Penetration and amplification of resonant perturbations in 3D ideal-MHD equilibria

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The 2016 International Sherwood Fusion Theory Conference

Abstract

1. Physically-meaningful, computationally-tractable solutions to $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ in 3D are fundamental.

2. In equilibria with nested magnetic surfaces and smooth pressure & transform profiles, resonant surfaces
   a. beget a non-analytic dependence on the boundary,
   b. create non-physical infinite currents.

3. Recently, for the first time, we computed the 1/x and δ-function current-densities. We introduced a new class of well-defined solutions that admit additional δ-function current-densities that produce a discontinuity in the rotational-transform that removes the singularities.

4. Our solutions yield predictions that RMPs penetrate past the rational surface and into the core of the plasma; and the perturbation is magnified by pressure inside the resonant surface, increasingly so as stability limits are approached.

Penetration and amplification of resonant perturbations in 3D ~~ideal-MHD equilibria~~ force balance, $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$

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Ideal MHD determines the plasma equilibrium and linear stability

"MHD represents the simplest self-consistent model describing the macroscopic equilibrium and stability properties of a plasma."

"The model describes how magnetic, inertial and pressure forces interact within an ideal perfectly conducting plasma in an arbitrary magnetic geometry."

"There is a general consensus that any configuration meriting consideration as a fusion reactor must satisfy the equilibrium and stability limits set by ideal MHD. If not, catastrophic termination of the plasma on a very short time scale .. is the usual consequence." *[True for tokamaks. There is growing evidence that linear stability may be irrelevant? for stellarators.]*

J.P. Freidberg, **Ideal Magnetohydrodynamics** Plenum Press, New York, 1987

Equilibrium
- Grad-Shafranov, VMEC, NSTAB,
- Reconstruction: e.g. EFIT, V3FIT, STELLOPT
- Experimental design, e.g. ITER, W7-X,
- Linearly Perturbed Equilibria, e.g. IPEC

Stability
- Kink
- Ballooning
- Peeling Ballooning (ELMs)

Transport
- Neoclassical & Turbulent transport

All presuppose a solution to $\nabla p = \mathbf{j} \times \mathbf{B}$ with nested surfaces and smooth profiles: given p and e.g. $\iota\!\!\!-$, find $\mathbf{B}$.

Topic of this talk:
Constructing well-defined solutions to $\nabla p = \mathbf{j} \times \mathbf{B}$ in 3D.

Resonant Magnetic Perturbations
- "first approximation" requires solutions to $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ in 3D geometry

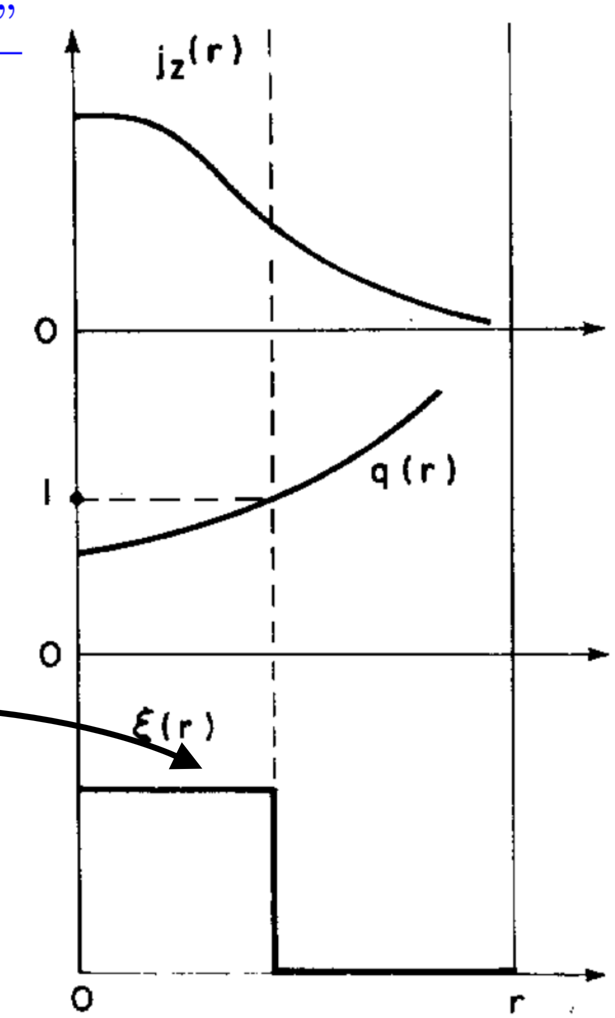
Problem: solutions to force balance with nested surfaces have a non-analytic dependence on 3D boundary

(with or without pressure)

Breakdown of perturbation theory:

Following Rosenbluth, Dagazian & Rutherford, [Phys. Fluids **16**, 1894 (1973)]

" .. we digress to discuss briefly the standard perturbation theory approach to such nonlinear problems, .. which is not applicable here due to the singular nature of the lowest order step function solution for $\boldsymbol{\xi}$"



" In the absence of such singularity we could formally expand .."

$$\boldsymbol{\xi} \;=\; \epsilon\,\boldsymbol{\xi}_1 + \epsilon^2\,\boldsymbol{\xi}_2 + \epsilon^3\,\boldsymbol{\xi}_3 + \dots$$

$$\delta\mathbf{B}[\boldsymbol{\xi}] \equiv \nabla\times(\boldsymbol{\xi}\times\mathbf{B}),$$
$$\delta p[\boldsymbol{\xi}] \equiv (\gamma-1)\boldsymbol{\xi}\cdot\nabla p - \gamma\nabla\cdot(p\boldsymbol{\xi})$$

Equilibrium and perturbed equations

$$\begin{array}{lcl}
\mathbf{F}[\mathbf{x}] & \equiv & \nabla\, p_0 - \mathbf{j}_0 \times \mathbf{B}_0 = 0 \\
\mathcal{L}_0[\boldsymbol{\xi}_1] & \equiv & \nabla\delta p[\boldsymbol{\xi}_1] - \delta\mathbf{j}[\boldsymbol{\xi}_1] \times \mathbf{B}_0 - \mathbf{j}_0 \times \delta\mathbf{B}[\boldsymbol{\xi}_1] = 0 \\
\mathcal{L}_0[\boldsymbol{\xi}_2] & = & \nabla\delta p[\boldsymbol{\xi}_2] - \delta\mathbf{j}[\boldsymbol{\xi}_2] \times \mathbf{B}_0 - \mathbf{j}_0 \times \delta\mathbf{B}[\boldsymbol{\xi}_2] = \delta\mathbf{j}[\boldsymbol{\xi}_1] \times \delta\mathbf{B}[\boldsymbol{\xi}_1] \\
\mathcal{L}_0[\boldsymbol{\xi}_3] & = & \dots \qquad = \dots
\end{array}$$

"However, since $\mathcal{L}_0$ is a singular operator .. this equation cannot, in general, be solved, .."

"leads, of course, to successively worse divergences in this perturbation theory approach which therefore breaks down .."

"we must abandon the perturbation theory approach.."

The singularity also affects Newton iterative solvers: $\mathbf{x}_{i+1} \equiv \mathbf{x}_i - \nabla\mathbf{F}^{-1}\cdot\mathbf{F}[\mathbf{x}_i]$

Problem: solutions to force-balance with nested surfaces have singularities in the parallel current-density.

$\nabla p = \mathbf{j} \times \mathbf{B}$ yields $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$. $\qquad$ $\mathbf{j}$ is current-*density*, $\quad$ current $= \int_{\mathcal{S}} \mathbf{j} \cdot d\mathbf{s}$.

Write $\mathbf{j} = \sigma \mathbf{B} + \mathbf{j}_\perp$, $\nabla \cdot \mathbf{j} = 0$ yields $\boxed{\mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp}$ (1)

Nested flux surfaces allows (ψ, θ, ζ) s.t.

$$\begin{array}{rcl} \mathbf{B} & = & \nabla\psi \times \nabla\theta + \iota \, \nabla\zeta \times \nabla\psi \\ \sqrt{g}\,\mathbf{B} \cdot \nabla & = & \partial_\zeta + \iota \, \partial_\theta \\ \sqrt{g}\,\mathbf{B} \cdot \nabla\zeta & = & 1 \end{array}$$

Fourier, $\sigma \equiv \sum_{m,n} \sigma_{m,n}(\psi) e^{i(m\theta - n\zeta)}$, Eqn(1) becomes $\boxed{(\iota m - n)\sigma_{m,n} = i(\sqrt{g} \nabla \cdot \mathbf{j}_\perp)_{m,n}}$ (2)

Resonant, parallel current-density : $\sigma_{m,n} = \underbrace{\dfrac{g_{m,n}(x)\, p'(x)}{x}}_{\text{Pfirsch-Schlüter}} + \Delta_{m,n} \underbrace{\delta_{m,n}(x)}_{\delta\text{-function}}$, where $x \equiv \iota - n/m$.

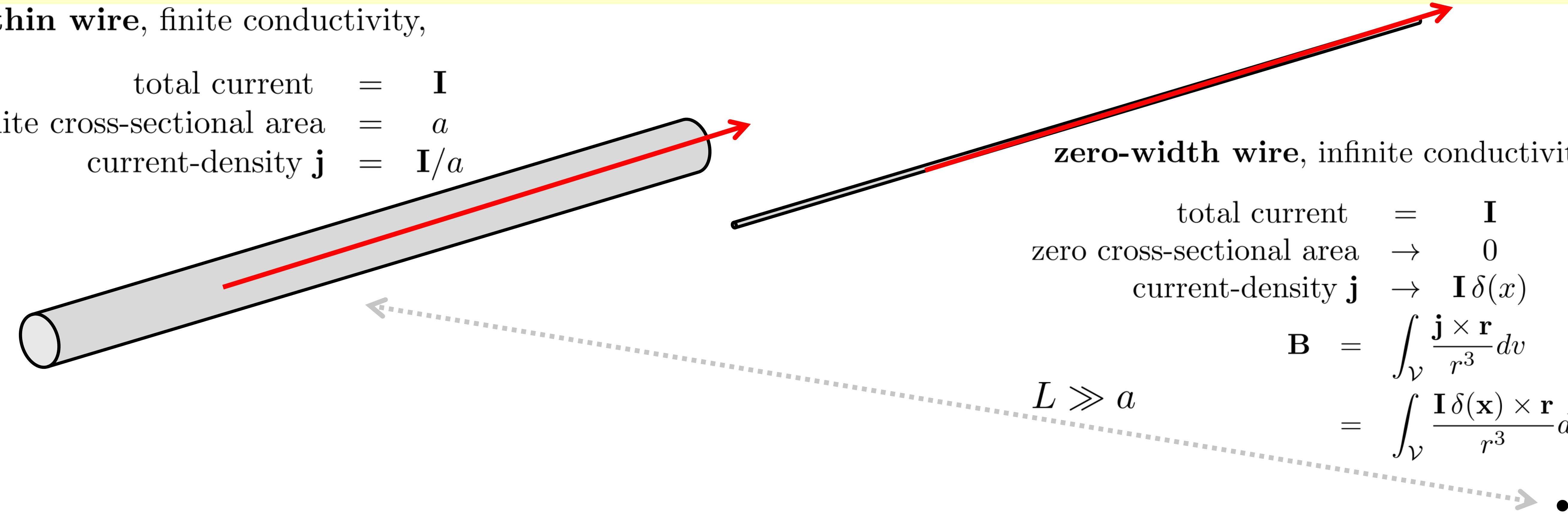
The δ-function current-density is integrable, e.g.

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0), \quad \int_{-\infty}^{\bar{x}} \delta(x)dx = H(\bar{x}) = \textit{Heaviside step function}, \; xH' = 0,$$

and is an acceptable mathematical idealization of localized currents.

thin wire, finite conductivity,

$$\begin{aligned} \text{total current} &= \mathbf{I} \\ \text{finite cross-sectional area} &= a \\ \text{current-density } \mathbf{j} &= \mathbf{I}/a \end{aligned}$$



zero-width wire, infinite conductivity,

$$\begin{aligned} \text{total current} &= \mathbf{I} \\ \text{zero cross-sectional area} &\rightarrow 0 \\ \text{current-density } \mathbf{j} &\rightarrow \mathbf{I}\,\delta(x) \\ \mathbf{B} &= \int_{\mathcal{V}} \frac{\mathbf{j} \times \mathbf{r}}{r^3} dv \\ &= \int_{\mathcal{V}} \frac{\mathbf{I}\,\delta(\mathbf{x}) \times \mathbf{r}}{r^3} dv \end{aligned}$$

Approximating a localized current-density by a δ-function current density

1. is acceptable for a **macroscopic** physical model that assumes **infinite conductivity**, and
2. is mathematically-tractable (one just needs to accommodate discontinuous solutions).

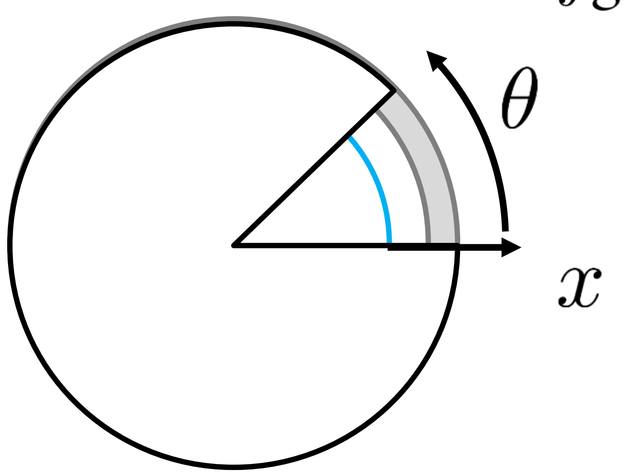
$$\begin{aligned} \text{Net current through cross-section} \quad \int_{\mathcal{S}} \mathbf{j} \cdot d\mathbf{s} &= \int d\psi \int d\theta \, \sqrt{g}\, \mathbf{j} \cdot \nabla\zeta \\ &= \int_{-\epsilon}^{+\epsilon} dx \int_0^{2\pi} d\theta \;\; \Delta_{m,n} \, \delta_{m,n}(x) \, e^{i(m\theta - n\zeta)} \sqrt{g}\, \mathbf{B} \cdot \nabla\zeta \\ \sqrt{g}\, \mathbf{B} \cdot \nabla\zeta = 1 \qquad &= 0 \\ i.e. \quad & \text{no discontinuity in rotational-transform} \end{aligned}$$

The pressure-driven 1/x current density gives infinite parallel currents through certain surfaces.

Parallel current-density $\qquad \mathbf{j}_{\parallel} = \sum_{m,n} \left[\frac{g_{m,n}\, p'}{x} + \Delta_{m,n} \delta_{m,n}(x) \right] e^{(im\theta - in\zeta)} \mathbf{B}.$

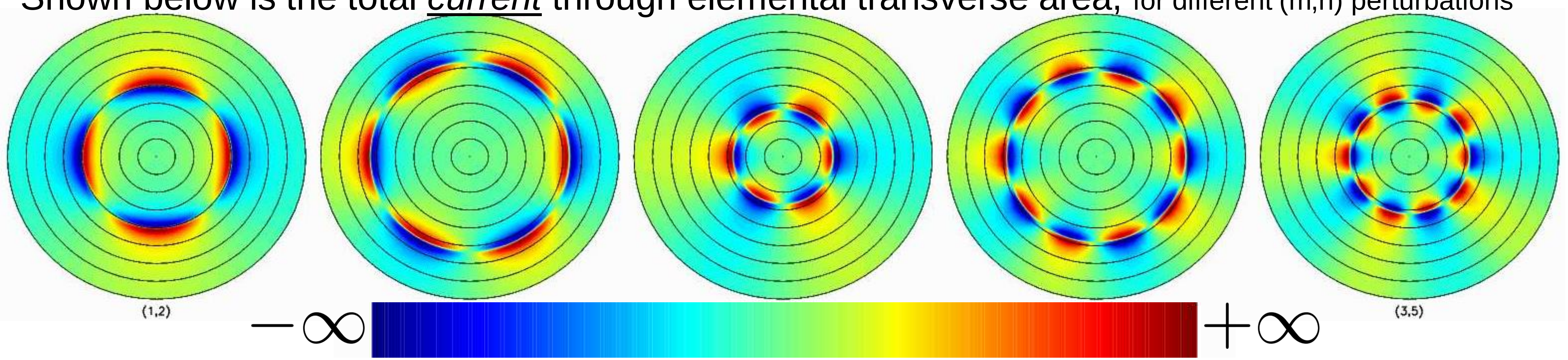
Parallel current through cross-section

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{j}_{\parallel} \cdot d\mathbf{s} &= \int d\psi \int d\theta \; \sqrt{g}\, \mathbf{j}_{\parallel} \cdot \nabla\zeta \\ &= \int_{\epsilon}^{\delta} dx \int_{0}^{\pi/m} d\theta \; \frac{g_{m,n}\, p'}{x} e^{i(m\theta - n\zeta)} \sqrt{g}\, \mathbf{B} \cdot \nabla\zeta \\ &= g_{m,n,0}\, p'_0 \frac{2}{m} \int_{\epsilon}^{\delta} dx \frac{1}{x} \\ &= g_{m,n,0}\, p'_0 \frac{2}{m} (\ln\delta - \ln\epsilon) \to \infty \text{ as } \epsilon \to 0. \end{aligned}$$



The problem is *NOT* a lack of numerical resolution.
Is a dense collection of alternating infinite currents physical?

