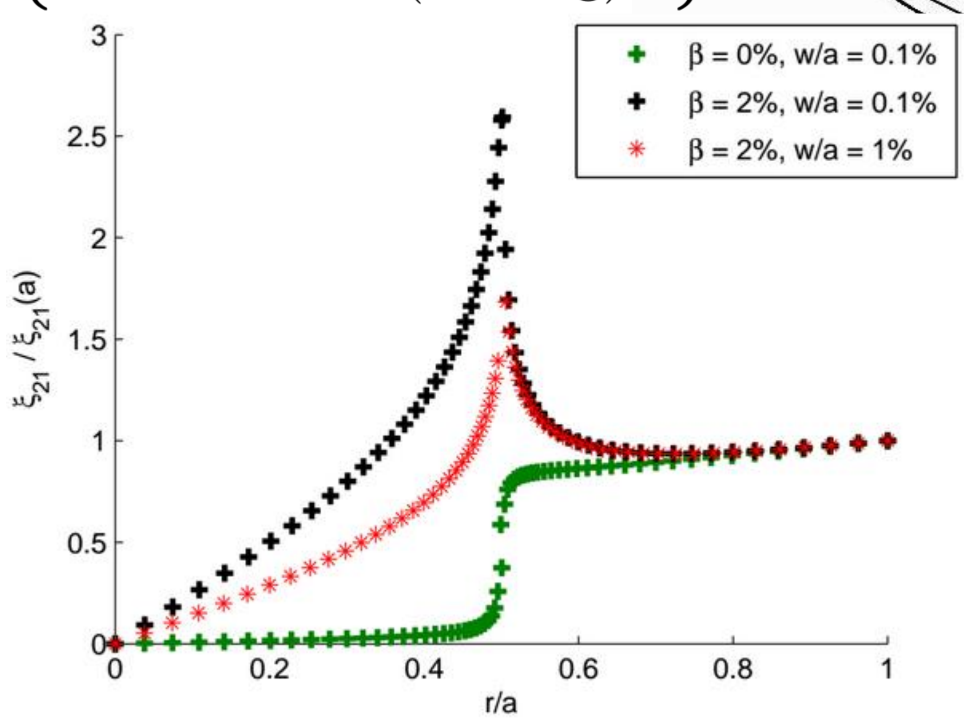
   iii. "partially" relaxed equilibrium computed with Stepped Pressure Equilibrium Code (SPEC)
   [Hudson, Dewar *et al*., Phys. Plasmas **19**, 112502 (2012)]



2. a magnetic island (tearing mode) arises.

$$\left\{ \begin{array}{c} \text{Discontinuous transform} \\ \text{with no island} \end{array} \right\} \approx \left\{ \begin{array}{c} \text{Continuous transform} \\ \text{with island (tearing)} \end{array} \right\}$$





The structure of the displacement is similar!

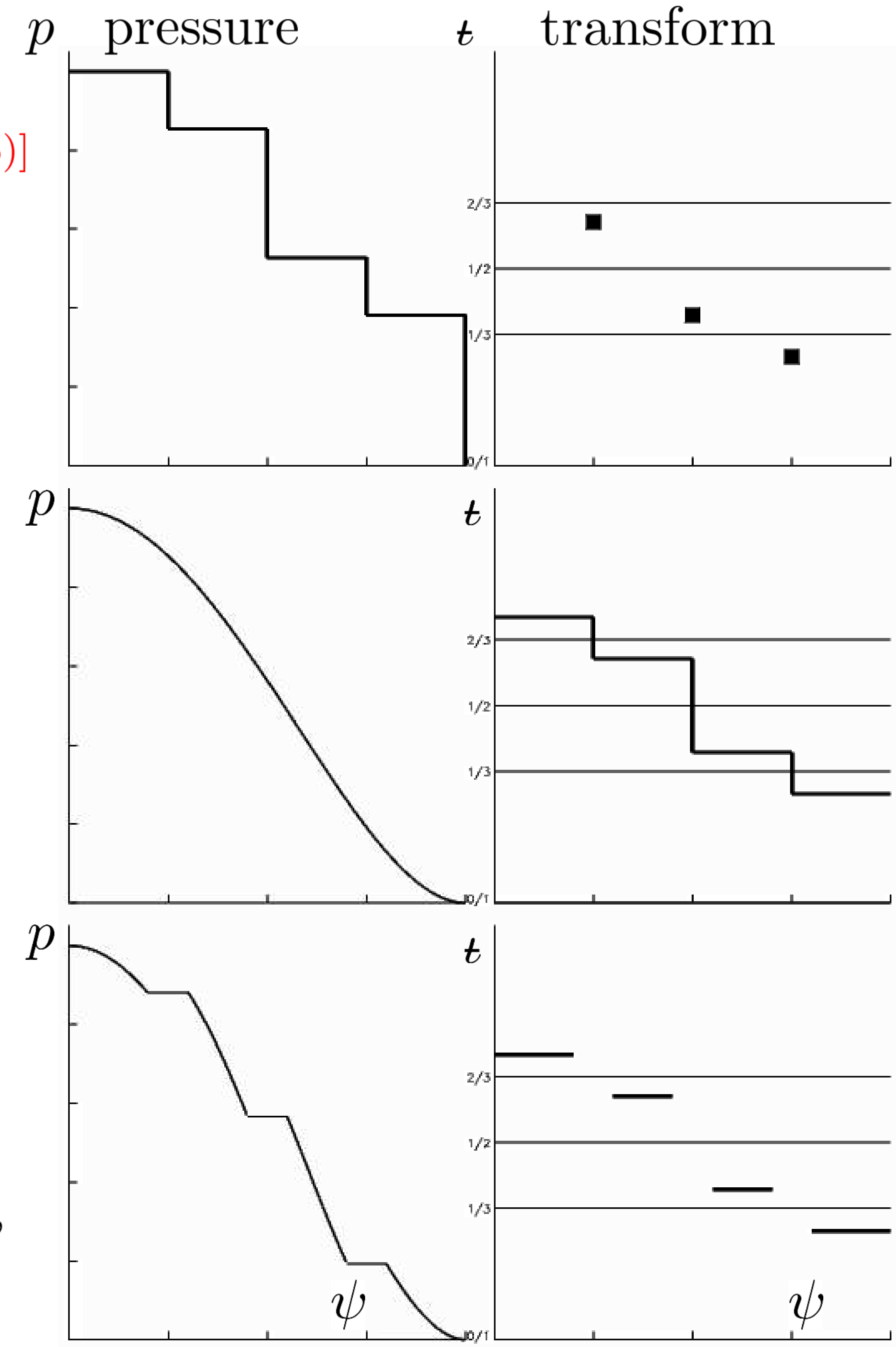
[Loizu, Hudson *et al.,* in preparation.]

Conclusion:
We have solved a fundamental theoretical problem.
The classes of general, tractable 3D MHD equilibria are

1. **Stepped-pressure equilibria,**
   i. [Bruno & Laurence, Commun. Pure Appl. Math. **49**, 717 (1996)]
   ii. transform constrained discretely
   iii. pressure discontinuity at $\iota\!\!\!-$ = irrational
   iv. allows for islands, magnetic fieldline chaos

2. **Stepped-transform equilibria,**
   i. [Loizu, Hudson *et al.*, Phys. Plasmas **22**, 090704 (2015)]
   ii. transform almost everywhere irrational
   iii. arbitrary, smooth pressure
   iv. continuously-nested flux surfaces

3. **Or, a combination of the above.**
   i. **each can be computed using SPEC**
   ii. suggests VMEC, NSTAB, IPEC, etc. should be modified to allow for discontinuous transform



Q. **How does a state with continuous transform "ideally evolve" into a 3D state with discontinuous transform?**
implications for ideal stability if no accessible 3D state exists?

Back up slides

Multi-Region Relaxed MHD, (MRxMHD),and the Stepped Pressure Equilibrium Code, (SPEC)

$$\mathcal{F} \equiv \sum_{i=1}^{N_V} \left[\int_{\mathcal{R}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv - \frac{\mu}{2} \left(\int_{\mathcal{R}_i} \mathbf{A} \cdot \mathbf{B} \, dv - H_i \right) \right]$$

Theoretical developments

i. Including flow, $\rho v^2/2$, into energy functional,
[Dennis, Hudson *et al.*, Phys. Plasmas **21**, 042501 (2014)]

ii. Including flow & pressure-anisotrophy,
[Dennis, Hudson *et al.*, Phys. Plasmas **21**, 072512 (2014)]

iii. Including flow & Hall-effect,
[Lingam, Abdelhamid & Hudson, Phys. Plasmas **23**, 082103 (2016)]

iv. Derivation from Lagrangian variational,
[Dewar, Yoshida *et al.*, J. Plasma Phys. **81**, 515810604 (2015)]

Numerical developments

i. finite-elements replaced by Chebshev polynomials,

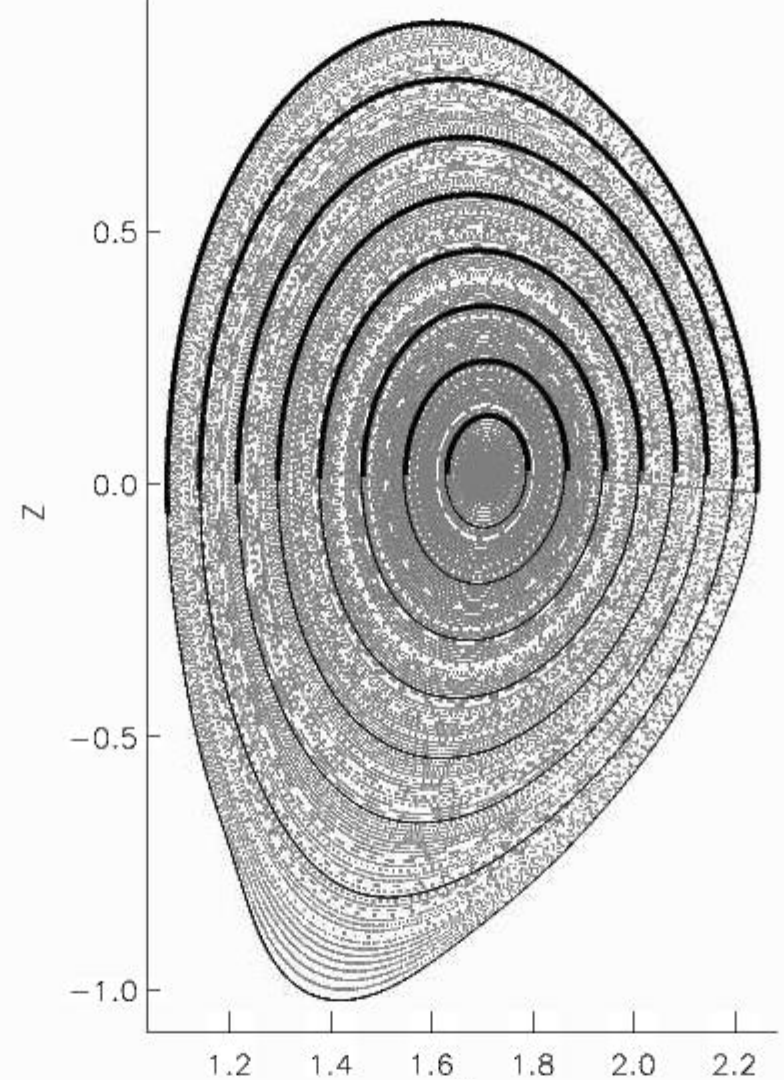
ii. arbitrary (non-stellarator-symmetric) geometry,

iii. Cartesian, cylindrical or toroidal,

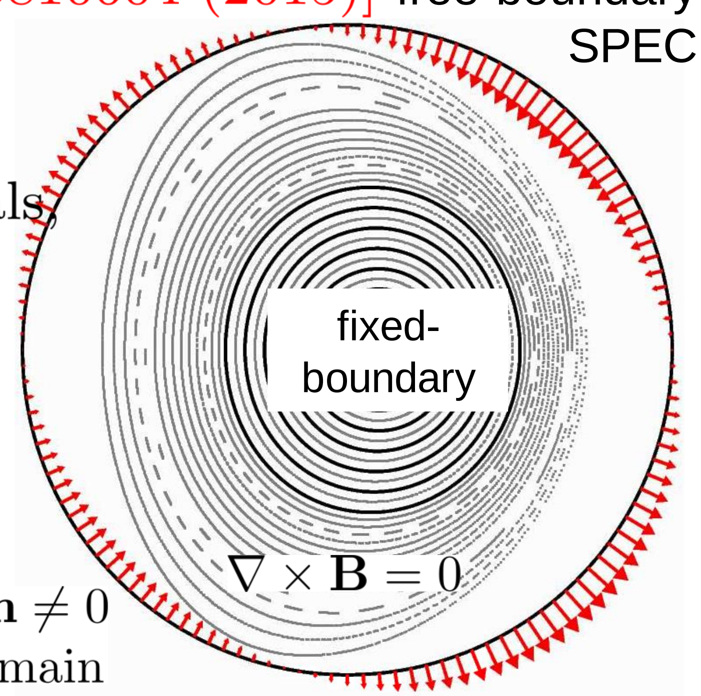
iv. linearized equations,

v. free-boundary equilibria.