Shown below is the total <u>*current*</u> through elemental transverse area, for different (m,n) perturbations



If there are rational surfaces, then we must choose:

1. flatten pressure near rationals, smooth pressure; ✗
2. flatten pressure near rationals, fractal pressure; ✗
3. flatten pressure near rationals, discontinuous pressure; ✓
4. restrict attention to "healed" configurations [Weitzner, PoP **21**, 022515, 2014], [Zakharov, JPP **81**, 515810609, 2015]

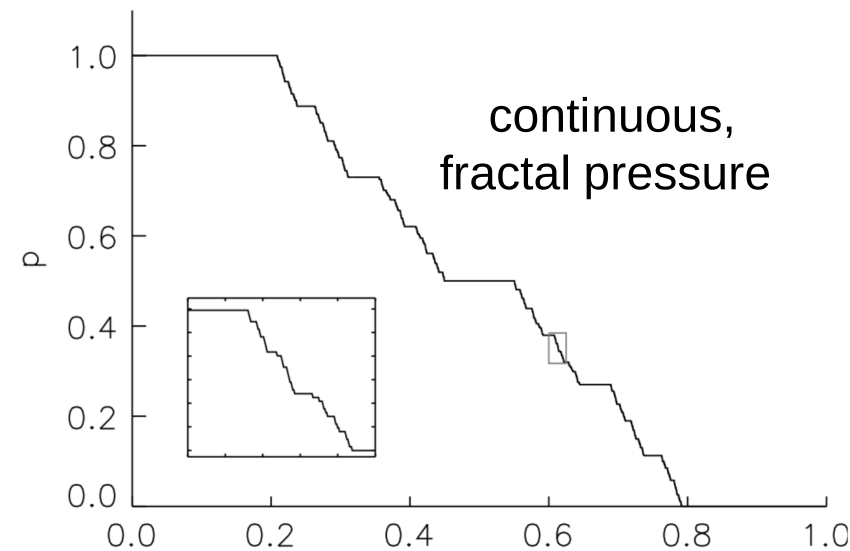
1. **Locally-flattened, smooth pressure:**

if $(i.)$ $p'(x) = 0$ if $|x - n/m| < \epsilon_{m,n}$, $\forall (n,m)$,

and $(ii.)$ $p'(x)$ is continuous, then $p'(x) = 0$, $\forall x$. **No pressure!**

2. **"Diophantine" pressure profile:** **e.g. from KAM theory**

$$p'(x) = \begin{cases} 1, & \text{if } |x - n/m| > r/m^k, \quad \forall (n,m), \text{ e.g. } r = 0.2,\ k = 2, \\ 0, & \text{if } |x - n/m| < r/m^k, \quad \exists (n,m), \end{cases}$$



$p'(x)$ is discontinuous on an uncountable infinity of points,

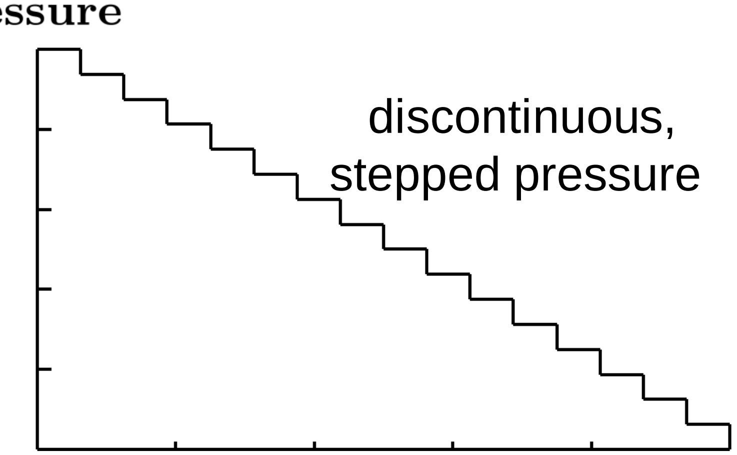
Not computationally tractable.

e.g. cannot constrain topology of non-integrable **B** to match fractal pressure

"The function p is continuous but its derivative is pathological." Grad, Phys. Fluids 10, 137 (1967)]

3. **"Stepped" pressure profile:** ✓

Existence of Three-Dimensional Toroidal MHD Equilibria with Nonconstant Pressure
[Bruno & Laurence, Commun. Pure Appl. Math. **49**, 717 (1996)]
" . . our theorems insure the existence of sharp boundary solutions for tori
whose departure from axisymmetry is sufficiently small;
they allow for solutions to be constructed with an arbitrary number of pressure jumps."



Culmination of long history of "waterbag" or "sharp-boundary" equilibria
[Potter, "Waterbag methods in magnetohydrodynamics", Methods in Computational Physics, **16**, 43 (1976)]
[Berk et al., Phys. Fluids, **29**, 3281 (1986)]
[Kaiser & Salat Phys. Plasmas **1**, 281 (1994)]

Relaxed MHD $\leftarrow$ Multi-Region relaxed MHD $\rightarrow$ Ideal MHD

[Taylor, Phys. Rev. Lett. **33**, 1139 (1974)] [Dewar, Hole, Hudson, et al., circa 2006] [Kruskal & Kulsrud, Phys. Fluids **1**, 265 (1958)]

$N_V = 1$ **Relaxed MHD**

$$\mathcal{F} \equiv \underbrace{\int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{energy} - \frac{\mu}{2} \underbrace{\int_{\mathcal{R}} \mathbf{A} \cdot \mathbf{B} \, dv}_{helicity},$$

$\delta \mathbf{B} \equiv \nabla \times \delta \mathbf{A}$ is arbitrary in $\mathcal{R}$

$(\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ on $\partial \mathcal{R})$

$+$ constrained flux

$$\delta \mathcal{F} = 0, \quad p = p_0, \quad \nabla \times \mathbf{B} = \mu \, \mathbf{B} \text{ in } \mathcal{R};$$

$N_V = \infty$ **Ideal MHD**

$$\mathcal{F} \equiv \int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv,$$

$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ in $\mathcal{R}$

(fluxes & helicity conserved)

$$\delta \mathcal{F} = 0, \quad p = p(\psi), \quad \nabla p = \mathbf{j} \times \mathbf{B} \text{ in } \mathcal{R}.$$

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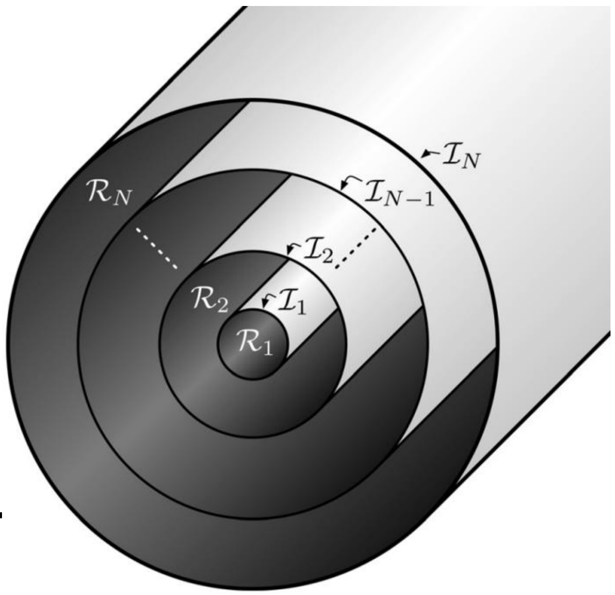
$N_V < \infty$ **MRx MHD**

$$\mathcal{F} \equiv \sum_{i=1}^{N_V} \left\{ \int_{\mathcal{R}_i} \left(\frac{p}{\gamma-1} + \frac{B^2}{2} \right) dv - \frac{\mu_i}{2} \int_{\mathcal{R}_i} \mathbf{A}\cdot\mathbf{B}\, dv \right\},$$

$\delta\mathbf{B}_i \equiv \nabla\times\delta\mathbf{A}_i$ is arbitrary in $\mathcal{R}_i$
$\delta\mathbf{B}_i = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}_i)$ on $\partial\mathcal{R}_i$
$+$ constrained fluxes in $\mathcal{R}_i$

$$\delta\mathcal{F} = 0, \quad p = p_i, \quad \nabla\times\mathbf{B} = \mu_i\,\mathbf{B} \ \text{ in } \mathcal{R}_i; \quad \left[\left[p + \frac{B^2}{2} \right]\right] = 0 \text{ across } \partial\mathcal{R}_i;$$



$\boxed{\textbf{S}\text{tepped } \textbf{P}\text{ressure } \textbf{E}\text{quilibrium } \textbf{C}\text{ode}}$

[Hudson, Dewar et al., Phys. Plasmas **19**, 112502 (2012)]

$N_V = \infty$ **Ideal MHD**

$$\mathcal{F} \equiv \int_{\mathcal{R}} \left(\frac{p}{\gamma-1} + \frac{B^2}{2} \right) dv,$$

$\delta\mathbf{B} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B})$ in $\mathcal{R}$
(fluxes & helicity conserved)

$$\delta\mathcal{F} = 0, \quad p = p(\psi), \quad \nabla p = \mathbf{j}\times\mathbf{B} \ \text{ in } \mathcal{R}.$$

Relaxed MHD ← Multi-Region relaxed MHD → Ideal MHD

[Taylor, Phys. Rev. Lett. **33**, 1139 (1974)] [Dewar, Hole, Hudson, et al., circa 2006] [Kruskal & Kulsrud, Phys. Fluids **1**, 265 (1958)]

$N_V = 1$ Relaxed MHD

$$\mathcal{F} \equiv \underbrace{\int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{energy} - \frac{\mu}{2} \underbrace{\int_{\mathcal{R}} \mathbf{A} \cdot \mathbf{B} \, dv}_{helicity},$$

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$(\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ on $\partial \mathcal{R})$

$+$ constrained flux

$$\delta \mathcal{F} = 0, \quad p = p_0, \quad \nabla \times \mathbf{B} = \mu \mathbf{B} \text{ in } \mathcal{R};$$

$N_V < \infty$ MRx MHD

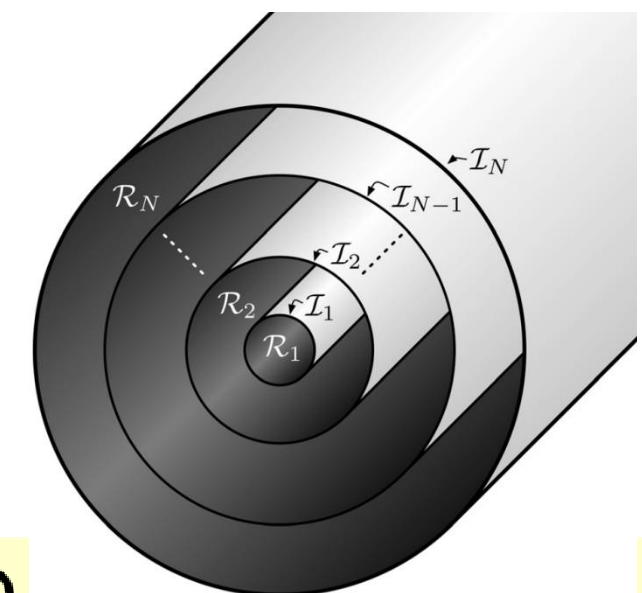
$$\mathcal{F} \equiv \sum_{i=1}^{N_V} \left\{ \int_{\mathcal{R}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv - \frac{\mu_i}{2} \int_{\mathcal{R}_i} \mathbf{A} \cdot \mathbf{B} \, dv \right\},$$

$\delta \mathbf{B}_i \equiv \nabla \times \delta \mathbf{A}_i$ is arbitrary in $\mathcal{R}_i$

$\delta \mathbf{B}_i = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_i)$ on $\partial \mathcal{R}_i$

$+$ constrained fluxes in $\mathcal{R}_i$

$$\delta \mathcal{F} = 0, \quad p = p_i, \quad \nabla \times \mathbf{B} = \mu_i \mathbf{B} \text{ in } \mathcal{R}_i; \quad \left[\left[p + \frac{B^2}{2} \right]\right] = 0 \text{ across } \partial \mathcal{R}_i;$$



$$\rightarrow \quad p(\psi), \nabla p = \mathbf{j} \times \mathbf{B} \quad \text{as} \quad N_V \rightarrow \infty,$$

[Dennis, Hudson et al., Phys. Plasmas **20**, 032509, 2013]

Our approach for computing "ideal" force-balance in 3D

1) multi-region relaxed MHD equilibria are well-defined in 3D,
2) take limit as $N_V \rightarrow \infty$ to study $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ with nested surfaces and smooth pressure

Compute the 1/x and δ-function current densities in perturbed geometry
Self-consistent solutions require **infinite shear**

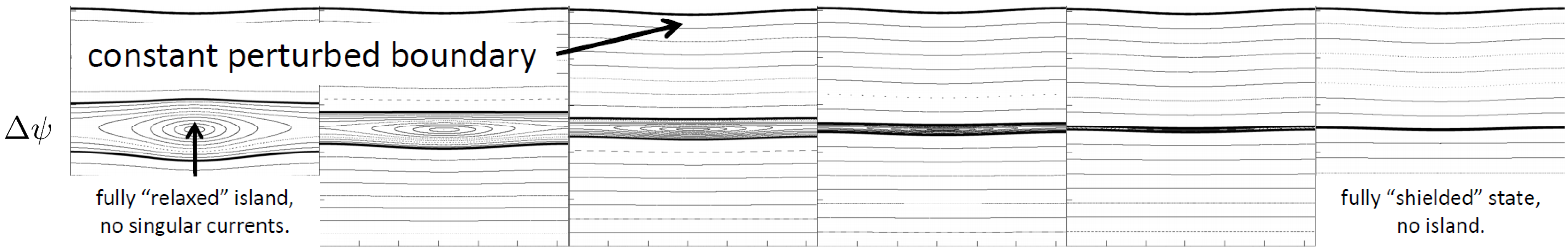
Cartesian, slab geometry with an $(m,n)=(1,0)$ resonantly-perturbed boundary

i. $N_V=3$ MRxMHD calculation, no pressure, $\iota(\psi)$ given discretely,

ii. take limit $\Delta\psi \equiv x^\beta$, $\iota_i=-x^\alpha/2$, $\iota_{i+1}=+x^\alpha/2$, shear $\equiv \Delta\iota/\Delta\psi = x^{\alpha-\beta}$, $\boxed{\beta>\alpha}$.

iii. island forced to vanish,

iv. resonant $\delta_{m,n}$-function current-density appears as tangential discontinuity in $\mathbf{B}$.