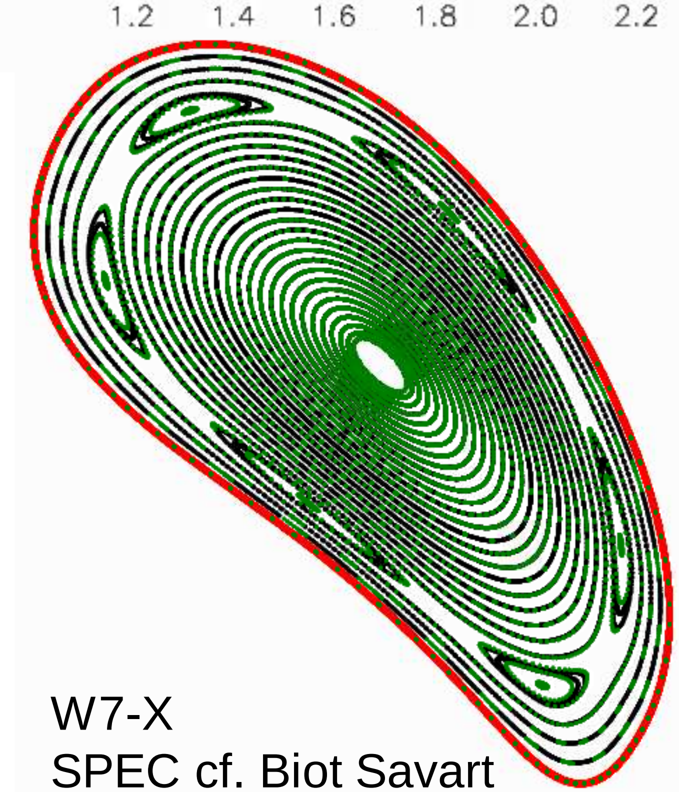
DIIID: SPEC cf. VMEC



free-boundary SPEC



$\mathbf{B} \cdot \mathbf{n} \neq 0$
on ∂Domain

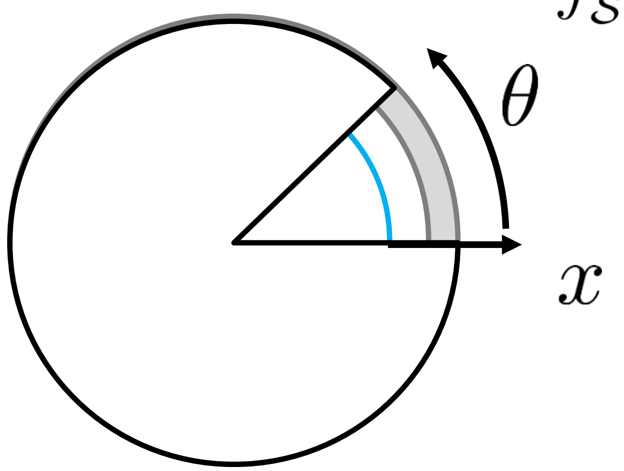


W7-X
SPEC cf. Biot Savart

Problem: the pressure-driven 1/x current density gives infinite parallel currents through certain surfaces.

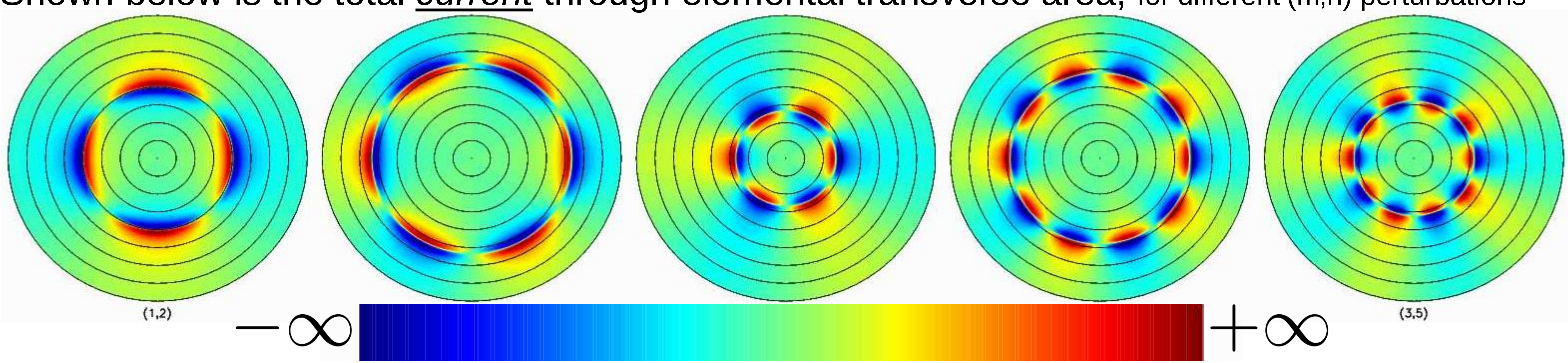
Parallel current-density
$$\mathbf{j}_\parallel = \sum_{m,n}\left[\frac{g_{m,n}\,p'}{x} + \Delta_{m,n}\delta_{m,n}(x)\right] e^{(im\theta - in\zeta)}\mathbf{B}.$$

Parallel current through cross-section
$$\int_{\mathcal{S}}\mathbf{j}_\parallel\cdot d\mathbf{s} = \int d\psi\int d\theta\;\sqrt{g}\,\mathbf{j}_\parallel\cdot\nabla\zeta$$
$$= \int_\epsilon^\delta dx\int_0^{\pi/m} d\theta\;\frac{g_{m,n}\,p'}{x}e^{i(m\theta-n\zeta)}\sqrt{g}\,\mathbf{B}\cdot\nabla\zeta$$
$$= g_{m,n,0}\,p_0'\,\frac{2}{m}\int_\epsilon^\delta dx\,\frac{1}{x}$$
$$= g_{m,n,0}\,p_0'\,\frac{2}{m}\,(\ln\delta - \ln\epsilon) \to \infty \text{ as } \epsilon\to 0.$$



The problem is *NOT* a lack of numerical resolution.
Is a dense collection of alternating infinite currents physical?

Shown below is the total <u>*current*</u> through elemental transverse area, for different (m,n) perturbations



If there are rational surfaces, then we must choose:

1. flatten pressure near rationals, smooth pressure; ✗
2. flatten pressure near rationals, fractal pressure; ✗
3. flatten pressure near rationals, discontinuous pressure; ✓
4. restrict attention to “healed” configurations [Weitzner, PoP **21**, 022515 (2014); Zakharov, JPP **81**, 515810609, (2015)]

1. Locally-flattened, smooth pressure:

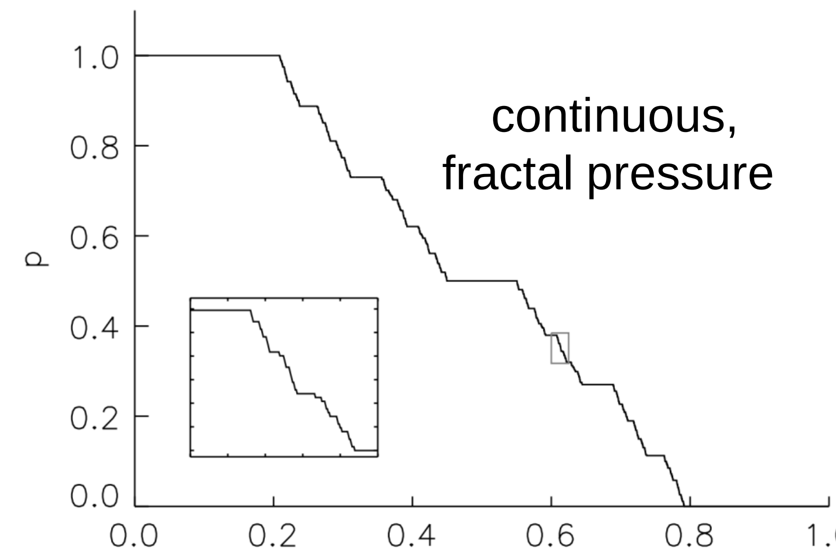
if (*i.*) $p'(x) = 0$ if $|x - n/m| < \epsilon_{m,n}$, $\forall(n,m)$,

and (*ii.*) $p'(x)$ is continuous, then $p'(x) = 0$, $\forall x$. **No pressure!**

2. “Diophantine” pressure profile: e.g. from KAM theory

$$p'(x) = \begin{cases} 1, & \text{if } |x - n/m| > r/m^k, \quad \forall(n,m), \text{ e.g. } r = 0.2,\ k = 2, \\ 0, & \text{if } |x - n/m| < r/m^k, \quad \exists(n,m), \end{cases}$$

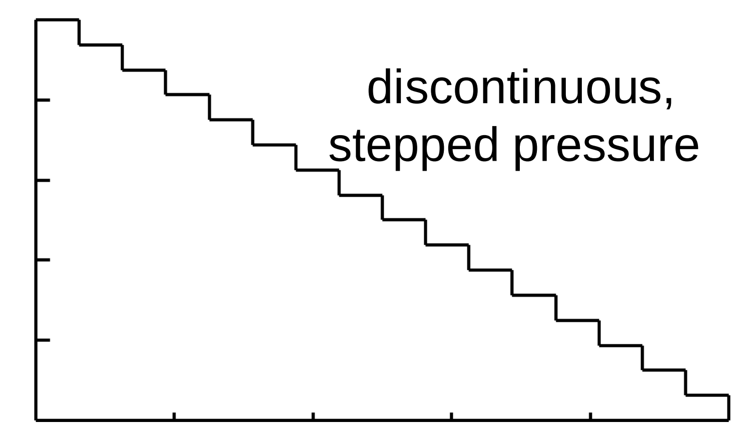
$p'(x)$ is discontinuous on an uncountable infinity of points,



“The function p is continuous but its derivative is pathological.” Grad, Phys. Fluids 10, 137 (1967)]

3. “Stepped” pressure profile: ✓

Existence of Three-Dimensional Toroidal MHD Equilibria with Nonconstant Pressure
[Bruno & Laurence, Commun. Pure Appl. Math. **49**, 717 (1996)]



Culmination of long history of “waterbag” or “sharp-boundary” equilibria:
[Potter, Methods Comp. Phys., **16**, 43 (1976); Berk et al., Phys. Fluids, **29**, 3281 (1986); Kaiser & Salat Phys. Plasmas **1**, 281 (1994)]

Relaxed MHD $\leftarrow$ Multi-Region relaxed MHD $\rightarrow$ Ideal MHD

[Taylor, Phys. Rev. Lett. **33**, 1139 (1974)] [Dewar, Hole, Hudson, et al., circa 2006] [Kruskal & Kulsrud, Phys. Fluids **1**, 265 (1958)]

$N_V = 1$ **Relaxed MHD**

$$\mathcal{F} \equiv \underbrace{\int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{energy} - \frac{\mu}{2} \underbrace{\int_{\mathcal{R}} \mathbf{A} \cdot \mathbf{B} \, dv}_{helicity},$$

$\delta\mathbf{B} \equiv \nabla \times \delta\mathbf{A}$ is arbitrary in $\mathcal{R}$

$(\delta\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ on $\partial\mathcal{R})$

$+$ constrained flux

$$\delta\mathcal{F} = 0, \quad p = p_0, \quad \nabla \times \mathbf{B} = \mu \mathbf{B} \text{ in } \mathcal{R};$$

$N_V = \infty$ **Ideal MHD**

$$\mathcal{F} \equiv \int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv,$$

$\delta\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ in $\mathcal{R}$

(fluxes & helicity conserved)

$$\delta\mathcal{F} = 0, \quad p = p(\psi), \quad \nabla p = \mathbf{j} \times \mathbf{B} \text{ in } \mathcal{R}.$$

Relaxed MHD $\leftarrow$ Multi-Region relaxed MHD $\rightarrow$ Ideal MHD

[Taylor, Phys. Rev. Lett. **33**, 1139 (1974)] [Dewar, Hole, Hudson, et al., circa 2006] [Kruskal & Kulsrud, Phys. Fluids **1**, 265 (1958)]

$N_V = 1$ Relaxed MHD

$$\mathcal{F} \equiv \underbrace{\int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{energy} - \frac{\mu}{2} \underbrace{\int_{\mathcal{R}} \mathbf{A} \cdot \mathbf{B} \, dv}_{helicity},$$

$\delta \mathbf{B} \equiv \nabla \times \delta \mathbf{A}$ is arbitrary in $\mathcal{R}$

$(\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ on $\partial \mathcal{R})$

$+$ constrained flux

$$\delta \mathcal{F} = 0, \quad p = p_0, \quad \nabla \times \mathbf{B} = \mu \mathbf{B} \text{ in } \mathcal{R};$$

$N_V < \infty$ MRx MHD

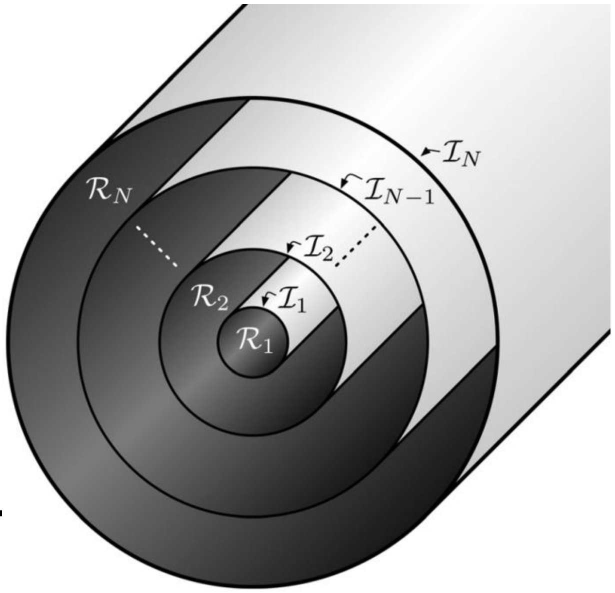
$$\mathcal{F} \equiv \sum_{i=1}^{N_V} \left\{ \int_{\mathcal{R}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv - \frac{\mu_i}{2} \int_{\mathcal{R}_i} \mathbf{A} \cdot \mathbf{B} \, dv \right\},$$

$\delta \mathbf{B}_i \equiv \nabla \times \delta \mathbf{A}_i$ is arbitrary in $\mathcal{R}_i$

$\delta \mathbf{B}_i = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_i)$ on $\partial \mathcal{R}_i$

$+$ constrained fluxes in $\mathcal{R}_i$

$$\delta \mathcal{F} = 0, \quad p = p_i, \quad \nabla \times \mathbf{B} = \mu_i \mathbf{B} \text{ in } \mathcal{R}_i; \quad \left[\left[p + \frac{B^2}{2} \right]\right] = 0 \text{ across } \partial \mathcal{R}_i;$$



$$\rightarrow \quad p(\psi), \nabla p = \mathbf{j} \times \mathbf{B} \quad \text{as} \quad N_V \rightarrow \infty,$$

[Dennis, Hudson et al., Phys. Plasmas **20**, 032509, 2013]

$N_V = \infty$ Ideal MHD

$$\mathcal{F} \equiv \int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv,$$

$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ in $\mathcal{R}$

(fluxes & helicity conserved)

$$\delta \mathcal{F} = 0, \quad p = p(\psi), \quad \nabla p = \mathbf{j} \times \mathbf{B} \text{ in } \mathcal{R}.$$

Relaxed MHD ← Multi-Region relaxed MHD → Ideal MHD

[Taylor, Phys. Rev. Lett. **33**, 1139 (1974)]   [Dewar, Hole, Hudson, et al., circa 2006]   [Kruskal & Kulsrud, Phys. Fluids **1**, 265 (1958)]

$N_V = 1$ Relaxed MHD

$$\mathcal{F} \equiv \underbrace{\int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{energy} - \frac{\mu}{2} \underbrace{\int_{\mathcal{R}} \mathbf{A} \cdot \mathbf{B} \, dv}_{helicity},$$

$\delta \mathbf{B} \equiv \nabla \times \delta \mathbf{A}$ is arbitrary in $\mathcal{R}$
$(\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ on $\partial \mathcal{R})$
+ constrained flux

$$\delta \mathcal{F} = 0, \quad p = p_0, \quad \nabla \times \mathbf{B} = \mu \mathbf{B} \text{ in } \mathcal{R};$$

$N_V < \infty$ MRx MHD

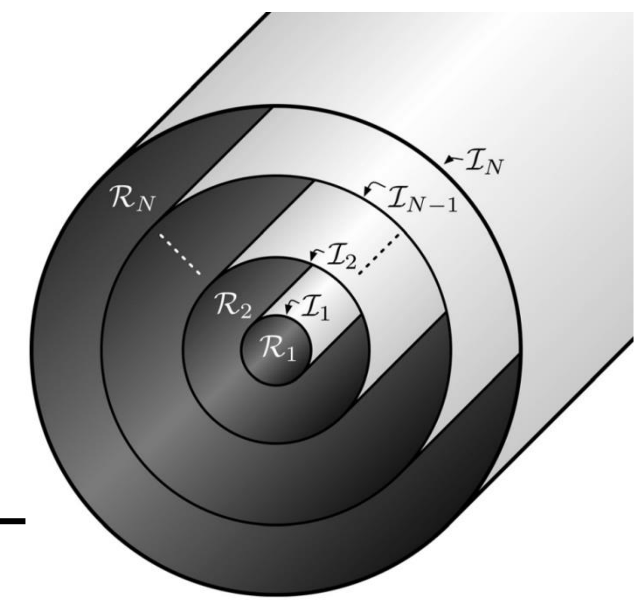
$$\mathcal{F} \equiv \sum_{i=1}^{N_V} \left\{ \int_{\mathcal{R}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv - \frac{\mu_i}{2} \int_{\mathcal{R}_i} \mathbf{A} \cdot \mathbf{B} \, dv \right\},$$

$\delta \mathbf{B}_i \equiv \nabla \times \delta \mathbf{A}_i$ is arbitrary in $\mathcal{R}_i$
$\delta \mathbf{B}_i = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_i)$ on $\partial \mathcal{R}_i$
+ constrained fluxes in $\mathcal{R}_i$

$$\delta \mathcal{F} = 0, \quad p = p_i, \quad \nabla \times \mathbf{B} = \mu_i \mathbf{B} \text{ in } \mathcal{R}_i; \quad \left[\left[p + \frac{B^2}{2} \right]\right] = 0 \text{ across } \partial \mathcal{R}_i;$$

Stepped **P**ressure **E**quilibrium **C**ode

[Hudson, Dewar et al., Phys. Plasmas **19**, 112502 (2012)]



$N_V = \infty$ Ideal MHD

$$\mathcal{F} \equiv \int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv,$$

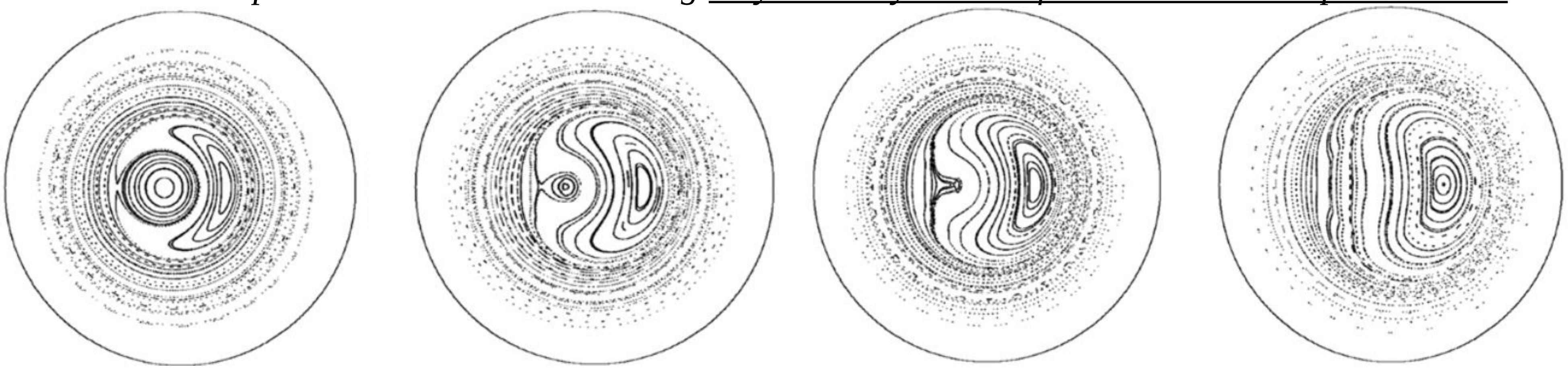
$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ in $\mathcal{R}$
(fluxes & helicity conserved)

$$\delta \mathcal{F} = 0, \quad p = p(\psi), \quad \nabla p = \mathbf{j} \times \mathbf{B} \text{ in } \mathcal{R}.$$

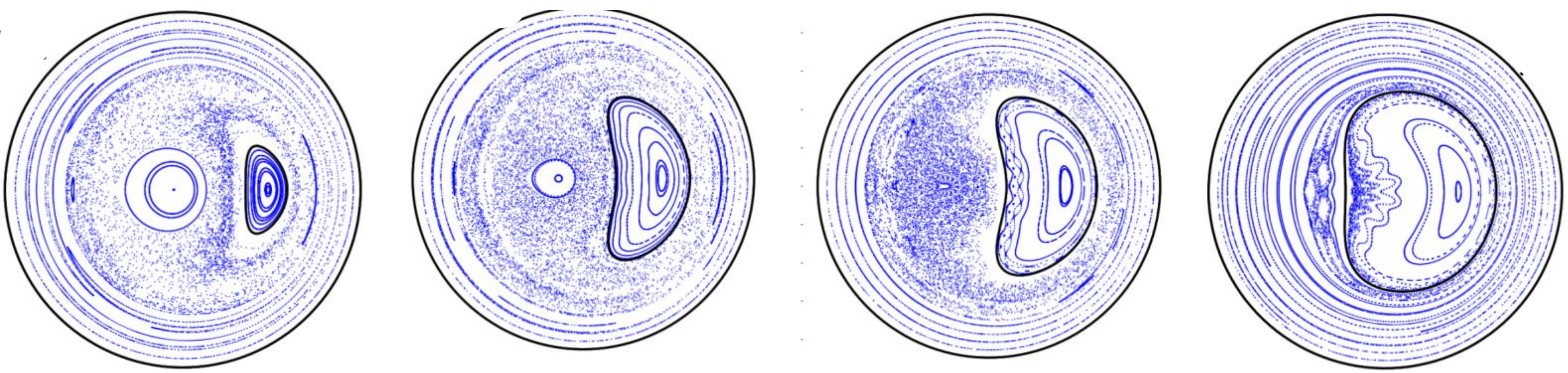
Nv=2, "double-Taylor" state with transport barrier; MRxMHD explains self-organization of RFP into helix.