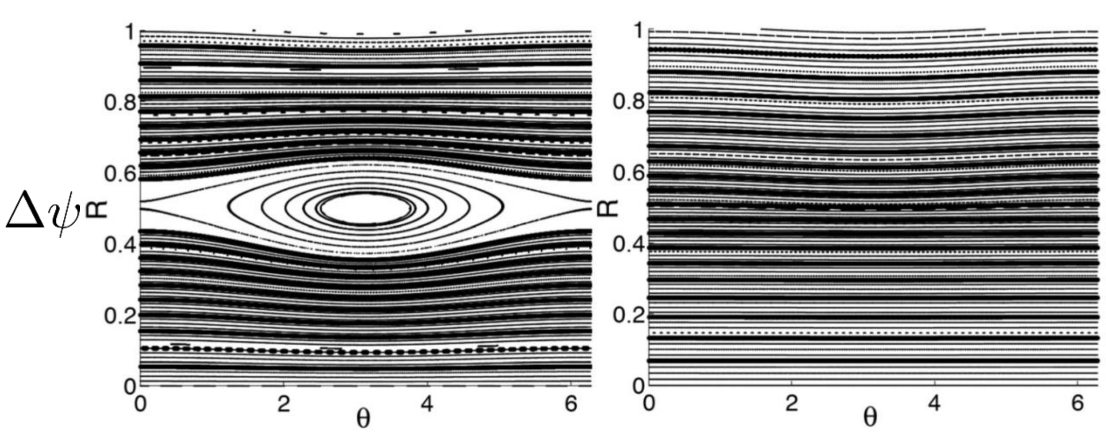


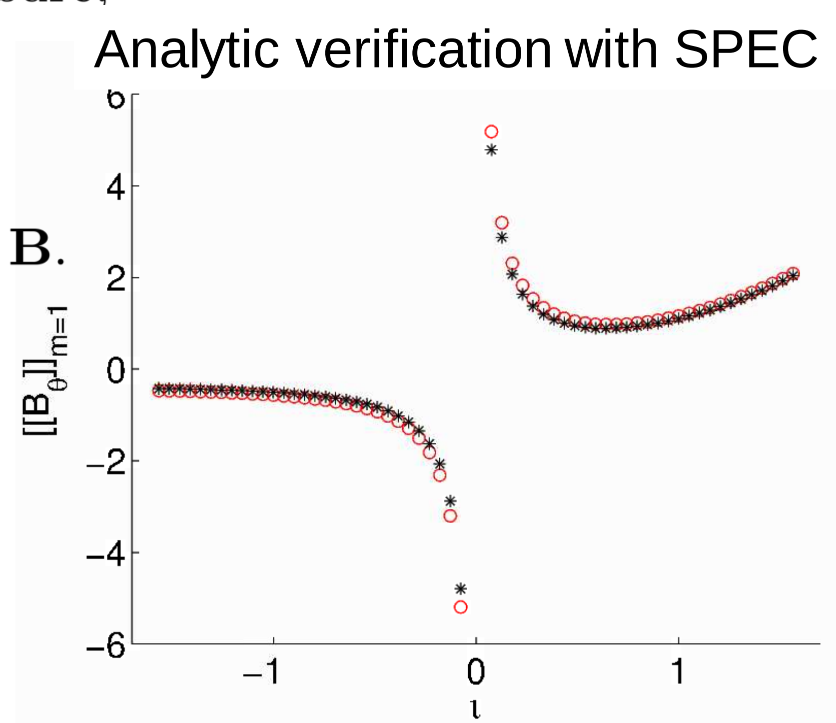
i. $N_V=$ large MRxMHD calculation, stepped pressure $\approx$ smooth pressure,

ii. take limit $\Delta\psi \equiv x^\beta$, $\iota_i=-x^\alpha/2$, $\iota_{i+1}=+x^\alpha/2$,

iii. island forced to vanish,

iv. resonant p'/x current-density appears as tangential discontinuity in $\mathbf{B}$.





[Loizu, Hudson et al., Phys. Plasmas **22**, 022501 (2015)]

Infinite gradient ≈ discontinuity.
Introduce new solutions to $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ with discontinuous transform

1. Cylindrical geometry with an $(m, n) = (2, 1)$ resonantly-perturbed boundary

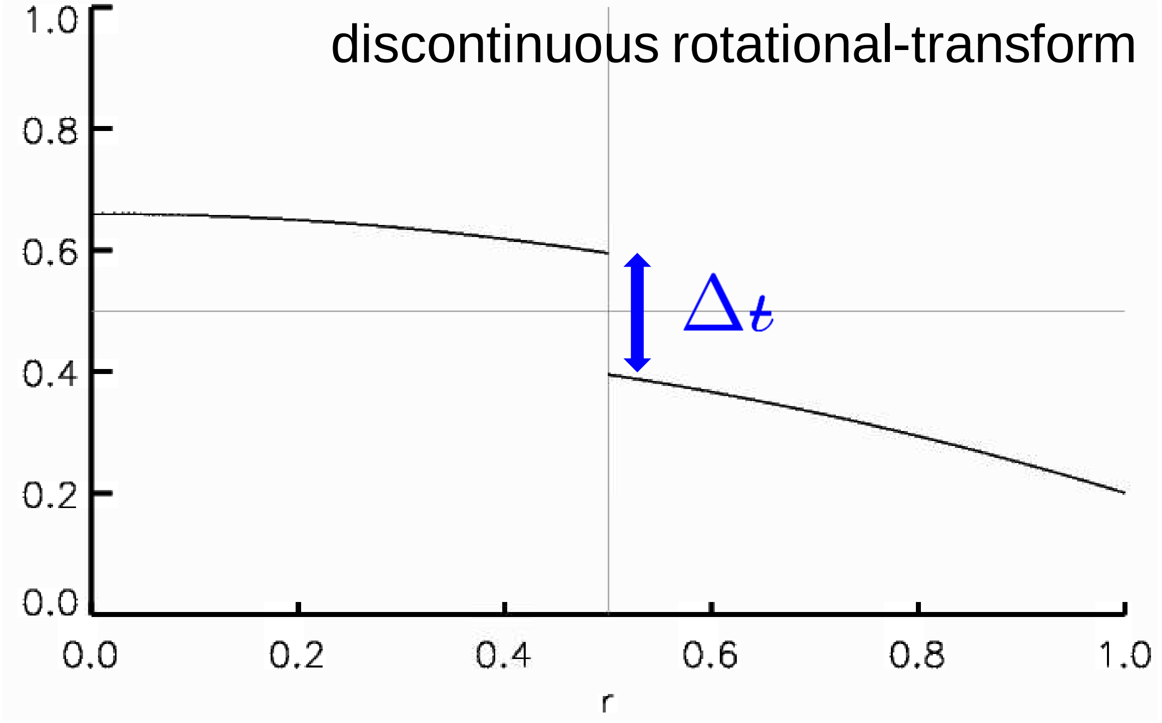
   i. $\boxed{p = 0,}$ $\iota(r) = \iota_0 - \iota_1 r^2 \pm \Delta\iota,$

   ii. compute cylindrically symmetric equilibrium
   $$\frac{dp}{dr} + \frac{1}{2}\frac{d}{dr}\left[B_z(1 + \iota^2 r^2)\right] + r\iota^2 B_z^2 = 0$$

   iii. compute linearly perturbed equilibrium:
   $$\mathcal{L}_0[\boldsymbol{\xi}] \equiv \boxed{} - \delta\mathbf{j}[\boldsymbol{\xi}] \times \mathbf{B}_0 - \mathbf{j}_0 \times \delta\mathbf{B}[\boldsymbol{\xi}] = 0$$
   for $\Delta\iota > 0$, $\mathcal{L}_0$ is non-singular,

   iv. solved analytically
   $$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\,\xi = 0$$

   v. for $\Delta\iota > \Delta\iota_{min}, \partial\xi/\partial r < 1$, non-overlapping perturbed surfaces
   for $\Delta\iota > 0$, $\boldsymbol{\xi}$ is continuous and smooth,
   for $\Delta\iota \to 0$, recover step-function solution



Perturbation penetrates into the core

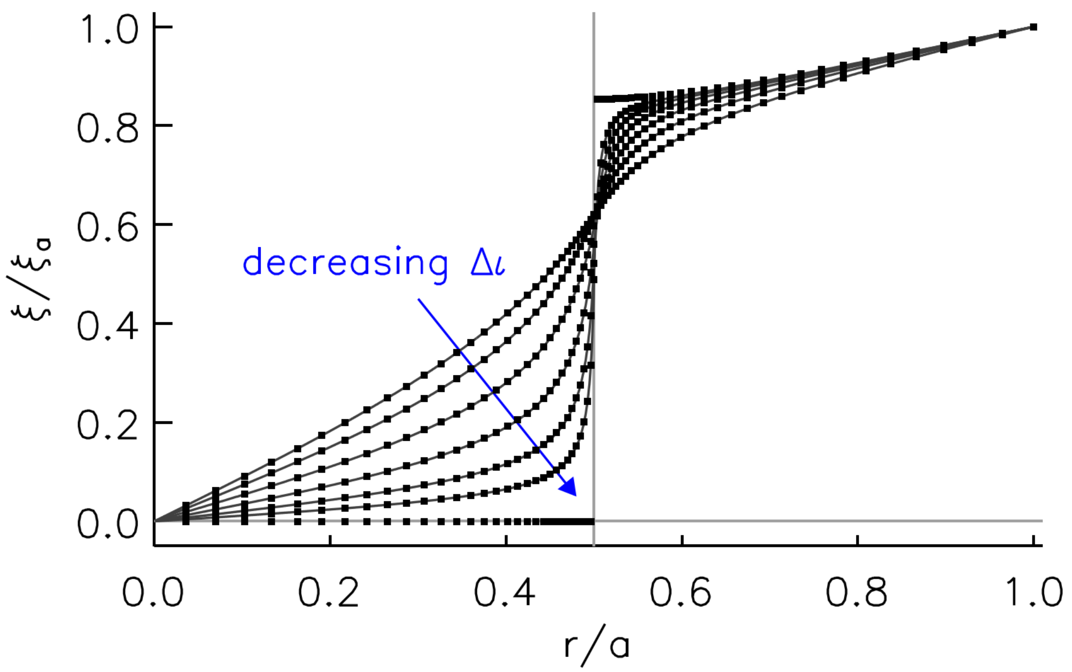
2. Comparison with SPEC

   i. construct large N_V MRxMHD calculation,

   ii. "linearized" SPEC calculation: $||\boldsymbol{\xi}_{exact} - \boldsymbol{\xi}_{linear}|| \sim N_V^{-1}$

   iii. nonlinear SPEC calculation: $||\boldsymbol{\xi}_{linear} - \boldsymbol{\xi}_{nonlinear}|| \sim \epsilon^2$

Analytic verification with SPEC



[Loizu, Hudson et al., Phys. Plasmas **22**, 090704 (2015)]

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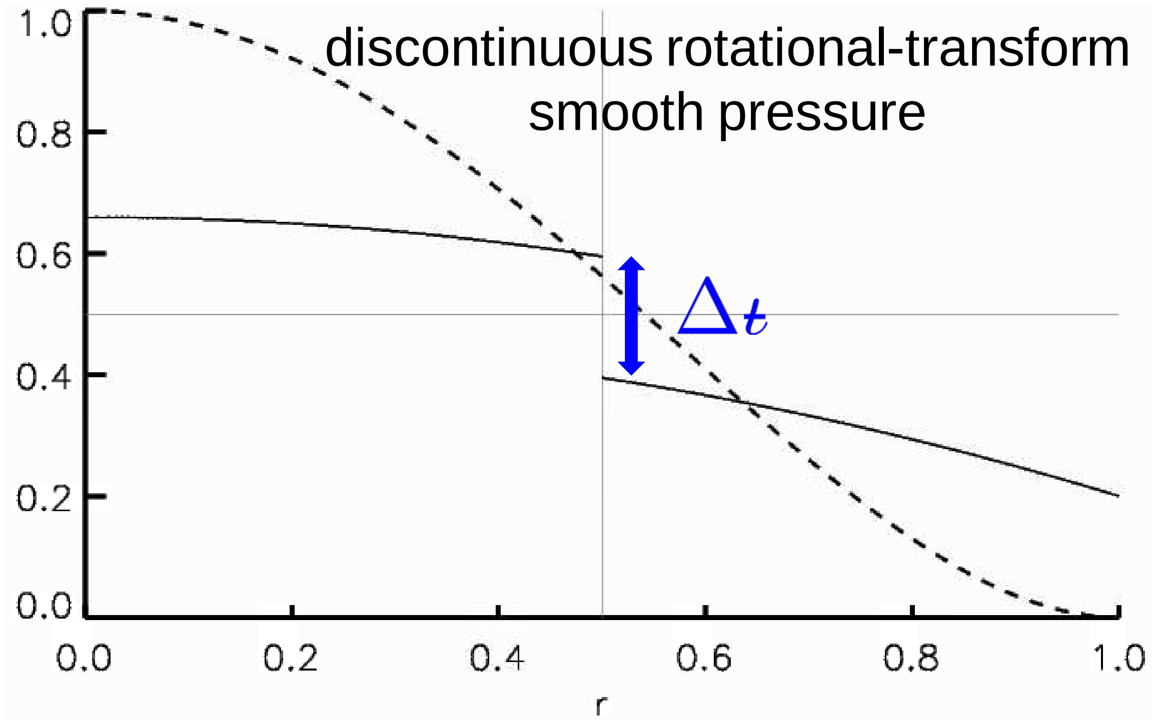
   i. $\boxed{p = p_0(1 - 2r^2 + r^4),}$ $\iota(r) = \iota_0 - \iota_1 r^2 \pm \Delta\iota,$

   ii. compute cylindrically symmetric equilibrium
   $$\frac{dp}{dr} + \frac{1}{2}\frac{d}{dr}\left[B_z(1 + \iota^2 r^2)\right] + r\iota^2 B_z^2 = 0$$

   iii. compute linearly perturbed equilibrium:
   $$\mathcal{L}_0[\boldsymbol{\xi}] \equiv \boxed{\nabla \delta p} - \delta\mathbf{j}[\boldsymbol{\xi}] \times \mathbf{B}_0 - \mathbf{j}_0 \times \delta\mathbf{B}[\boldsymbol{\xi}] = 0$$
   for $\Delta\iota > 0$, $\mathcal{L}_0$ is non-singular,

   iv. solved analytically
   $$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\,\xi = 0$$

   v. for $\Delta\iota > \Delta\iota_{min}, \partial\xi/\partial r < 1$, non-overlapping perturbed surfaces
   for $\Delta\iota > 0$, $\boldsymbol{\xi}$ is continuous and smooth,
   for $\Delta\iota \to 0$, recover step-function solution



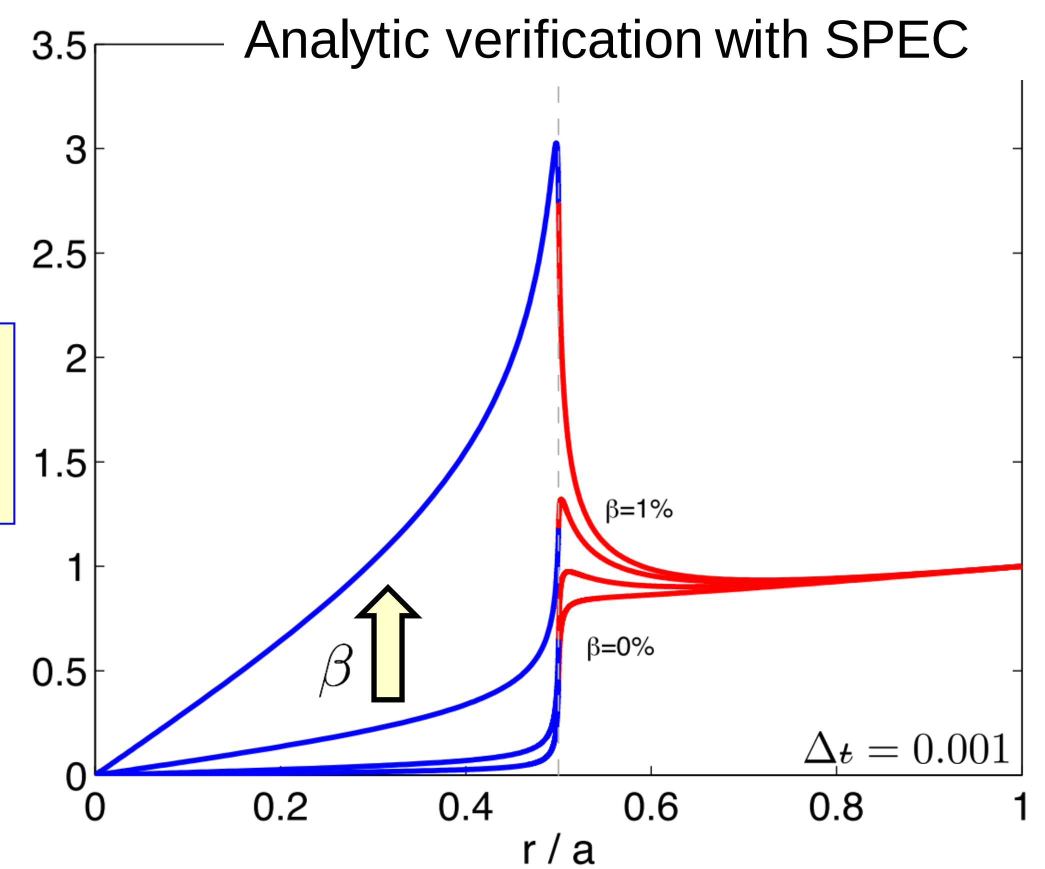
Perturbation amplified by pressure near and inside "resonant" surface

2. Comparison with SPEC

   i. construct large N_V MRxMHD calculation,

   ii. "linearized" SPEC calculation: $||\boldsymbol{\xi}_{exact} - \boldsymbol{\xi}_{linear}|| \sim N_V^{-1}$

   iii. nonlinear SPEC calculation: $||\boldsymbol{\xi}_{linear} - \boldsymbol{\xi}_{nonlinear}|| \sim \epsilon^2$

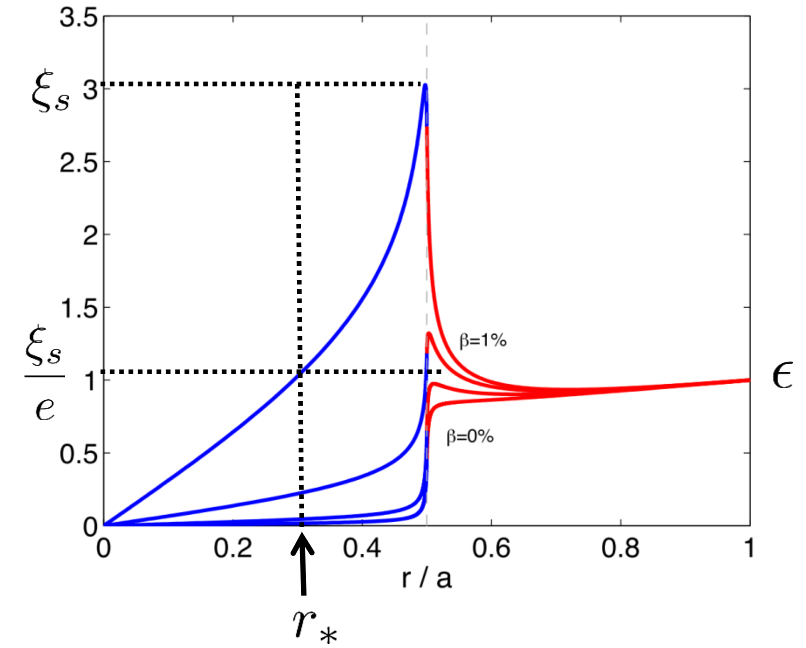


[Loizu, Hudson et al., Phys. Plasmas **23**, 055703 (2016)]

Amplification and penetration as stability boundary is approached

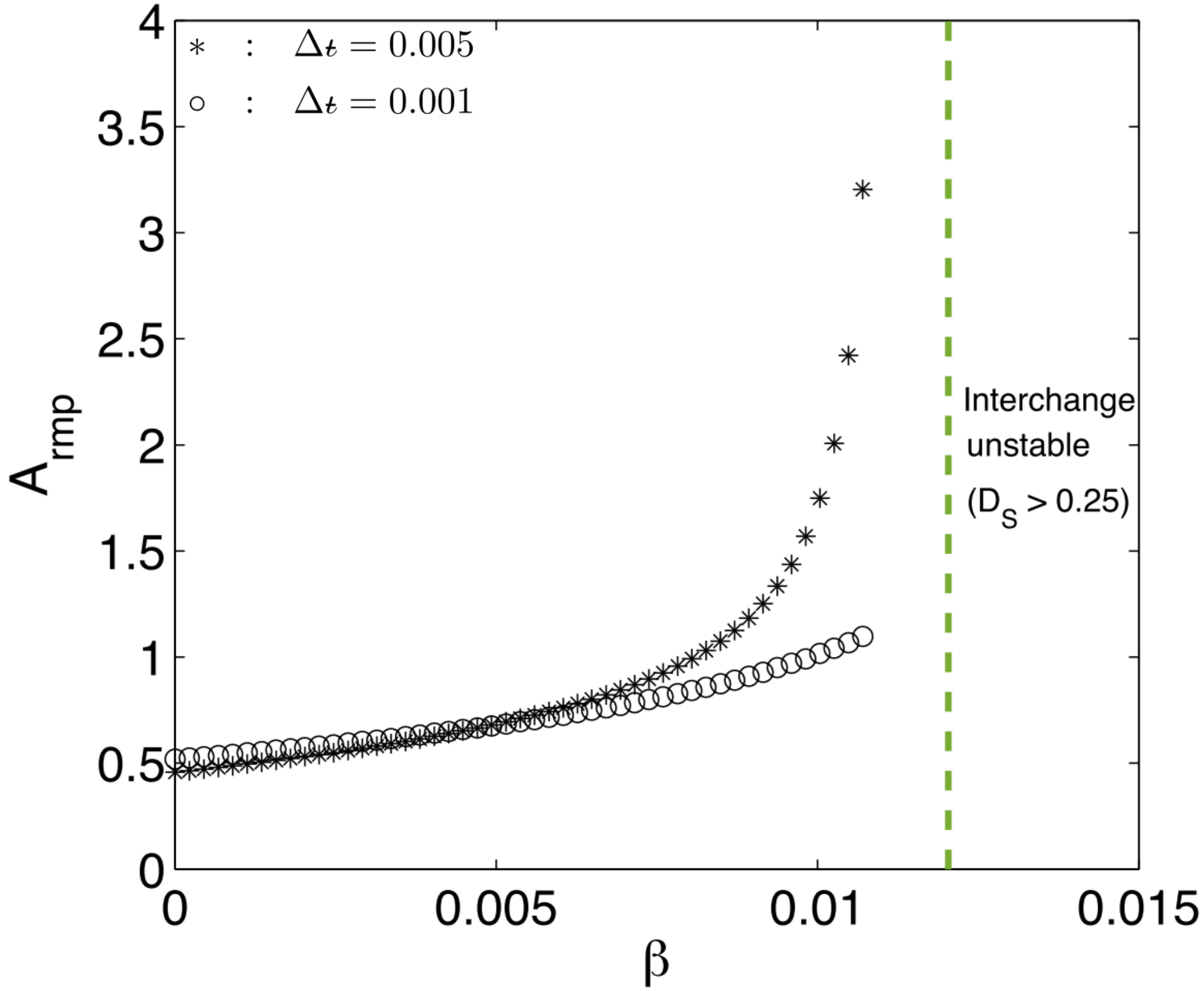
1. Can define a measure of

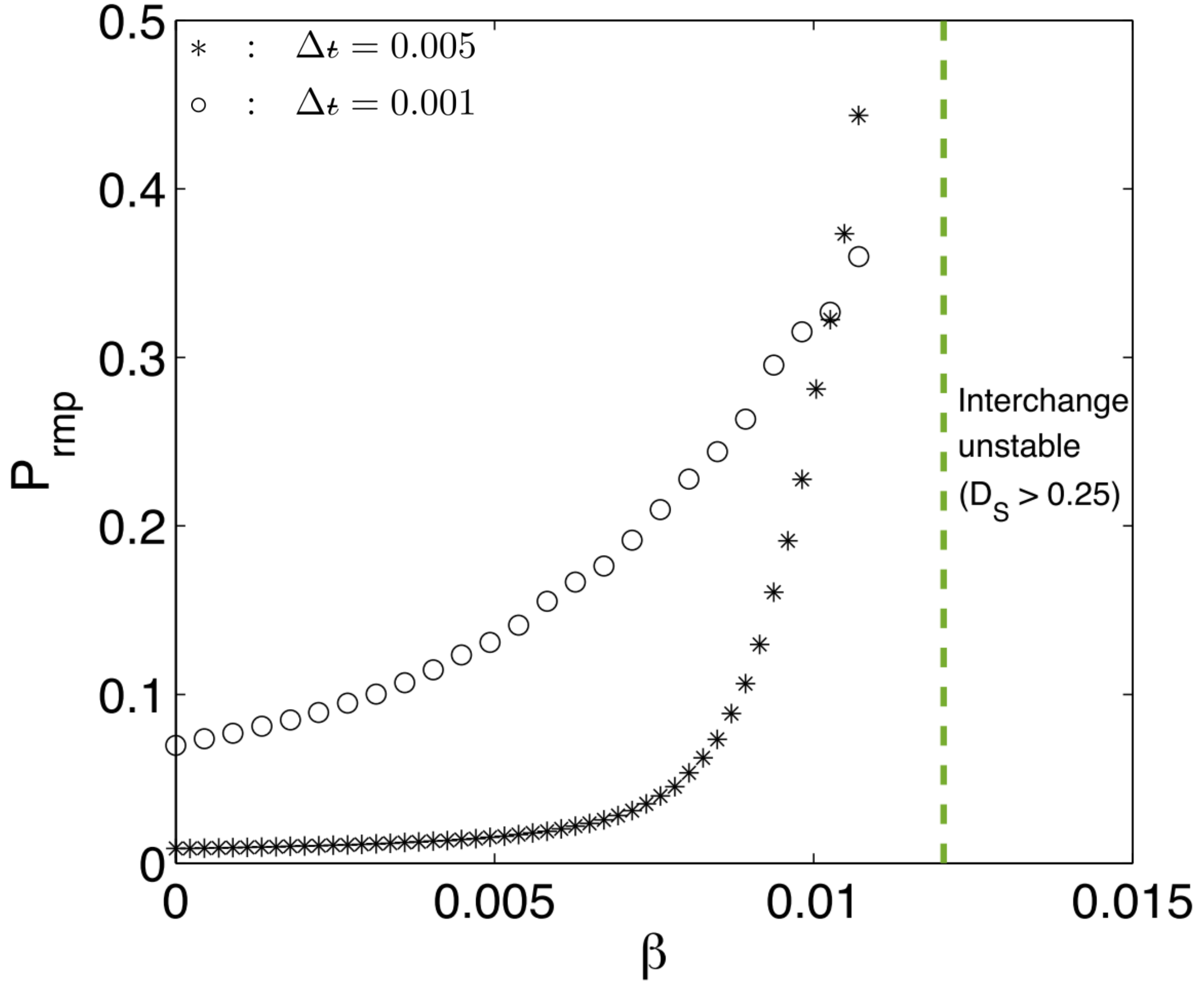
$$\text{“Amplification”} \quad A_{rmp} = \xi_s/\epsilon, \quad \text{where } \epsilon \equiv \text{boundary deformation}$$
$$\text{“Penetration”} \quad P_{rmp} = 1 - r_*/r_s, \quad \text{where } \xi(r_*) \equiv \xi_s/e$$



2. A necessary condition for interchange stability in a screw pinch is given by the Suydam criterion, $\boxed{D_S \equiv -\left(\frac{2p'\iota^2}{rB_z^2\iota'^2}\right)_s < \frac{1}{4}.}$

3. Amplification and penetration of RMP **fantastically increased** as stability limit approached.





[Loizu, Hudson et al., Phys. Plasmas **23**, 055703 (2016)]

Now, including pressure and an island . . .
Amplification and penetration of the RMP is still present.

1. Now, include a "relaxed" region,

   i. $\Delta\psi_t \equiv$ toroidal flux in relaxed region.

   ii. $\Delta \iota \equiv$ jump in transform across relaxed region.

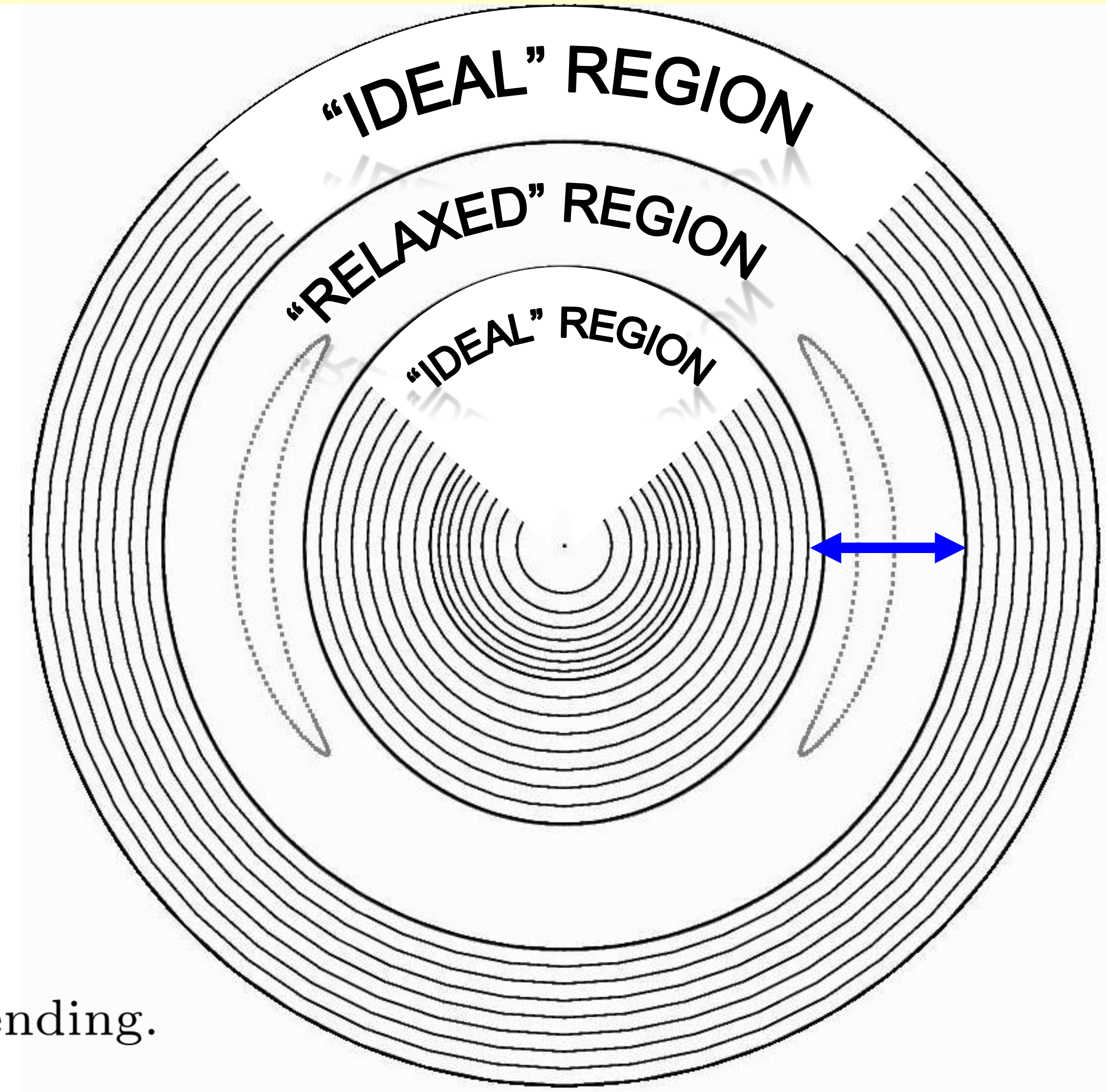
   so that an island is allowed to form.

2. SPEC calculations indicate that

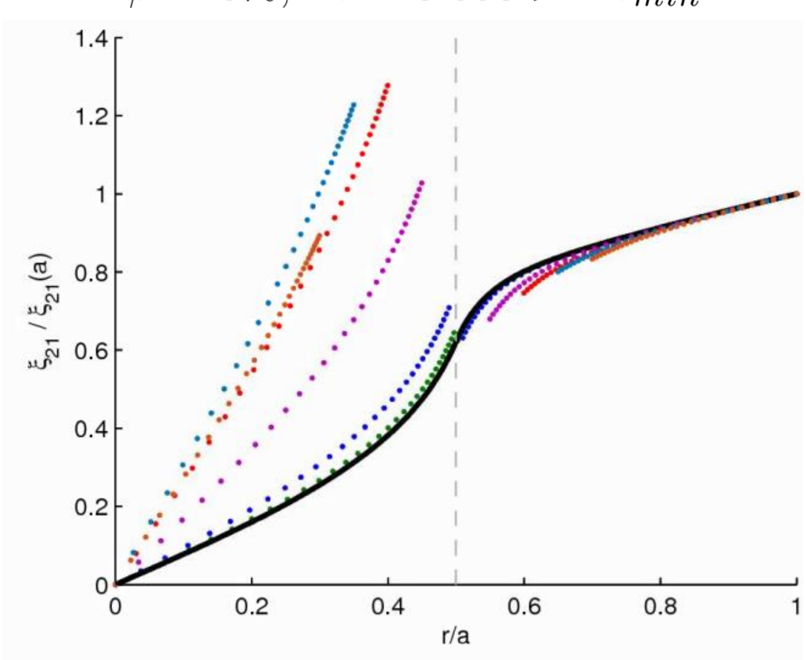
   i. The perturbation still penetrates.

   ii. The perturbation is still amplified by pressure.