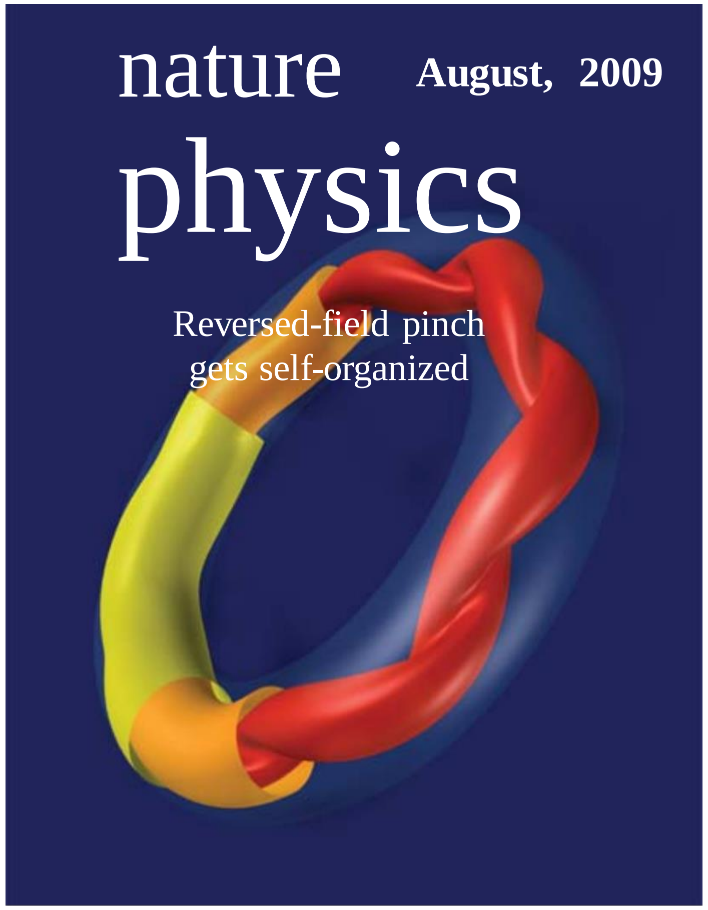
EXPERIMENTAL RESULTS
Overview of RFX-mod results
P. Martin et al., *Nucl. Fusion*, ***49*** *104019 (2009)*

Fig.6. Magnetic flux surfaces in the transition from a QSH state . . to a fully developed SHAx state . . The Poincaré plots are obtained considering only the axisymmetric field and dominant perturbation"



NUMERICAL CALCULATION USING STEPPED PRESSURE EQUILIBRIUM CODE
Minimally Constrained Model of Self-Organized Helical States in Reversed-Field Pinches
G. Dennis, S. Hudson et al. Phys. Rev. Lett. **111**, 055003 (2013)





Excellent qualitative agreement between numerical calculation and experiment.

Compute the 1/x and δ-function current densities in perturbed geometry
Self-consistent solutions require **infinite shear**

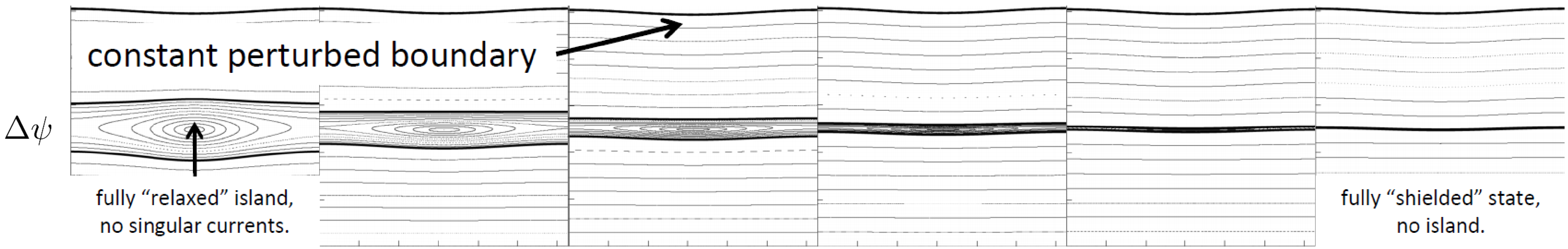
Cartesian, slab geometry with an $(m,n)=(1,0)$ resonantly-perturbed boundary

i. $N_V = 3$ MRxMHD calculation, no pressure, $\iota(\psi)$ given discretely,

ii. take limit $\Delta\psi \equiv x^\beta$, $\iota_i = -x^\alpha/2$, $\iota_{i+1} = +x^\alpha/2$, shear $\equiv \Delta\iota/\Delta\psi = x^{\alpha-\beta}$, $\boxed{\beta > \alpha}$.

iii. island forced to vanish,

iv. resonant $\delta_{m,n}$-function current-density appears as tangential discontinuity in $\mathbf{B}$.

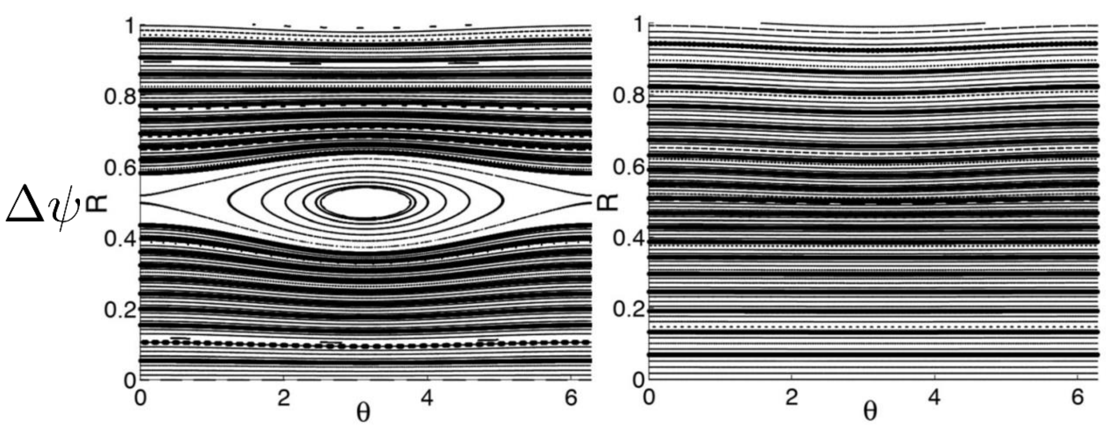


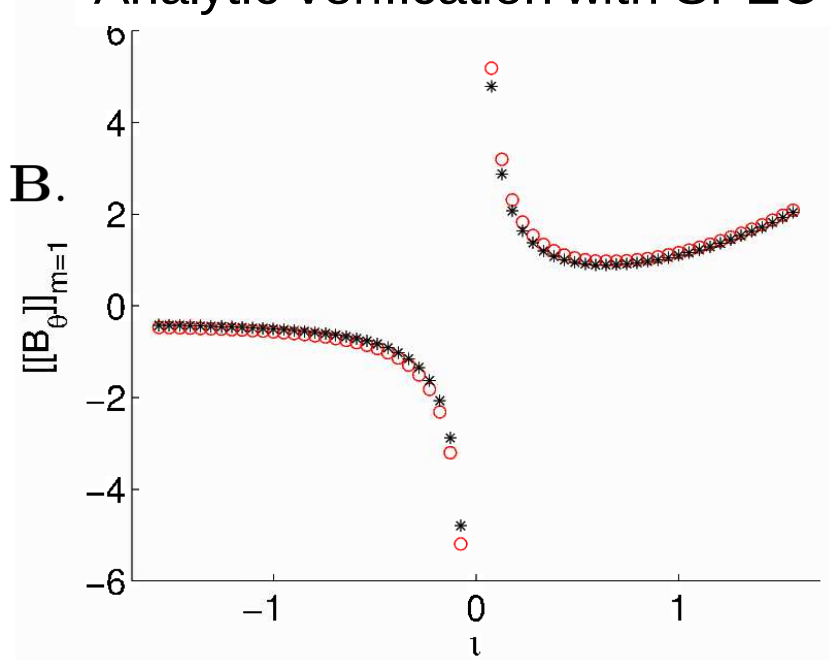
i. $N_V =$ large MRxMHD calculation, stepped pressure $\approx$ smooth pressure,

ii. take limit $\Delta\psi \equiv x^\beta$, $\iota_i = -x^\alpha/2$, $\iota_{i+1} = +x^\alpha/2$,

iii. island forced to vanish,

iv. resonant p'/x current-density appears as tangential discontinuity in $\mathbf{B}$.



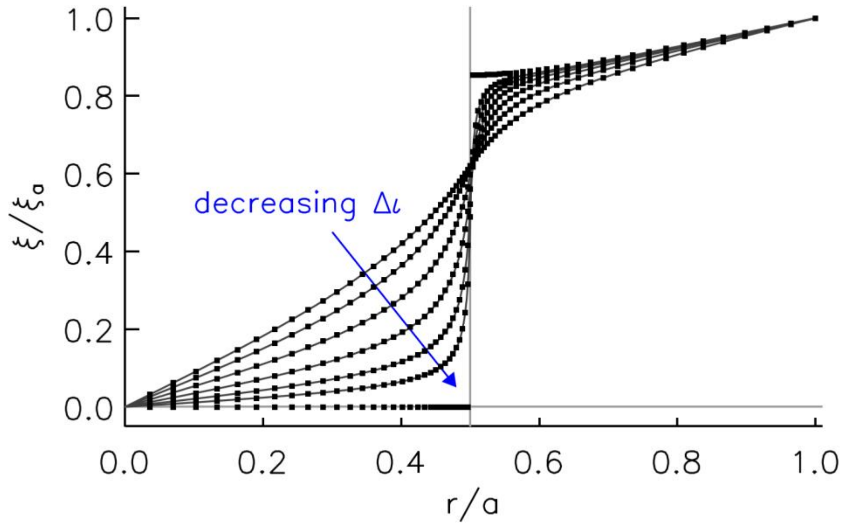


[Loizu, Hudson et al., Phys. Plasmas **22**, 022501 (2015)]

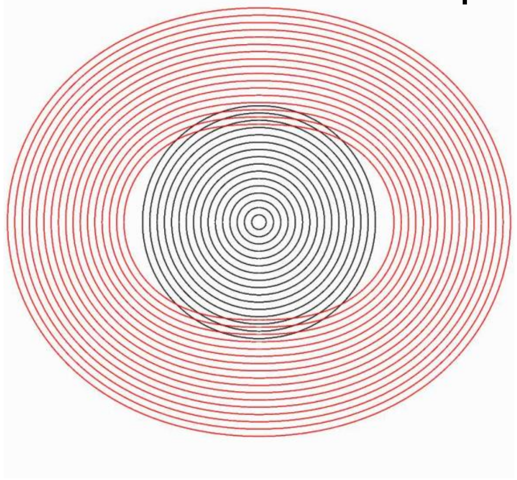
Necessary condition for non-overlapping of perturbed surfaces
Existence of non-linear solutions

1. **Condition for non-overlapping perturbed surfaces**

$$\left|\frac{\partial \xi}{\partial r}\right|_{max} < 1$$



Discontinuously-perturbed flux surfaces overlap!