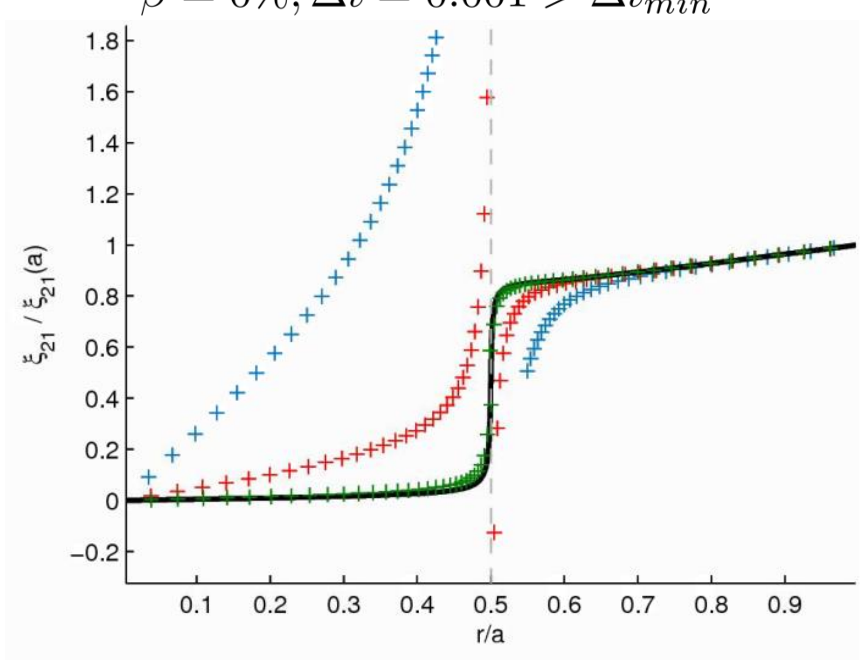
3. Precise comparison of SPEC cf. tearing mode theory pending.

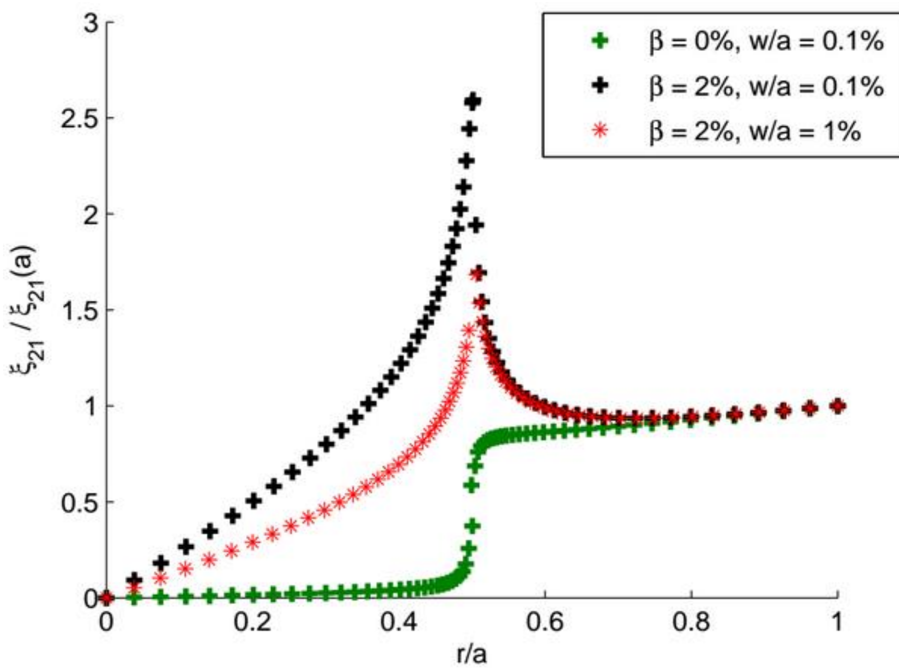


$\beta = 0\%, \Delta \iota = 0.050 > \Delta \iota_{min}$



$\beta = 0\%, \Delta \iota = 0.001 > \Delta \iota_{min}$





SPEC	allows	discontinuous	profiles: exact	agreement
VMEC	assumes	smooth	profiles: approximate	agreement

1. VMEC assumes smooth profiles

   and smooth profiles imply discontinuous displacement

2. but, VMEC enforces nested flux surfaces

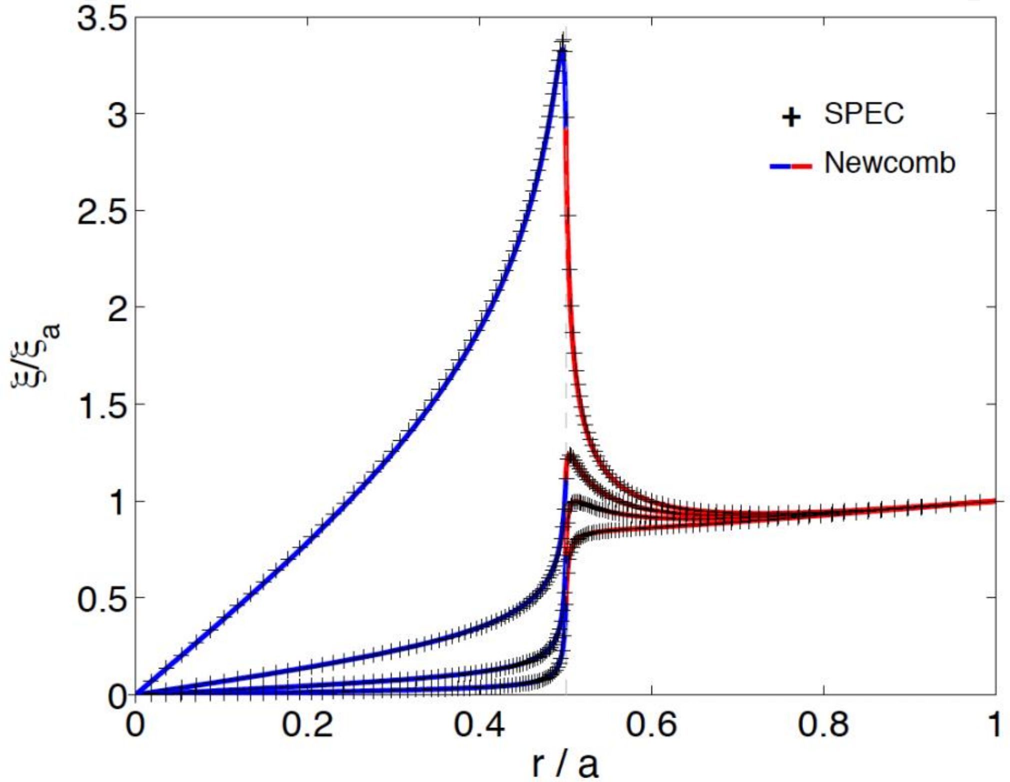
   nested flux surfaces in 3D imply $\dfrac{\partial \xi}{dr} < 1$ displacement from 2D

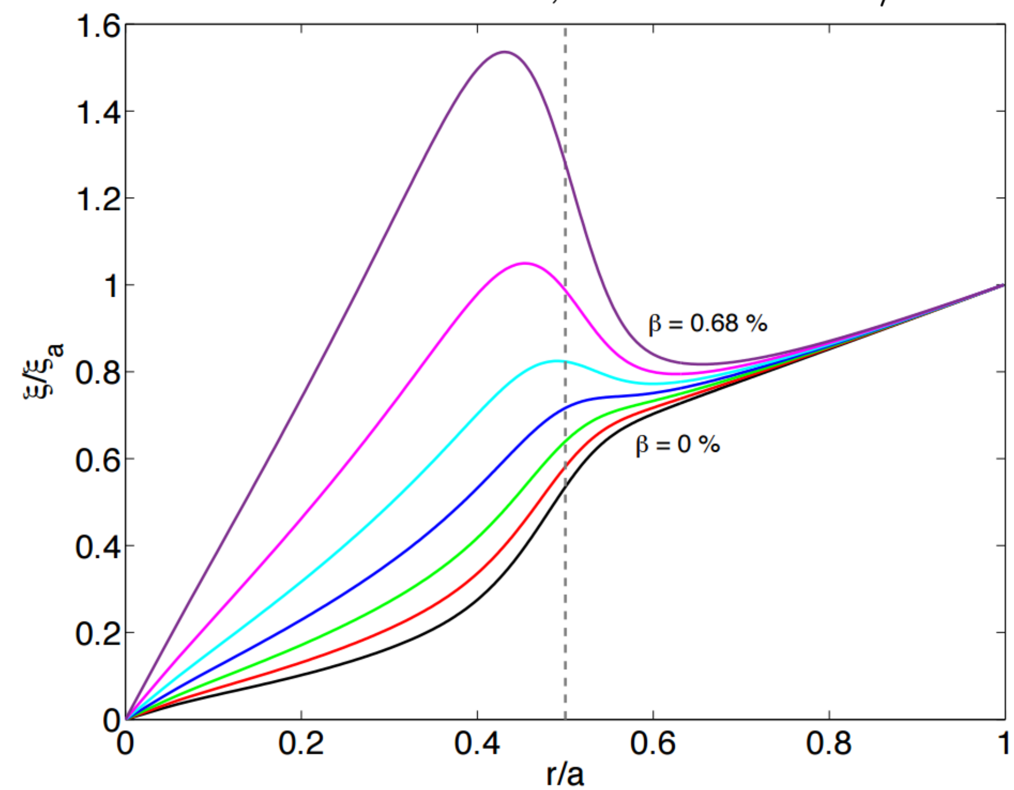
   and this is consistent only with discontinuous transform with $\Delta\iota > \Delta\iota_{min}$

3. Empirical study (i.e. radial convergence) shows that

   VMEC qualitatively reproduces self-consistent, perturbed solution

   interpretation: finite radial resolution implies an "effective" $\Delta\iota \sim \iota' h$, where $h \equiv 1/N$?





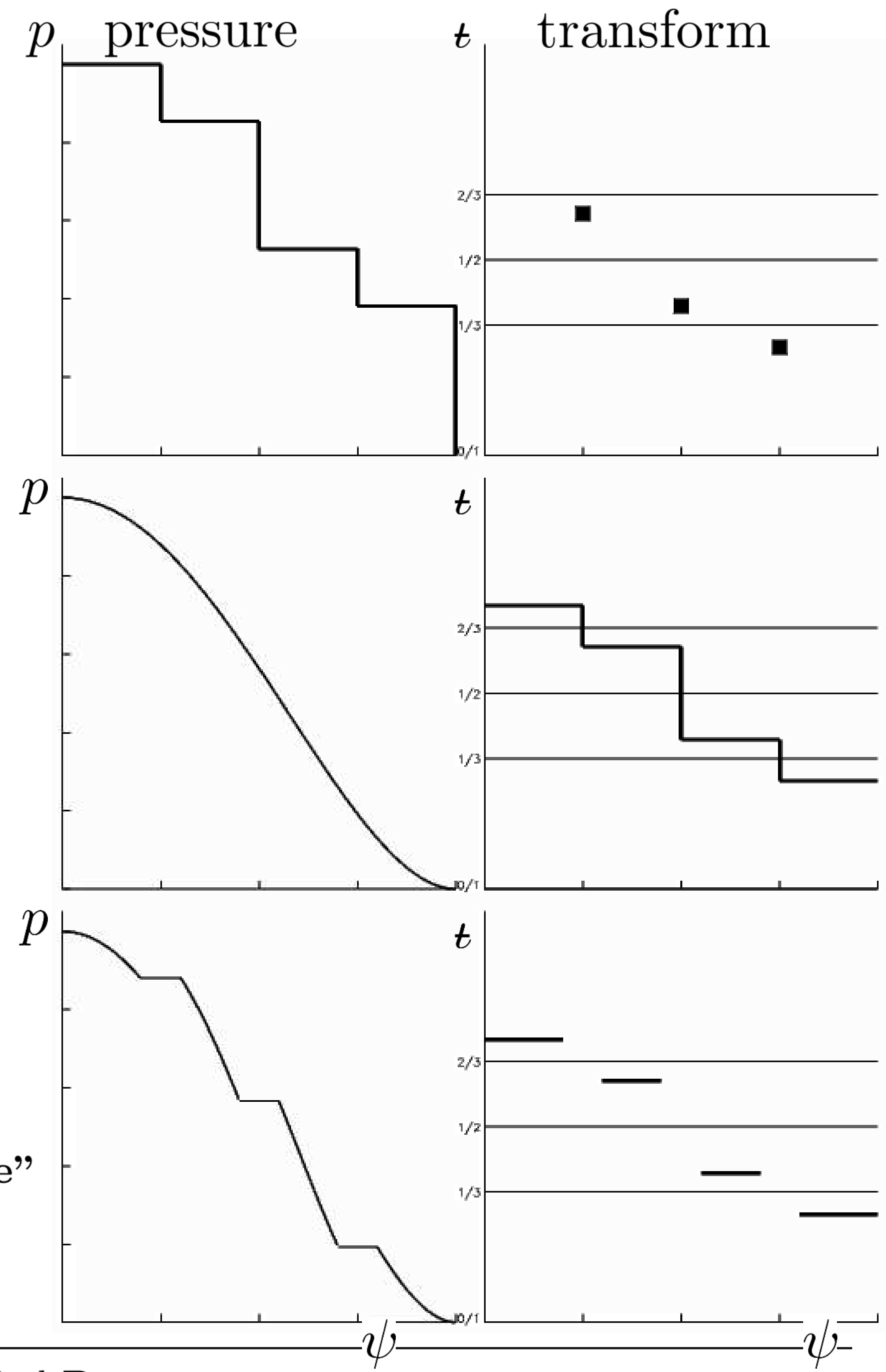
[Loizu, Hudson, …, Lazerson, ..., Phys. Plasmas **23**, 055703 (2016)]

Conclusion:
the two classes of general, relevant, tractable 3D MHD equilibria* are:

1. **Stepped-pressure equilibria,**
   i. Bruno & Laurence states
   ii. extrema of MRxMHD energy functional
   iii. transform constrained discretely
   iv. pressure discontinuity at $\iota\!\!-$ = irrational
   v. allows for islands, magnetic fieldline chaos
2. **Stepped-transform equilibria,**
   i. introduced by Loizu, Hudson et al.
   ii. extrema of ideal MHD energy functional
   iii. transform (almost) everywhere irrational
   iv. arbitrary, smooth pressure
   v. continuously-nested flux surfaces
3. **Or, a combination of the above.**
4. **Each class of equilibria can be computed using SPEC**
   i. in continuous-pressure regions, requires taking N_V large
   ii. suggests VMEC, NSTAB, should be modified
   to allow for discontinuous transform

Q. **How does a state with continuous transform "ideally evolve" into a 3D state with discontinuous transform?**
implications for ideal stability if no accessible 3D state exists?

!!! **See Dewar's talk on "putting the D into MRxMHD"**



* Equilibrium Code: given pressure, and given e.g. rotational-transform, find **B**.
[Difficult, if not impossible, to constrain non-trivial topology of **B** to match continuous-but-fractal pressure.]

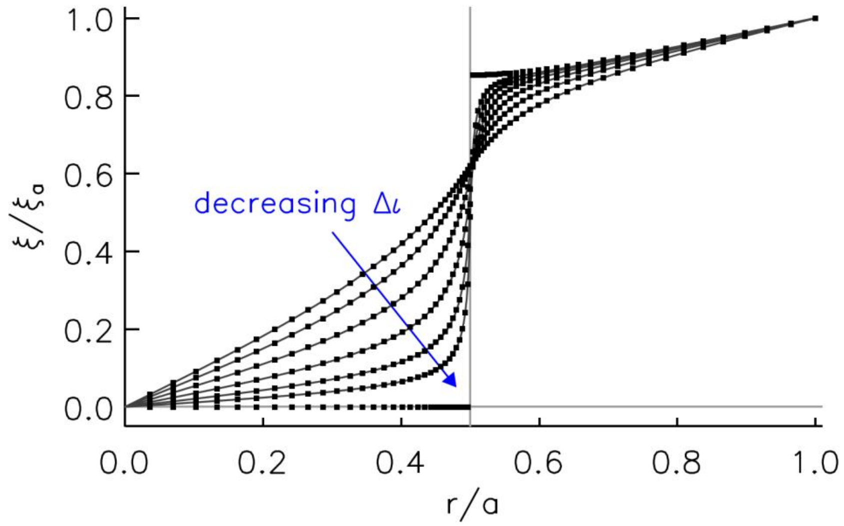
* An initial value code:= evolve pressure, **B** in "time"; becomes singular as $\eta \to 0$.

Back up slides

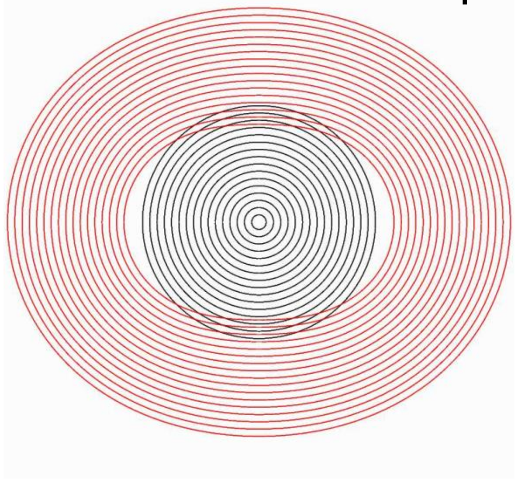
Necessary condition for non-overlapping of perturbed surfaces
Existence of non-linear solutions

1. **Condition for non-overlapping perturbed surfaces**

$$\left|\frac{\partial \xi}{\partial r}\right|_{max} < 1$$



Discontinuously-perturbed flux surfaces overlap!



2. **An asymptotic analysis near the rational surface**
   gives the *sine-qua-non* condition *(an indispensable condition, element, or factor; something essential)*

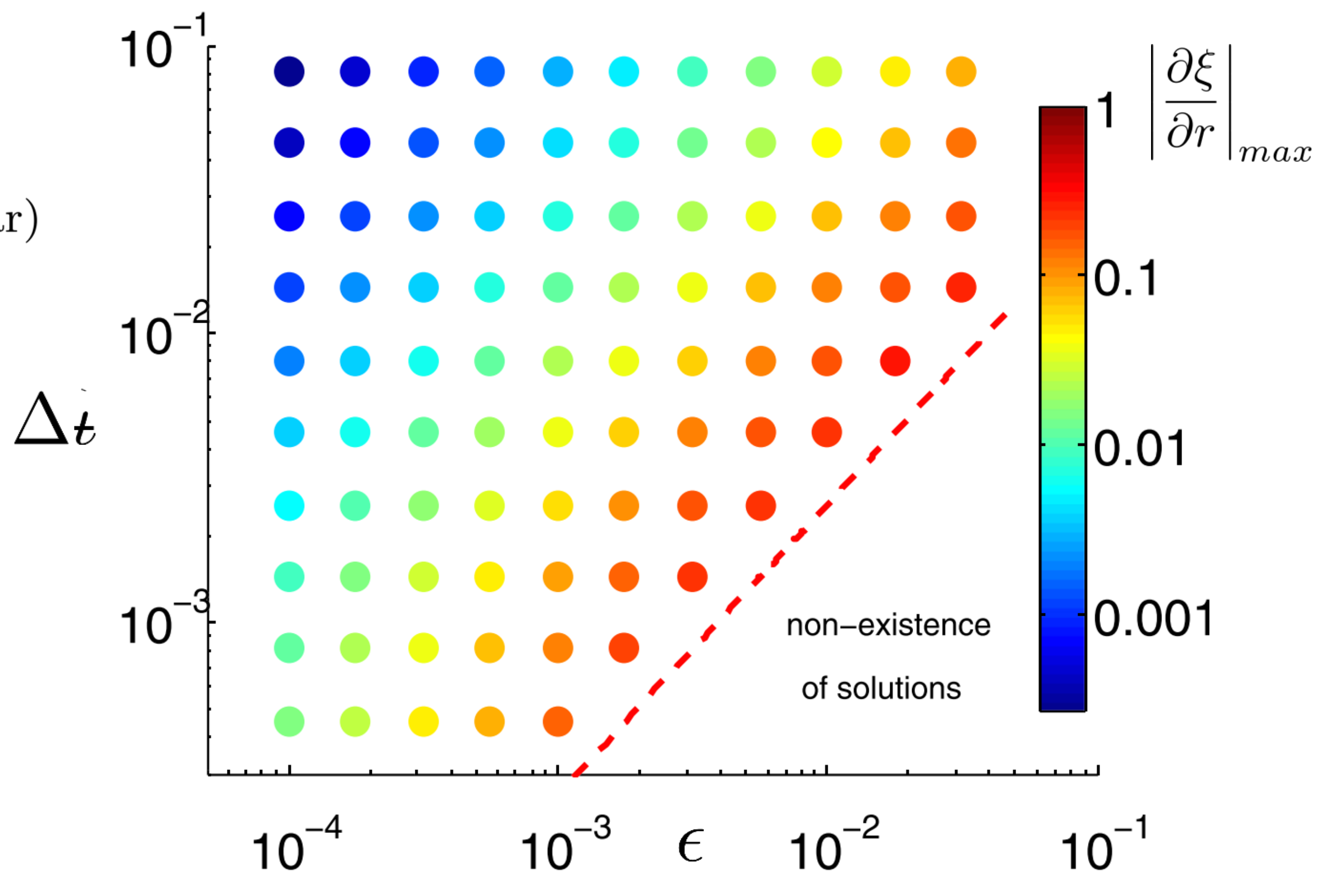
$$\Delta\iota > \Delta\iota_{min}, \quad \text{where } \Delta\iota_{min} \equiv 2\iota'_s \xi_s$$

(analysis for cylindrical, zero-β; general result probably similar)

3. **If this condition is violated, non-linear solutions do not exist.**

   i. Shown is ξ', as computed using non-linear SPEC calculations, as a function of $(\epsilon, \Delta\iota)$

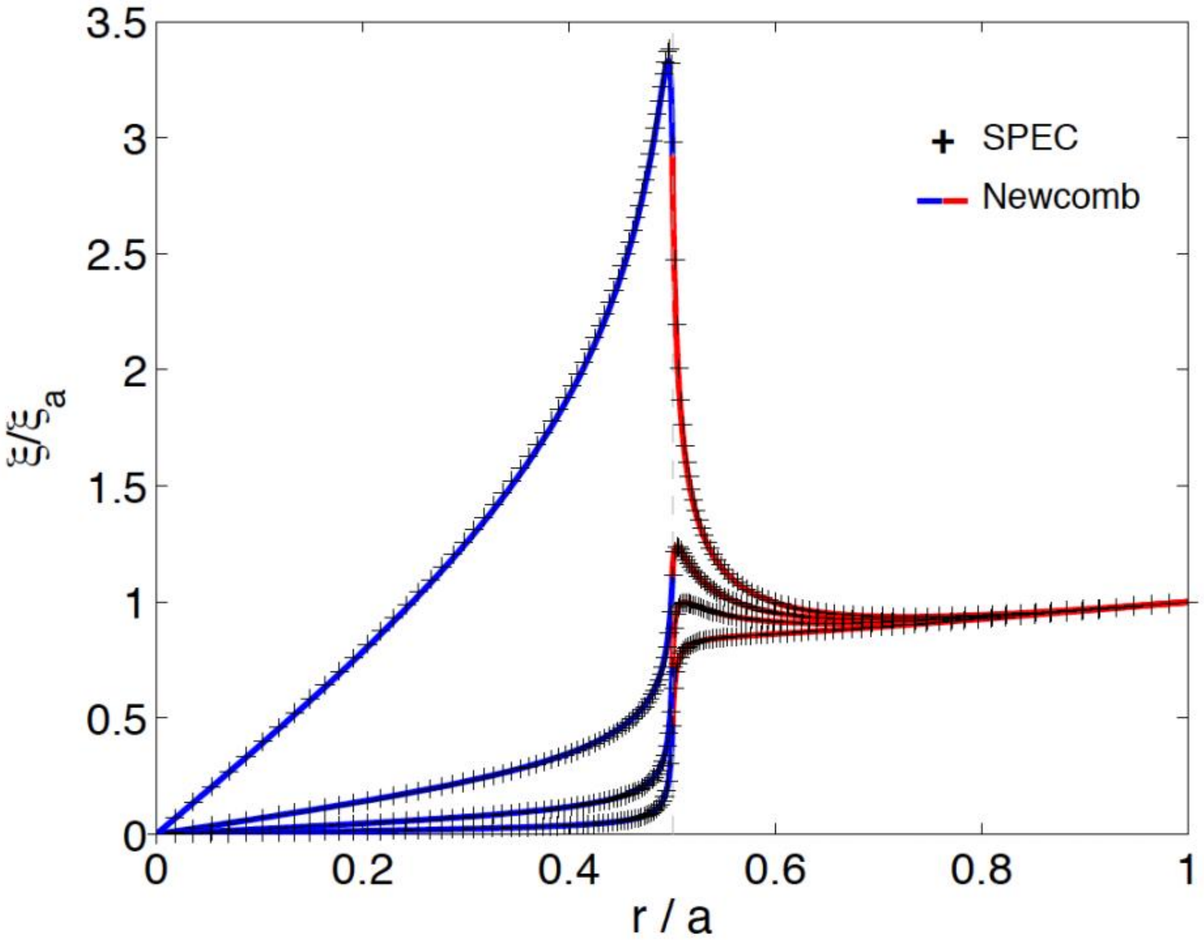
   ii. SPEC fails in ideal-limit, i.e. $N_V \to \infty$, when $\Delta\iota < \Delta\iota_{min}$



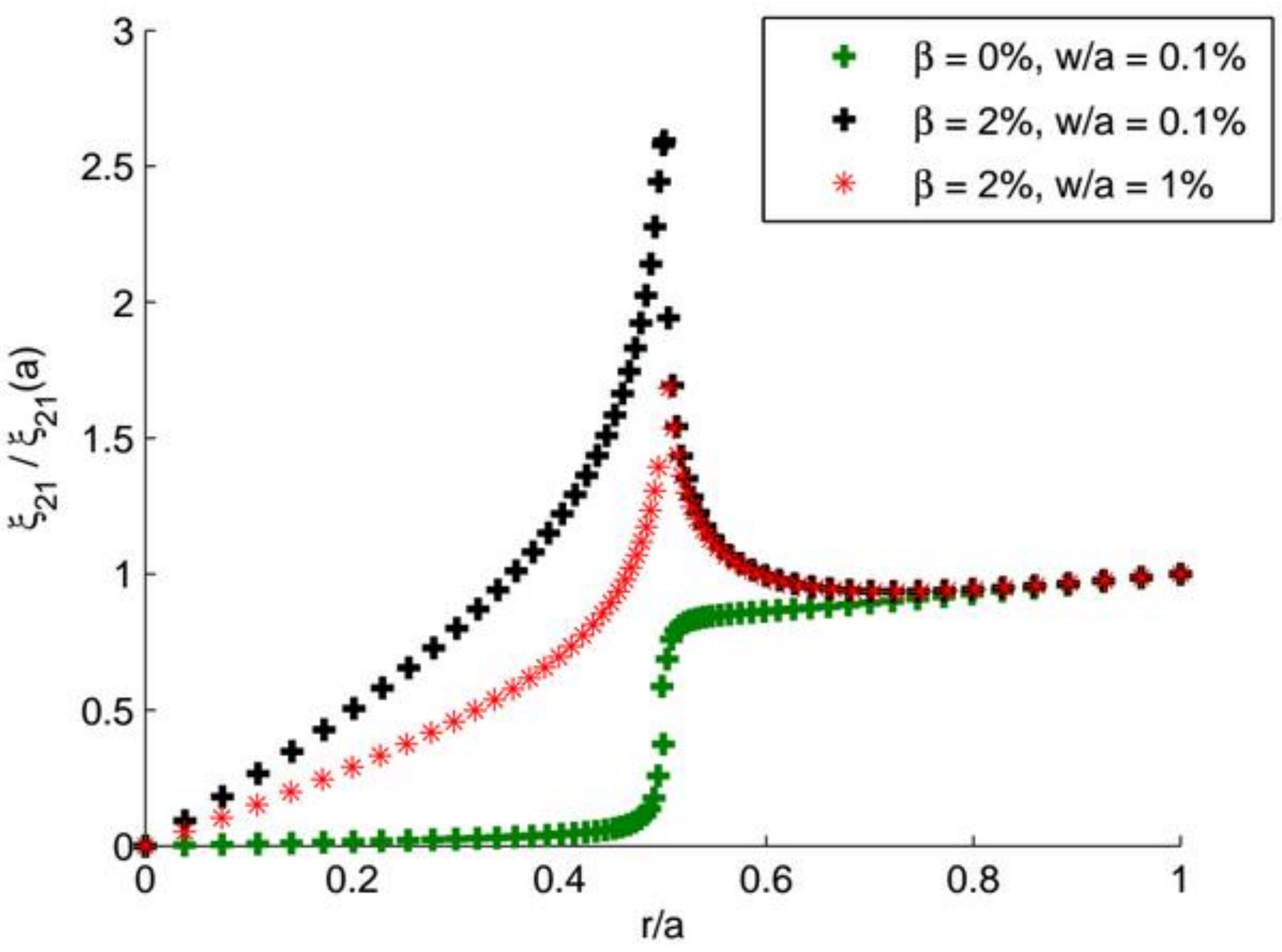
[Loizu, Hudson et al., Phys. Plasmas **22**, 090704 (2015)]

Discontinuous transform solution cf. “Tearing” solution

Discontinuous transform
with no island (ideal)



Continuous transform
with island (tearing)



Given continuous, non-integrable $\boldsymbol{B}$, $\boldsymbol{B}.\nabla p = 0$ implies p is fractal. Given fractal p, what is continuous, non-integrable $\boldsymbol{B}$?

- **Defn.** An equilibrium code computes the magnetic field consistent with a given p and e.g. given ι.
- **Theorem.** The topology of $\mathbf{B}$ is partially dictated by p.

  $\hookrightarrow$ Where $p' \neq 0$, $\mathbf{B} \cdot \nabla p = 0$ implies $\mathbf{B}$ must have flux surfaces.

  $\hookrightarrow$ Where $p' = 0$, $\mathbf{B}$ can have islands, chaos and/or flux surfaces.

TRANSPORT: given $\mathbf{B}$, solve for p.

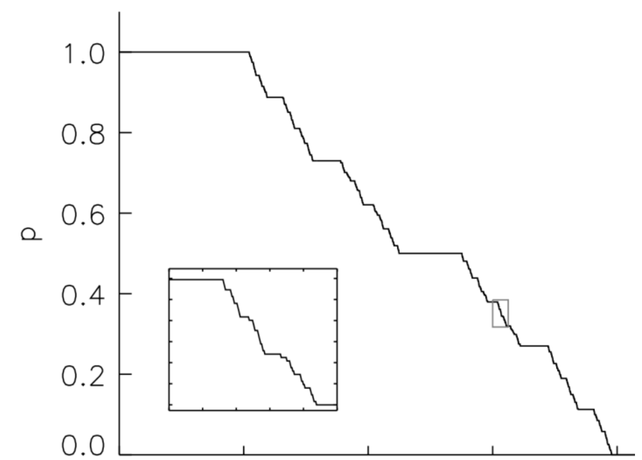
1. Given general, non-integrable magnetic field, $\mathbf{B} = \nabla \times [\psi \nabla \theta - \chi(\psi, \theta, \zeta) \nabla \zeta]$

   i. fieldline Hamiltonian: $\chi(\psi, \theta, \zeta) = \chi_0(\psi) + \sum_{m,n} \chi_{m,n}(\psi) e^{i(m\theta - n\zeta)}$

2. KAM theorem: for suff. small perturbation, "sufficiently irrational" flux surfaces survive

   i. if ι satisfies a "Diophantine" condition, $|\iota - n/m| > r/m^k, \forall (n,m)$, **excluded interval about every rational**

   ii. need e.g. Greene's residue criterion to determine if flux-surface$_{\iota}$ exists; lot's of work;



3. With $\mathbf{B} \cdot \nabla p = 0$, i.e. infinite parallel transport, pressure profile must be fractal:

$$p'(\iota) = \begin{cases} 1, & \text{if } |\iota - n/m| > r/m^k, \quad \forall (n,m), \text{ e.g. } r = 0.2,\ k = 2, \\ 0, & \text{if } |\iota - n/m| < r/m^k, \quad \exists (n,m), \end{cases}$$

$p'(x)$ is discontinuous on an uncountable infinity of points; impossible to discretize accurately;

EQUILIBRIUM: given p, solve for $\mathbf{B}$.