2. **An asymptotic analysis near the rational surface** gives the *sine-qua-non* condition *(an indispensable condition, element, or factor; something essential)*

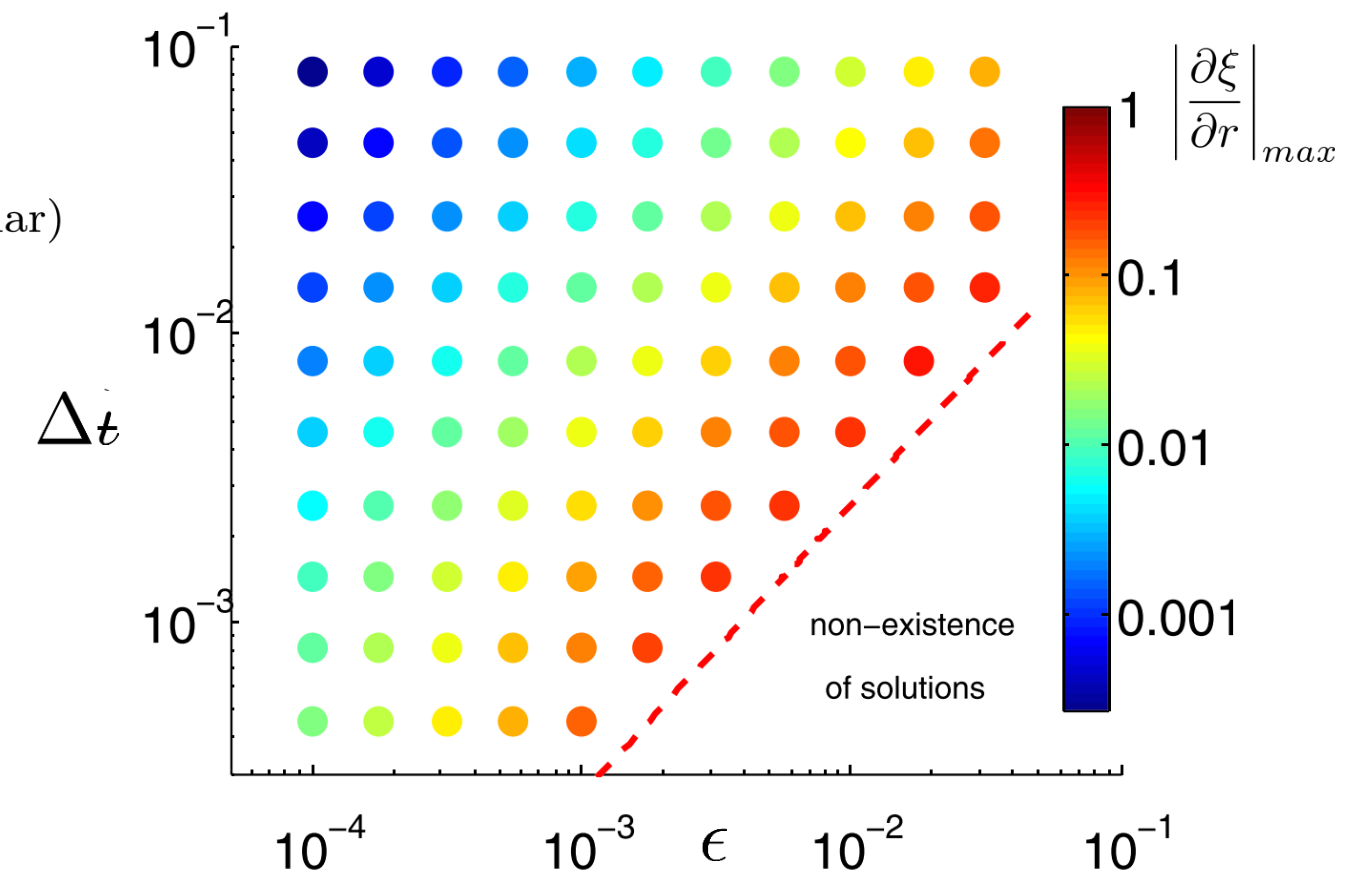
$$\Delta\iota > \Delta\iota_{min}, \quad \text{where } \Delta\iota_{min} \equiv 2\iota'_s \xi_s$$

(analysis for cylindrical, zero-β; general result probably similar)

3. **If this condition is violated, non-linear solutions do not exist.**

   i. Shown is ξ', as computed using non-linear SPEC calculations, as a function of $(\epsilon, \Delta\iota)$

   ii. SPEC fails in ideal-limit, i.e. $N_V \rightarrow \infty$, when $\Delta\iota < \Delta\iota_{min}$

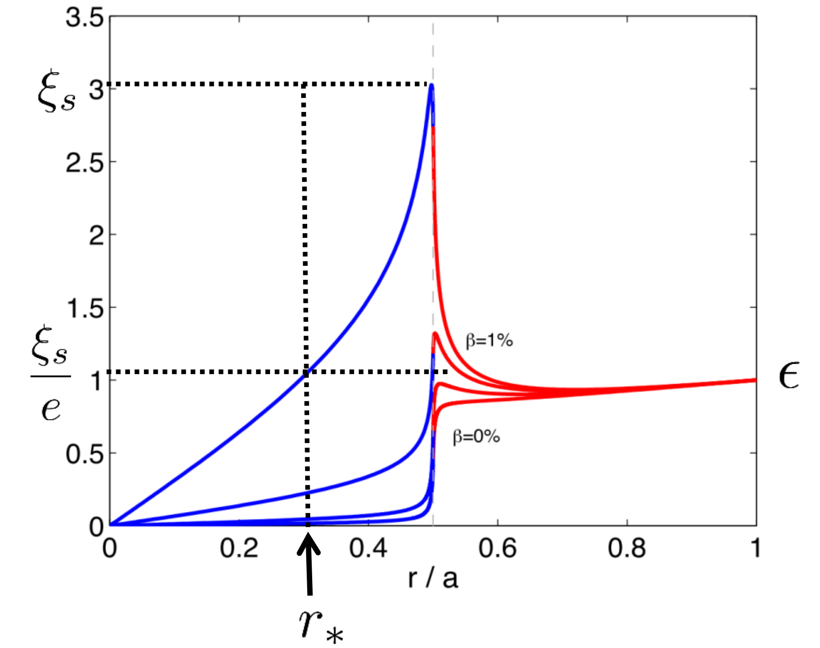


[Loizu, Hudson et al., Phys. Plasmas **22**, 090704 (2015)]

Amplification and penetration as stability boundary is approached

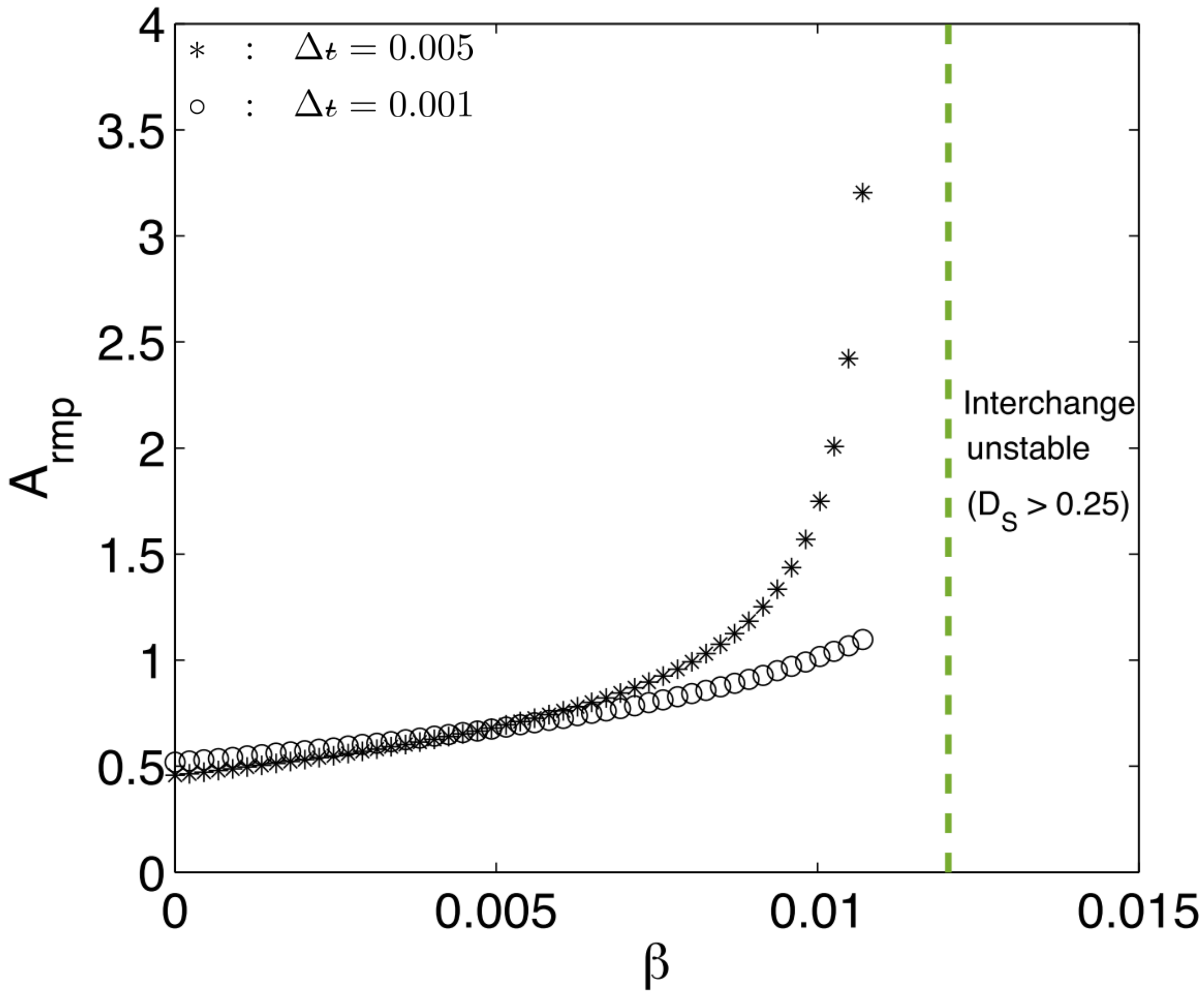
1. Can define a measure of

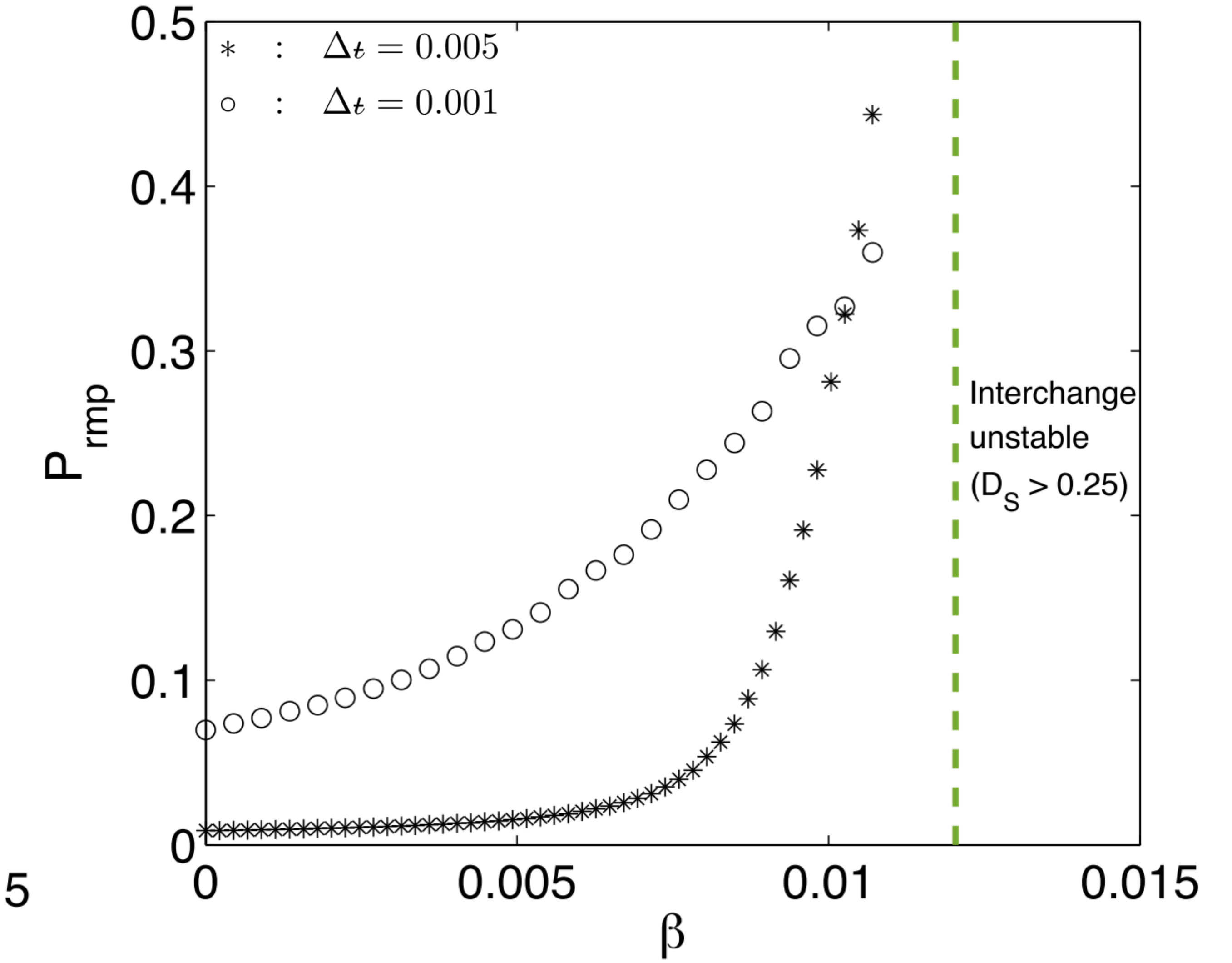
$$\text{“Amplification”} \quad A_{rmp} = \xi_s/\epsilon, \quad \text{where } \epsilon \equiv \text{boundary deformation}$$
$$\text{“Penetration”} \quad P_{rmp} = 1 - r_*/r_s, \quad \text{where } \xi(r_*) \equiv \xi_s/e$$



2. A necessary condition for interchange stability in a screw pinch is given by the Suydam criterion, $\boxed{D_S \equiv -\left(\frac{2p'\bar{\iota}^2}{rB_z^2\bar{\iota}'^2}\right)_s < \frac{1}{4}.}$

3. Amplification and penetration of RMP **fantastically increased** as stability limit approached.

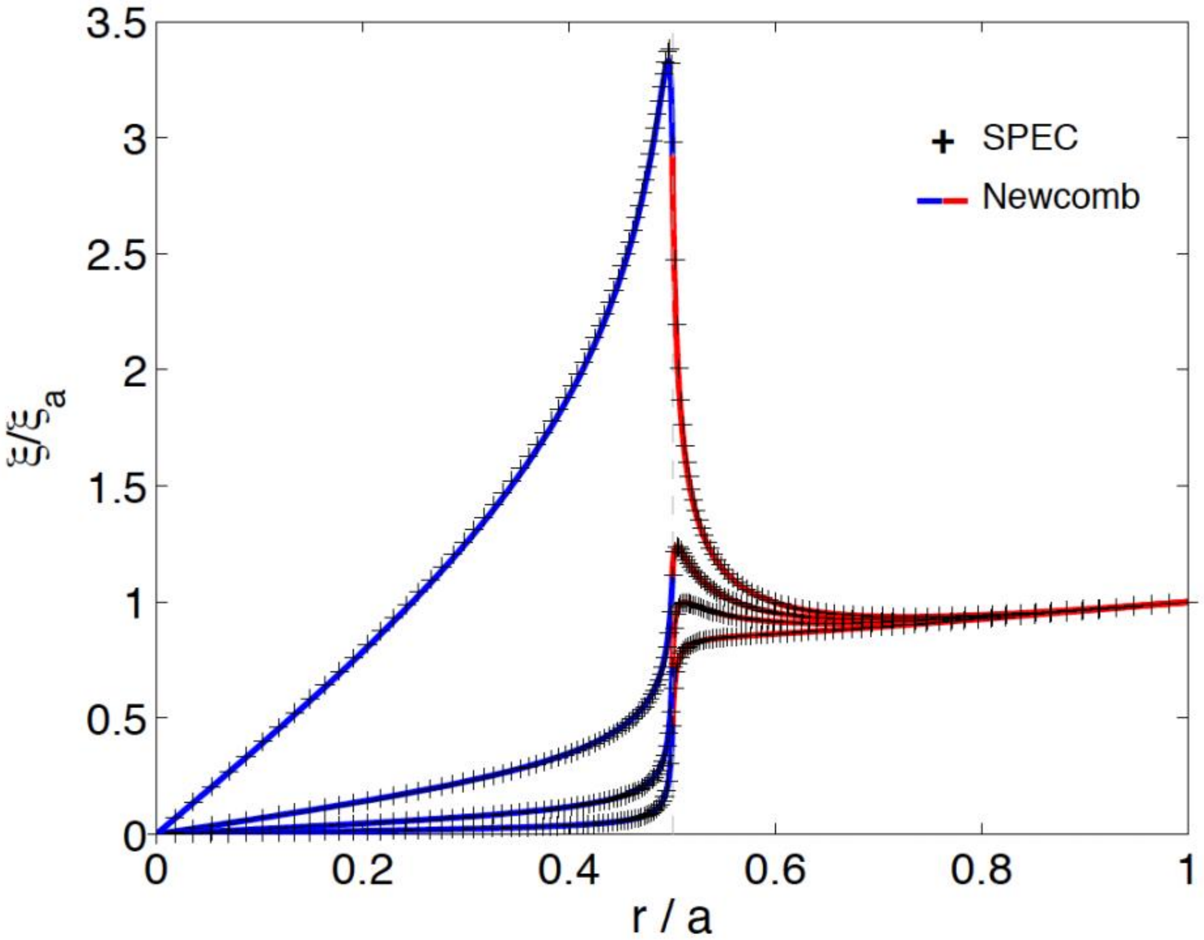




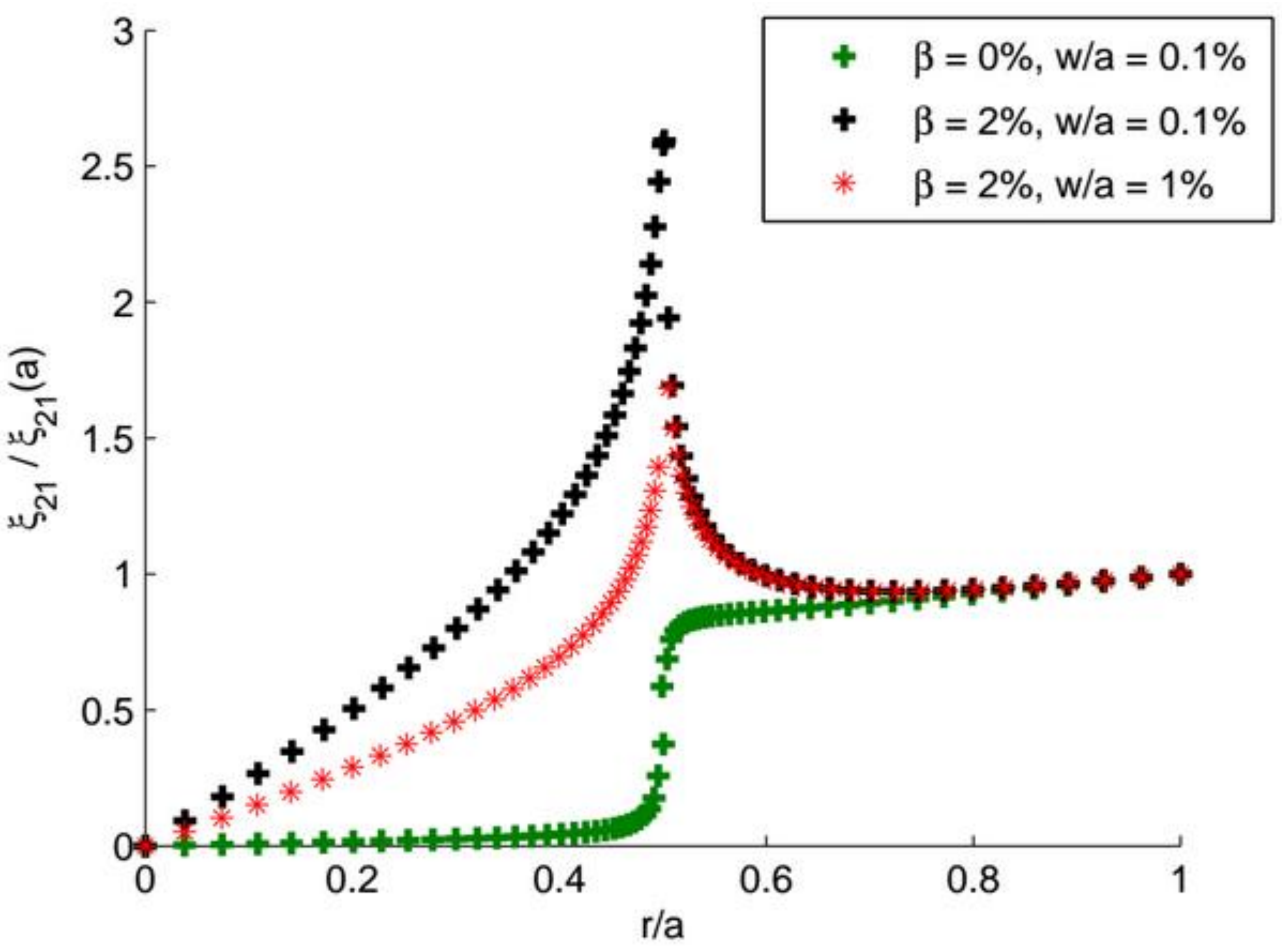
[Loizu, Hudson et al., Phys. Plasmas **23**, 055703 (2016)]

Discontinuous transform solution cf. “Tearing” solution

Discontinuous transform with no island (ideal)



Continuous transform with island (tearing)



SPEC	allows	discontinuous	profiles: exact	agreement
VMEC	assumes	smooth	profiles: approximate	agreement

1. VMEC assumes smooth profiles

   and smooth profiles imply discontinuous displacement

2. but, VMEC enforces nested flux surfaces

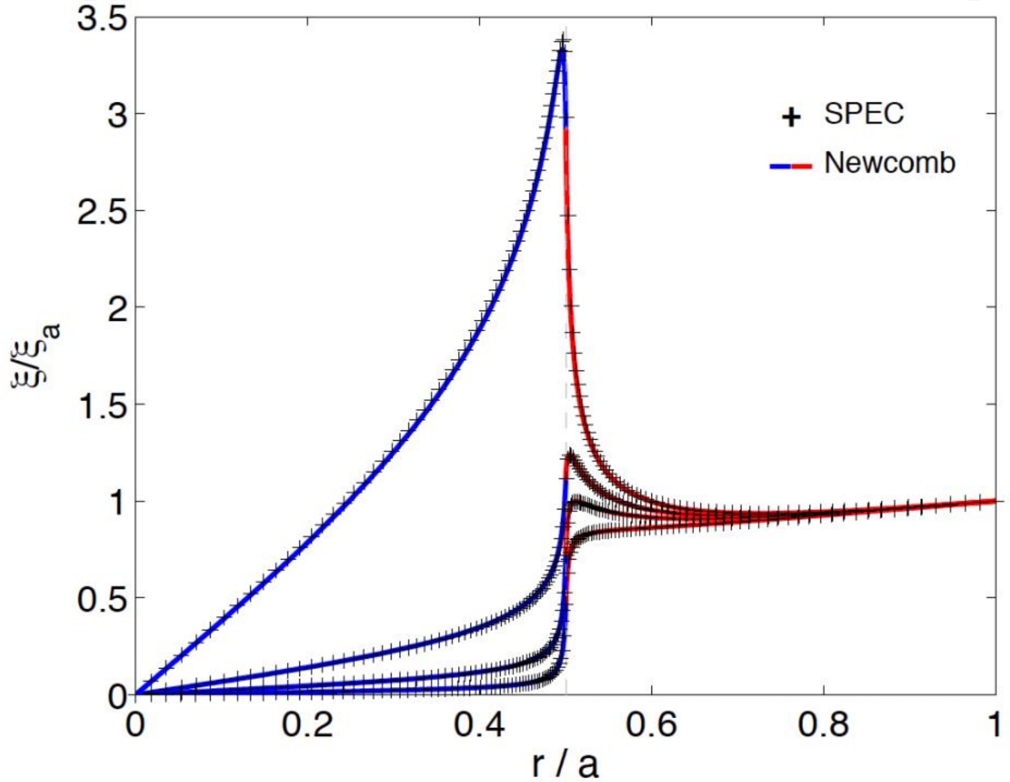
   nested flux surfaces in 3D imply $\frac{\partial \xi}{dr} < 1$ displacement from 2D