Q. <u>*Given*</u> a fractal p', how can the topology of $\mathbf{B}$ be constrained to enforce $\mathbf{B} \cdot \nabla p = 0$?

   i. e.g. if $p(\psi)$ is continuous and smooth, nowhere zero, then $\mathbf{B}$ *must* be integrable, i.e. $\chi_{m,n}(\psi) = 0$

   ii. if $p'(\psi)$ is fractal, then what are $\chi_{m,n}(\psi) =$?

Ongoing development of SPEC

1. Code improvements:

   i. finite-elements replaced by Chebshev polynomials

   e.g. $$\mathbf{A} \equiv \sum_{l,m,n}^{L,M,N} [\alpha_{l,m,n} T_l(s) \cos(m\theta - n\zeta) \nabla\theta + \beta_{l,m,n} T_l(s) \cos(m\theta - n\zeta) \nabla\zeta]$$

   ii. linearized equations

   iii. Cartesian, cylindrical, toroidal geometry

   iv. detailed online documentation,
   `http://w3.pppl.gov/~shudson/Spec/spec.html`

   v. easy-to-use, easy-to-edit, graphical user interface

2. Physics applications

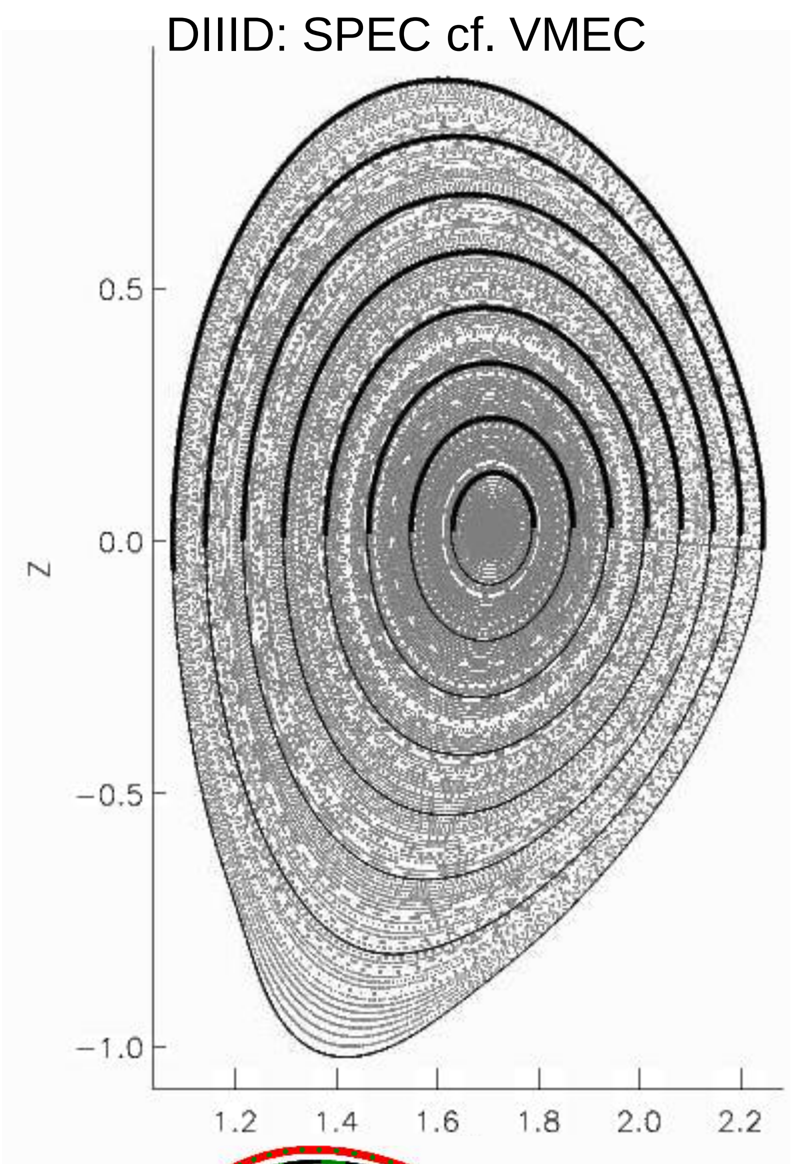
   i. W7-X vacuum verification calculations, OP1.1 [completed]

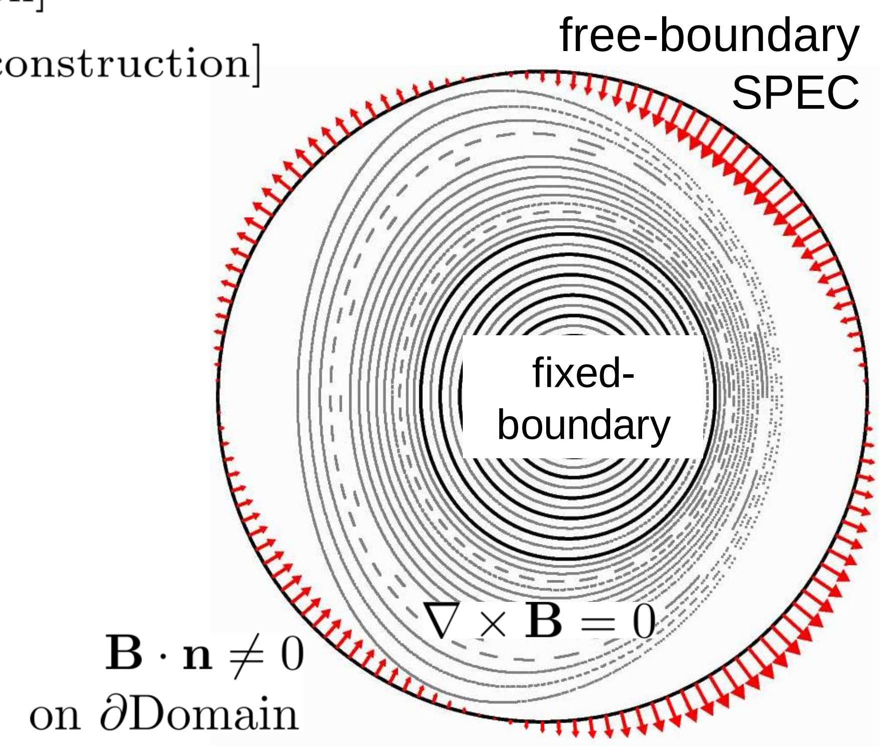
   ii. non-stellarator symmetric, e.g. DIIID, [completed]

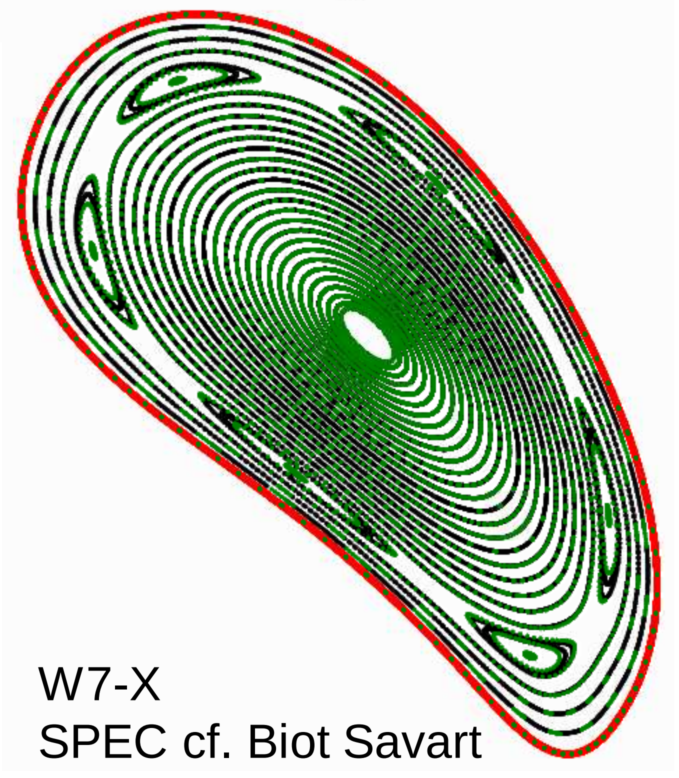
   iii. free-boundary, [completed]

   iv. including flow, [under construction]

   v. MRxMHD linear stability, [under construction]







Convergence studies using VMEC

[Lazerson, Loizu et al., Phys. Plasmas **23**, 012507 (2016)]

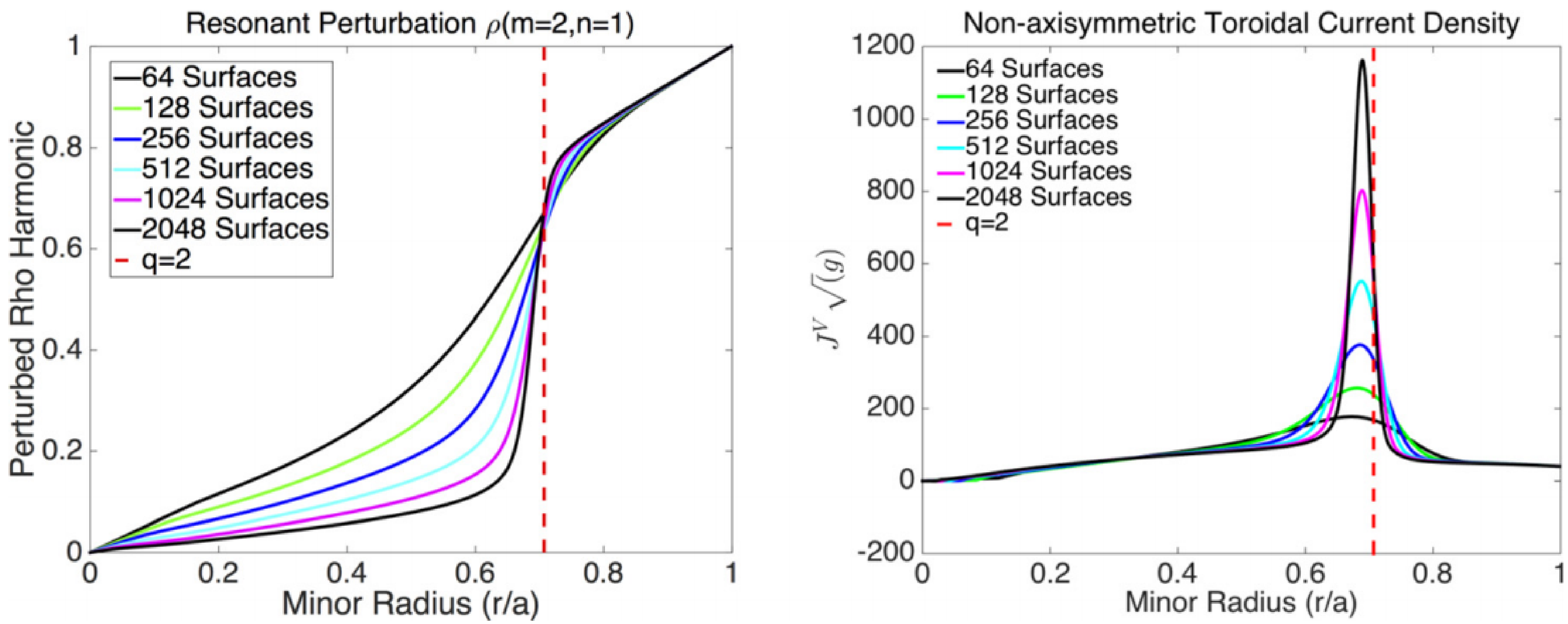


FIG. 2. Profile of the perturbed ρ harmonic (left) and the $m=2\ n=1$ component of the toroidal current density (right) showing dependence on radial resolution at fixed shear. Boundary perturbation 1×10^{-4} of minor radius. The $q=2$ surface is located at $s=0.5$ ($r/a\sim0.7$) in this plot. Note that the toroidal current density includes a Jacobian factor.

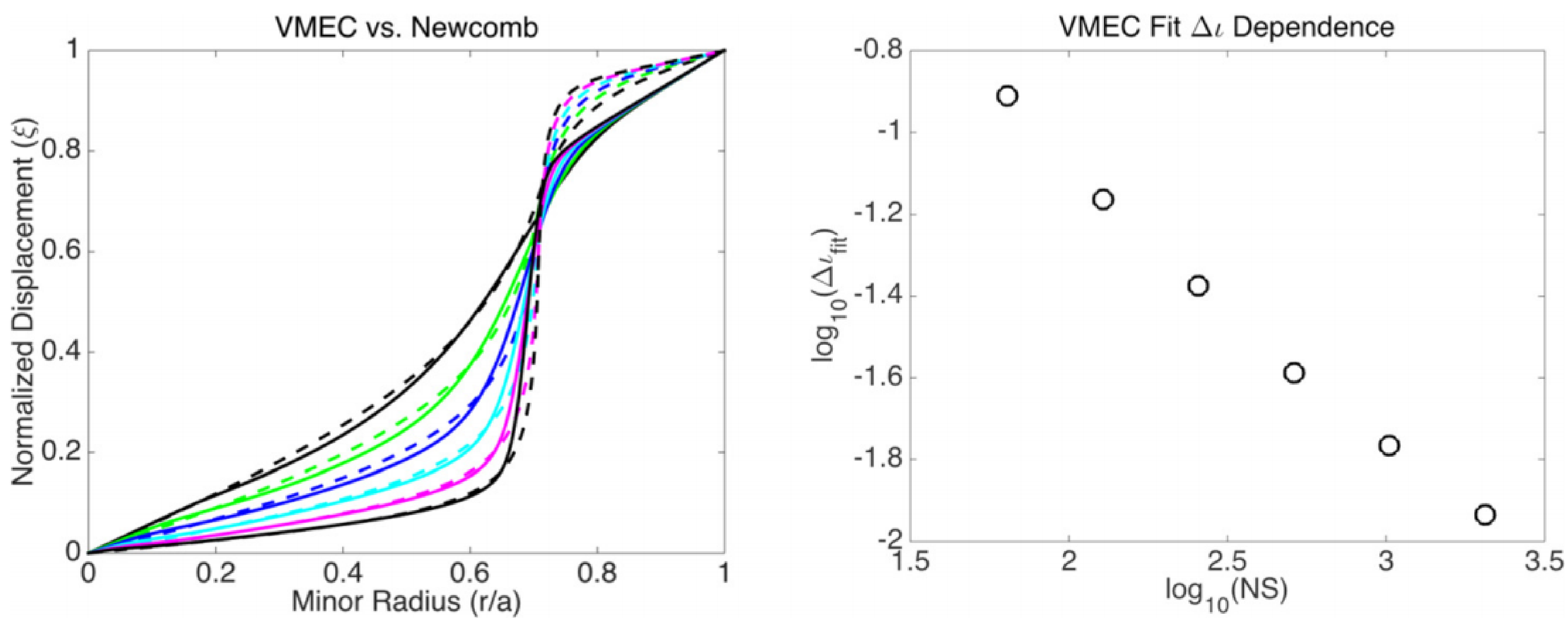


FIG. 5. Comparison of VMEC response (solid) to Loizu's solution to Newcomb's equation (dotted) (left) and the effective $\Delta\iota$ necessary to fit each curve (right). The colors are the same as those in Figure 2, and NS refers to the number of radial grid points.