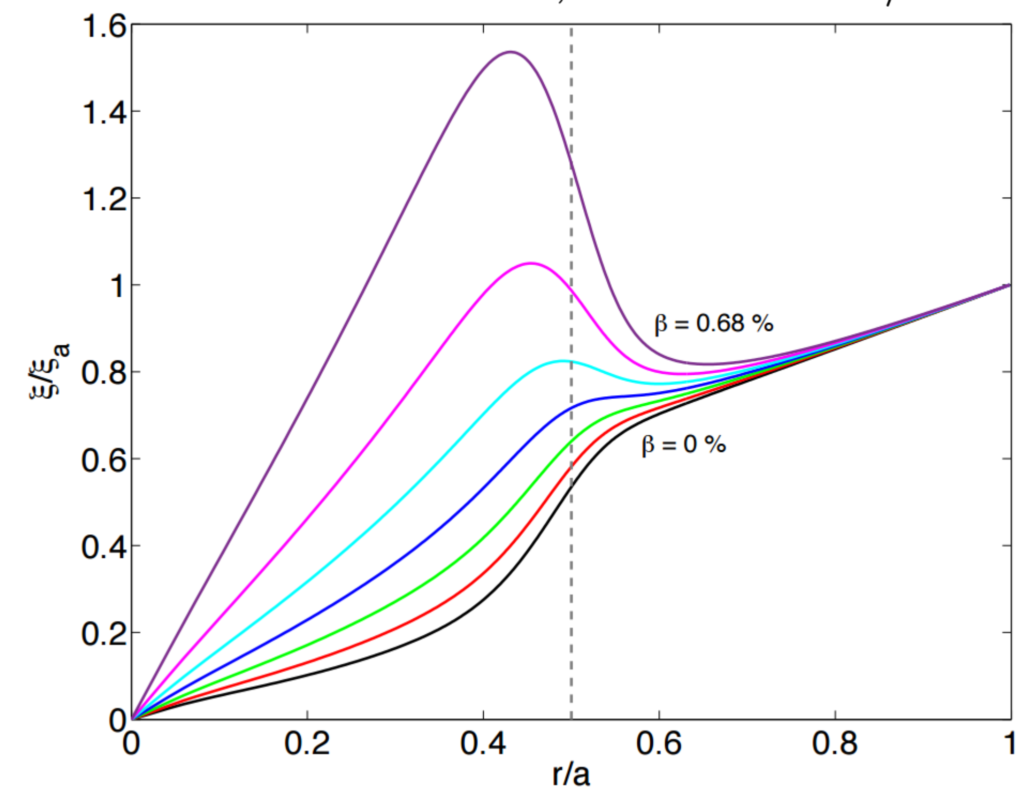
   and this is consistent only with discontinuous transform with $\Delta \iota\!\!\!- > \Delta \iota\!\!\!-_{min}$

3. Empirical study (i.e. radial convergence) shows that

   VMEC qualitatively reproduces self-consistent, perturbed solution

   interpretation: finite radial resolution implies an "effective" $\Delta \iota\!\!\!- \sim \iota\!\!\!-' h$, where $h \equiv 1/N$?





[Loizu, Hudson, …, Lazerson, ..., Phys. Plasmas **23**, 055703 (2016)]

Given continuous, non-integrable $\boldsymbol{B}$, $\boldsymbol{B}.\nabla p = 0$ implies p is fractal. Given fractal p, what is continuous, non-integrable $\boldsymbol{B}$?

- **Defn.** An equilibrium code computes the magnetic field consistent with a given p and e.g. given ι.
- **Theorem.** The topology of $\mathbf{B}$ is partially dictated by p.
  - $\hookrightarrow$ Where $p' \neq 0$, $\mathbf{B} \cdot \nabla p = 0$ implies $\mathbf{B}$ must have flux surfaces.
  - $\hookrightarrow$ Where $p' = 0$, $\mathbf{B}$ can have islands, chaos and/or flux surfaces.

TRANSPORT: given $\mathbf{B}$, solve for p.

1. Given general, non-integrable magnetic field, $\mathbf{B} = \nabla \times [\psi \nabla \theta - \chi(\psi, \theta, \zeta) \nabla \zeta]$

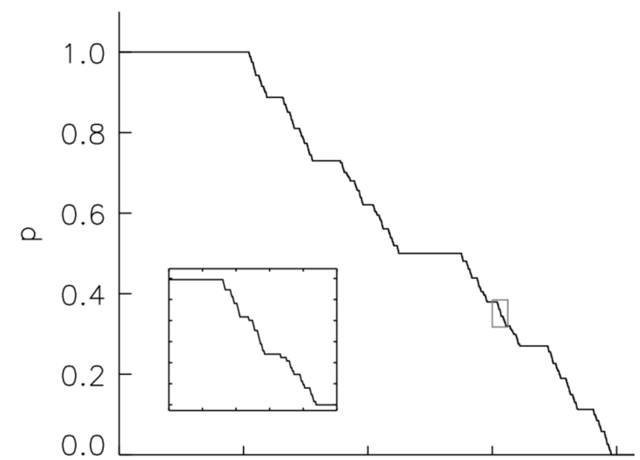
   i. fieldline Hamiltonian: $\chi(\psi, \theta, \zeta) = \chi_0(\psi) + \sum_{m,n} \chi_{m,n}(\psi) e^{i(m\theta - n\zeta)}$

2. KAM theorem: for suff. small perturbation, "sufficiently irrational" flux surfaces survive

   i. if ι satisfies a "Diophantine" condition, $|\iota - n/m| > r/m^k, \forall (n, m)$, **excluded interval about every rational**

   ii. need e.g. Greene's residue criterion to determine if flux-surface$_\iota$ exists; lot's of work;

3. With $\mathbf{B} \cdot \nabla p = 0$, i.e. infinite parallel transport, pressure profile must be fractal:



$$p'(\iota) = \begin{cases} 1, & \text{if } |\iota - n/m| > r/m^k, \quad \forall (n, m), \text{ e.g. } r = 0.2,\ k = 2, \\ 0, & \text{if } |\iota - n/m| < r/m^k, \quad \exists (n, m), \end{cases}$$

$p'(x)$ is discontinuous on an uncountable infinity of points; impossible to discretize accurately;

EQUILIBRIUM: given p, solve for $\mathbf{B}$.

Q. *Given* a fractal p', how can the topology of $\mathbf{B}$ be constrained to enforce $\mathbf{B} \cdot \nabla p = 0$?

   i. e.g. if $p(\psi)$ is continuous and smooth, nowhere zero, then $\mathbf{B}$ *must* be integrable, i.e. $\chi_{m,n}(\psi) = 0$

   ii. if $p'(\psi)$ is fractal, then what are $\chi_{m,n}(\psi) =$?

Convergence studies using VMEC

[Lazerson, Loizu et al., Phys. Plasmas **23**, 012507 (2016)]

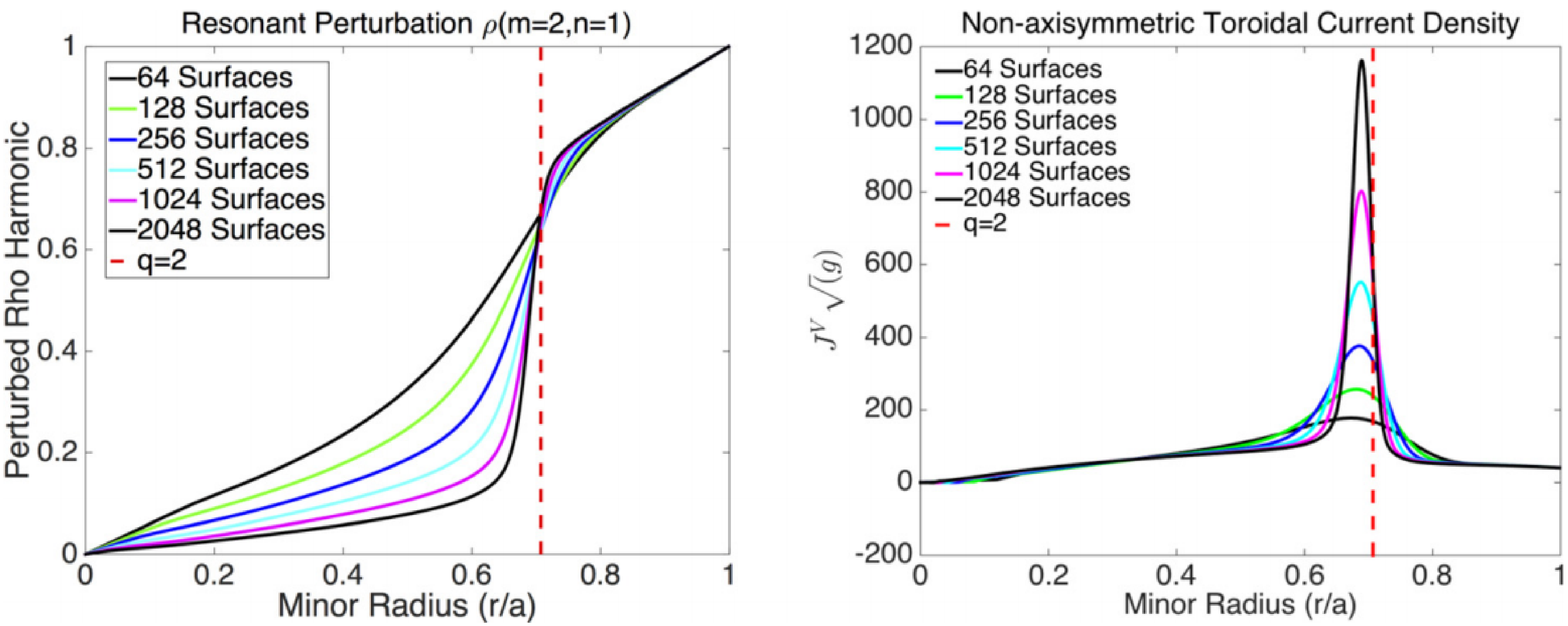


FIG. 2. Profile of the perturbed ρ harmonic (left) and the $m=2\,n=1$ component of the toroidal current density (right) showing dependence on radial resolution at fixed shear. Boundary perturbation 1×10^{-4} of minor radius. The $q=2$ surface is located at $s=0.5$ $(r/a\sim0.7)$ in this plot. Note that the toroidal current density includes a Jacobian factor.

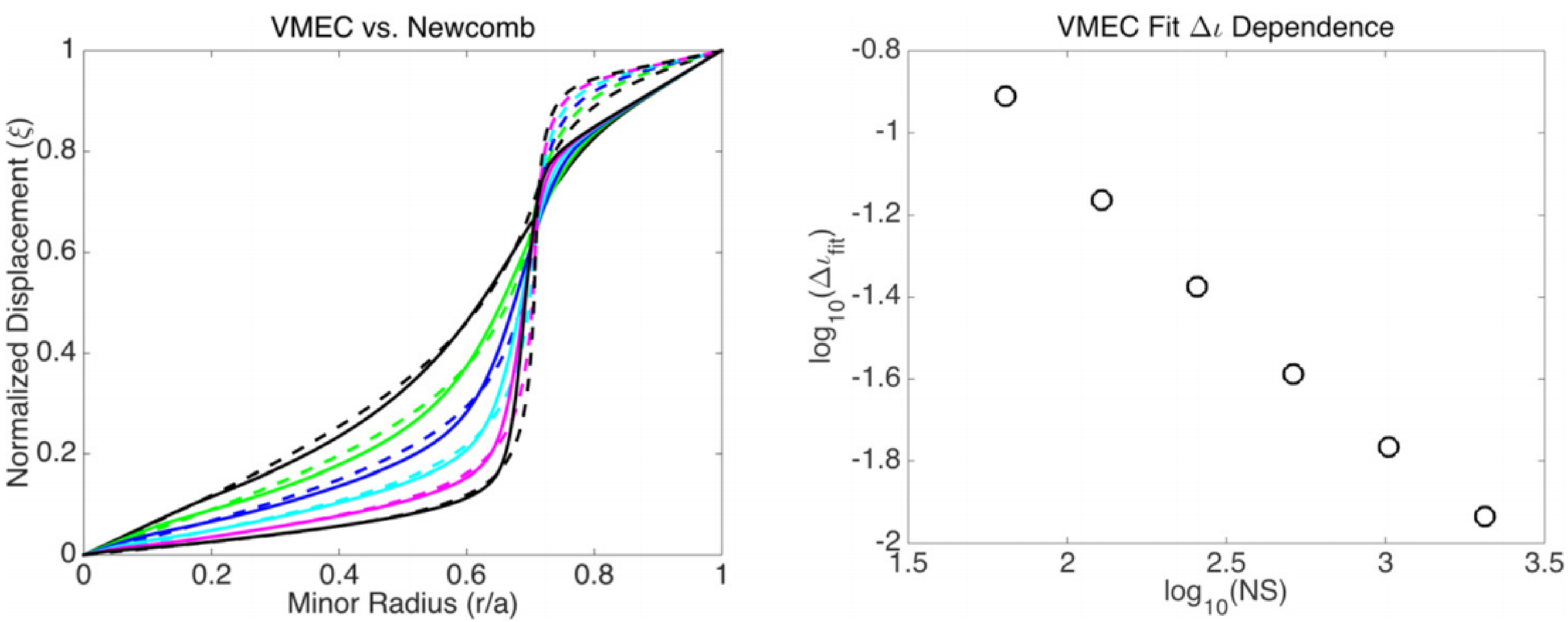


FIG. 5. Comparison of VMEC response (solid) to Loizu's solution to Newcomb's equation (dotted) (left) and the effective $\Delta\iota$ necessary to fit each curve (right). The colors are the same as those in Figure 2, and NS refers to the number of radial grid points.

Published SPEC convergence / verification calculations

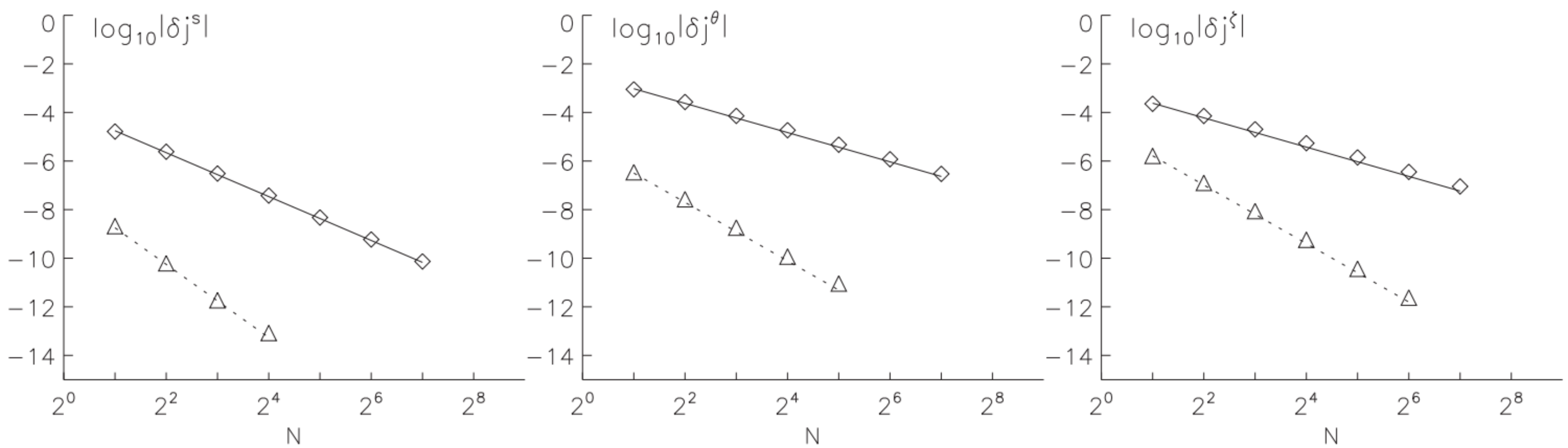


FIG. 2. Scaling of components of error, $\delta\mathbf{j} \equiv \mathbf{j} - \mu\mathbf{B}$, with respect to radial resolution. The diamonds are for the $n = 3$ (cubic) basis functions, the triangles are for the $n = 5$ (quintic) basis functions. The solid lines have gradient -3, -2, and -2, and the dotted lines have gradient -5, -4, and -4.

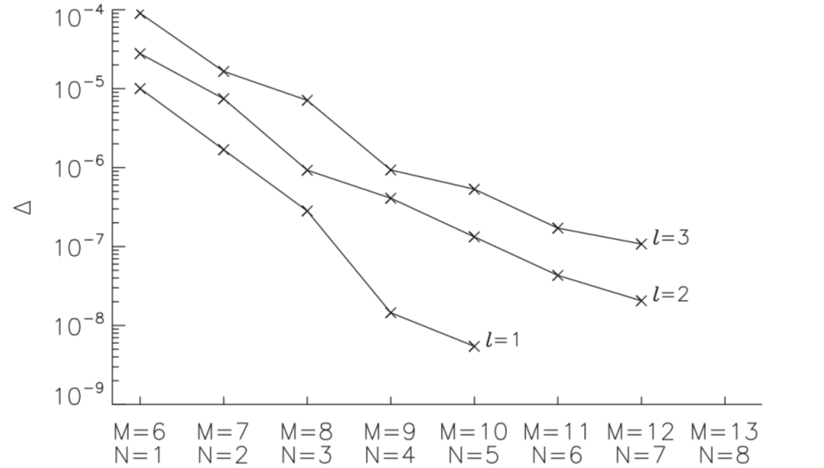


FIG. 6. Difference between finite M, N approximation to interface geometry, and a high-resolution reference approximation (with $M = 13$ and $N = 8$), plotted against Fourier resolution.

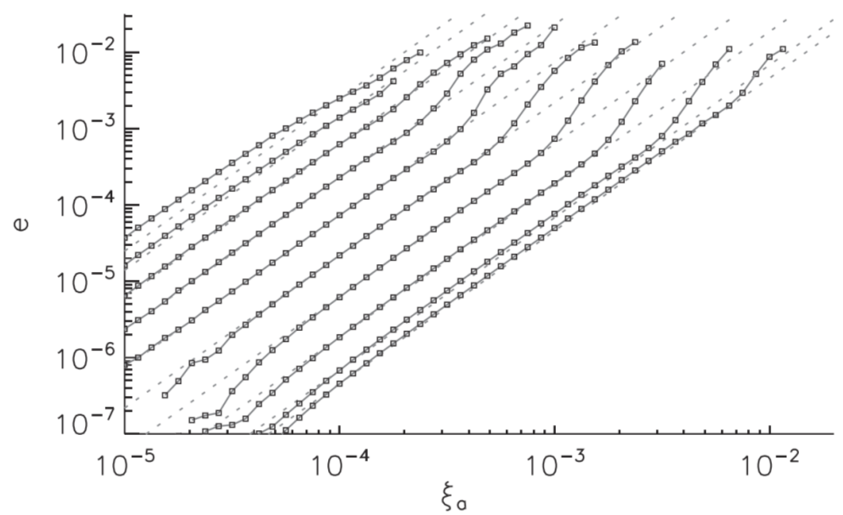


FIG. 2. Convergence of the error between linear and nonlinear SPEC equilibria as ξ_a is decreased, and for different values of $\Delta_{\imath}$, ranging from 10^{-4} (upper curve) to 10^{-1} (lower curve).

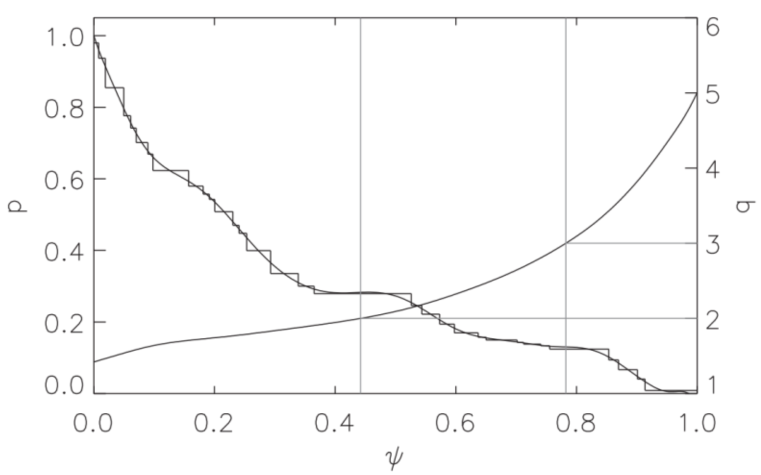


FIG. 7. Pressure profile (smooth) from a DIIID reconstruction using STELLOPT and stepped-pressure approximation. Also, shown is the inverse rotational transform $\equiv$ safety factor.

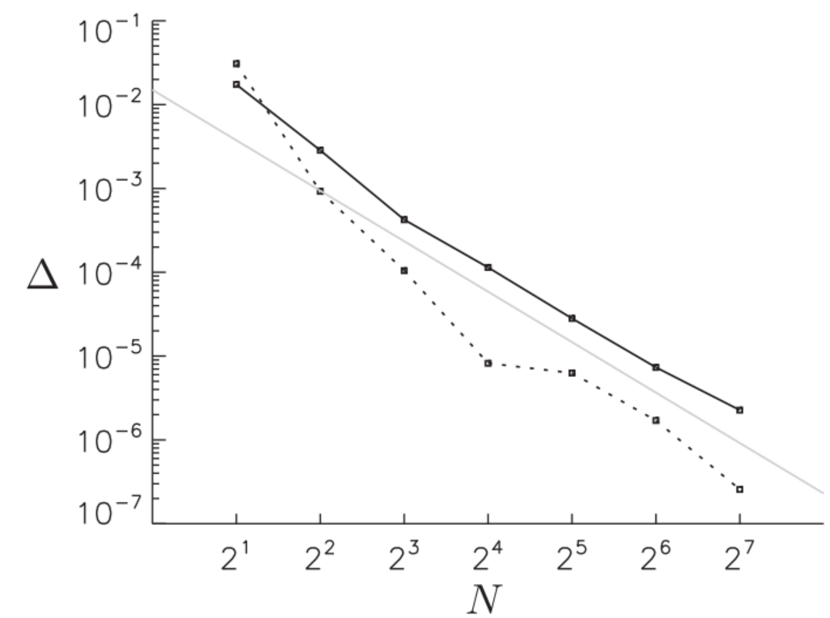


FIG. 5. Convergence: the error (Δ) between the continuous pressure (VMEC) and stepped pressure (SPEC) solutions are shown as a function of the number of plasma regions N for the $s = 1/4$ SPEC interface. The dotted line shows the zero-beta case ($p_0 = 0$), and the solid line shows the high-beta case ($p_0 = 16$). The grey line has a slope -2, the expected rate of convergence. These simulations were run on a single 3 GHz Intel Xeon 5450 CPU with the longest (the $N = 128$ case) taking 10.1 min using 20 poloidal Fourier harmonics and 768 fifth-order polynomial finite elements in the radial direction.

Early and recent publications

Hole, Hudson & Dewar,	PoP,	2006	*(theoretical model)*
Hudson, Hole & Dewar,	PoP,	2007	*(theoretical model)*
Dewar, Hole et al.,	Entropy,	2008	*(theoretical model)*
Hudson, Dewar et al.,	PoP,	2012	*(SPEC)*
Dennis, Hudson et al.,	PoP,	2013	*(MRxMHD $\rightarrow$ ideal as $N_R \rightarrow \infty$)*
Dennis, Hudson et al.,	PRL,	2013	*(helical states in RFP = double Taylor state)*
Dennis, Hudson et al.,	PoP,	2014	*(MRxMHD+flow)*
Dennis, Hudson et al.,	PoP,	2014	*(MRxMHD+flow+pressure anisotrophy)*
Loizu, Hudson et al.,	PoP,	2015	*(first ever computation of 1/x & δ current-densities in ideal-MHD)*
Loizu, Hudson et al.,	PoP,	2015	*(well-defined, 3D MHD with discontinuous transform)*
Dewar, Yoshida et al.,	JPP,	2015	*(variational formulation of MRxMHD dynamics)*
Loizu, Hudson et al.,	PoP,	2016	*(pressure amplification of RMPs)*

Recent and upcoming invited talks

Hudson, Dewar, et al.,	2012	International Sherwood Fusion Theory Conference
Dennis, Hudson, et al.,	2013	International Sherwood Fusion Theory Conference
Dennis, Hudson, et. al.,	2013	International Stellarator Heliotron Workshop
Hole, Dewar, et al.,	2014	International Congress on Plasma Physics
Loizu, Hudson, et al.,	2015	International Sherwood Fusion Theory Conference
Loizu, Hudson, et al.,	2015	APS-DPP
Hudson, Loizu et al.,	2016	International Sherwood Fusion Theory Conference
Hudson, Loizu, et al.,	2016	Asia Pacific Plasma Theory Conference, 2016
Loizu, Hudson, et al.,	2016	Varenna Fusion Theory Conference