Published SPEC convergence / verification calculations

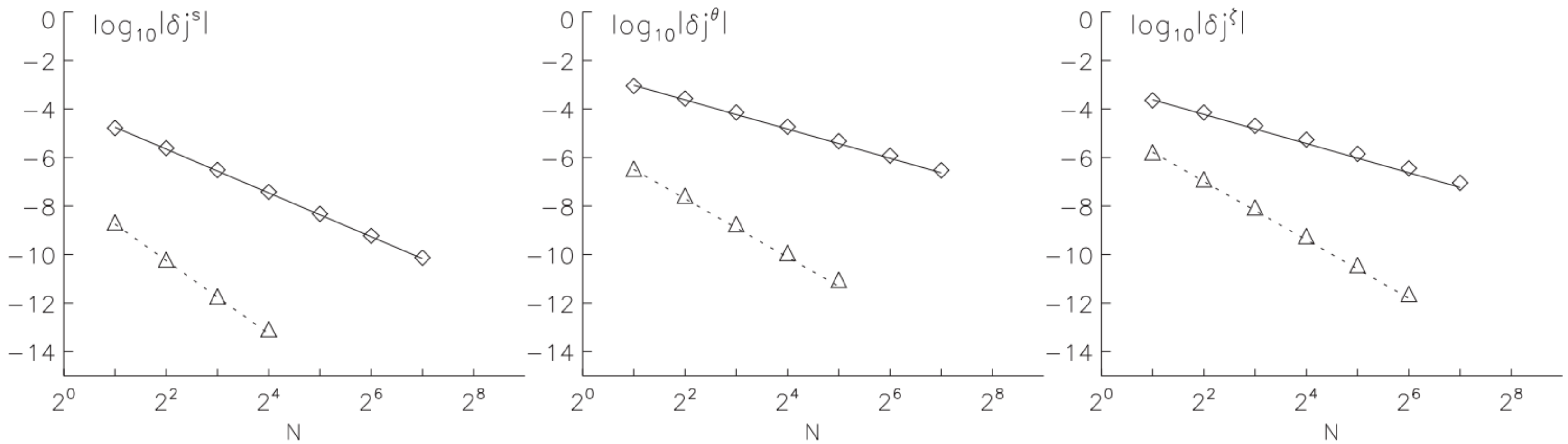


FIG. 2. Scaling of components of error, $\delta\mathbf{j} \equiv \mathbf{j} - \mu\mathbf{B}$, with respect to radial resolution. The diamonds are for the $n=3$ (cubic) basis functions, the triangles are for the $n=5$ (quintic) basis functions. The solid lines have gradient −3, −2, and −2, and the dotted lines have gradient −5, −4, and −4.

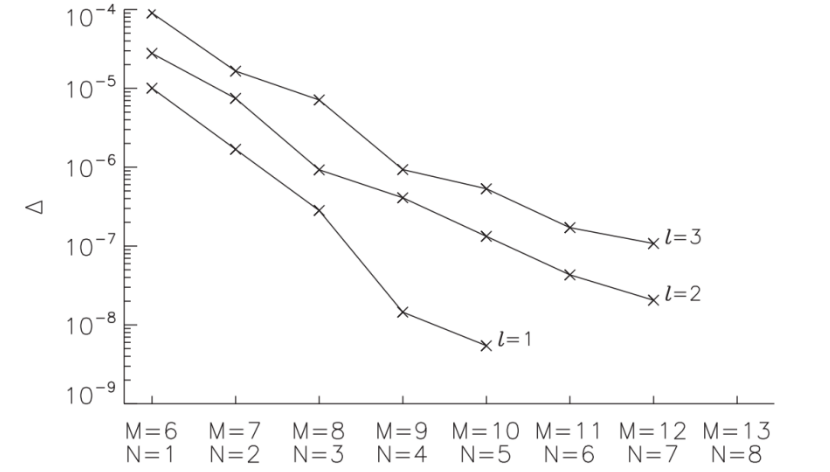


FIG. 6. Difference between finite M, N approximation to interface geometry, and a high-resolution reference approximation (with $M=13$ and $N=8$), plotted against Fourier resolution.

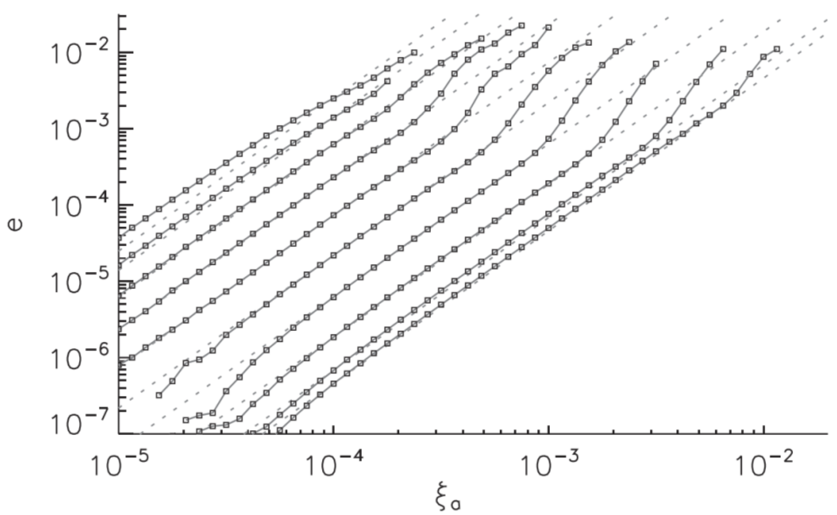


FIG. 2. Convergence of the error between linear and nonlinear SPEC equilibria as ξ_a is decreased, and for different values of $\Delta_{\dagger}$, ranging from 10^{-4} (upper curve) to 10^{-1} (lower curve).

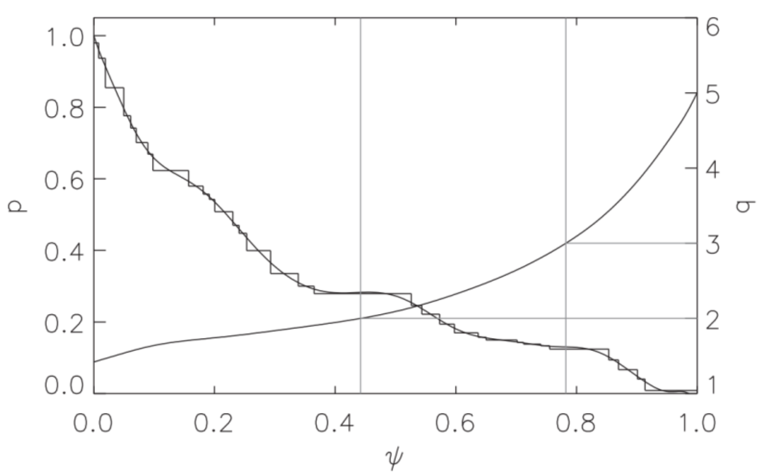


FIG. 7. Pressure profile (smooth) from a DIIID reconstruction using STELLOPT and stepped-pressure approximation. Also, shown is the inverse rotational transform ≡ safety factor.

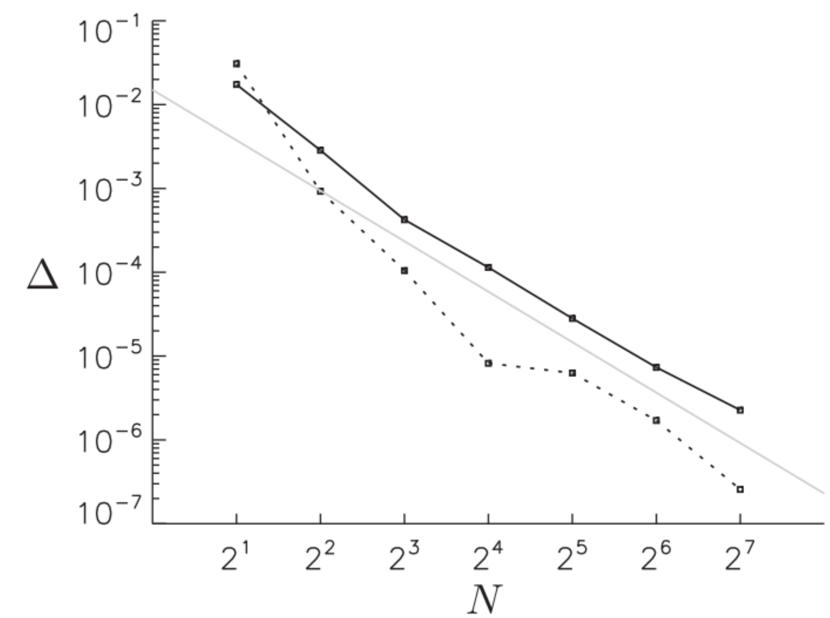
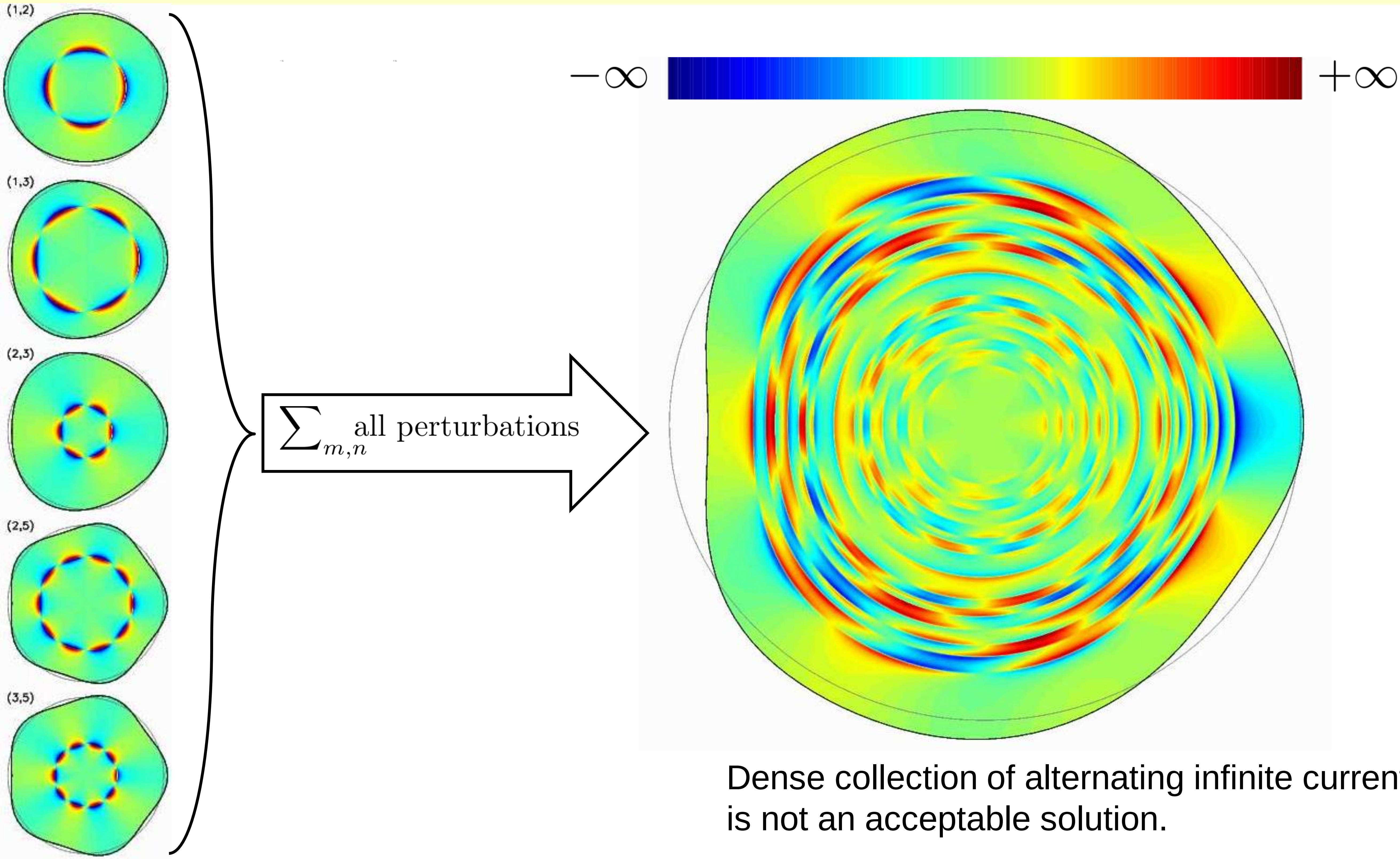


FIG. 5. Convergence: the error (Δ) between the continuous pressure (VMEC) and stepped pressure (SPEC) solutions are shown as a function of the number of plasma regions N for the $s=1/4$ SPEC interface. The dotted line shows the zero-beta case ($p_0=0$), and the solid line shows the high-beta case ($p_0=16$). The grey line has a slope −2, the expected rate of convergence. These simulations were run on a single 3 GHz Intel Xeon 5450 CPU with the longest (the $N=128$ case) taking 10.1 min using 20 poloidal Fourier harmonics and 768 fifth-order polynomial finite elements in the radial direction.

In arbitrary, three-dimensional geometry, “solutions” to $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ with smooth profiles and nested surfaces are nonsense.

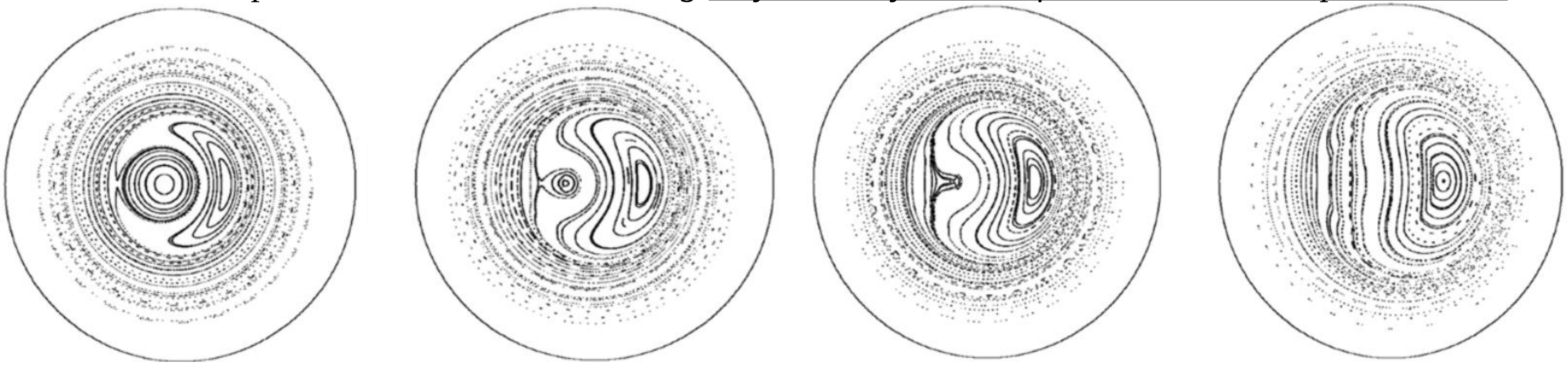


Dense collection of alternating infinite currents is not an acceptable solution.

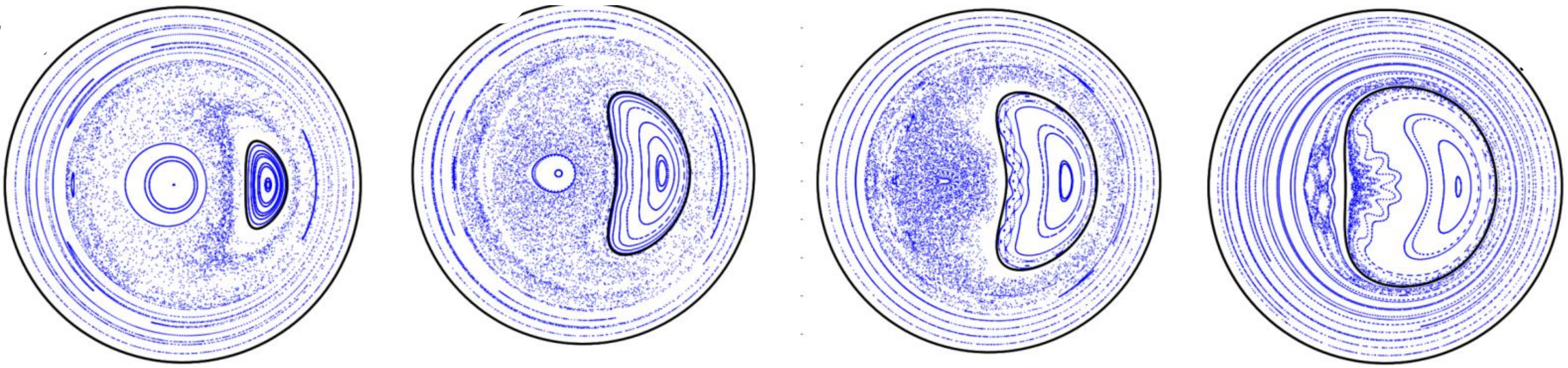
MRxMHD explains self-organization of Reversed Field Pinch into internal helical state

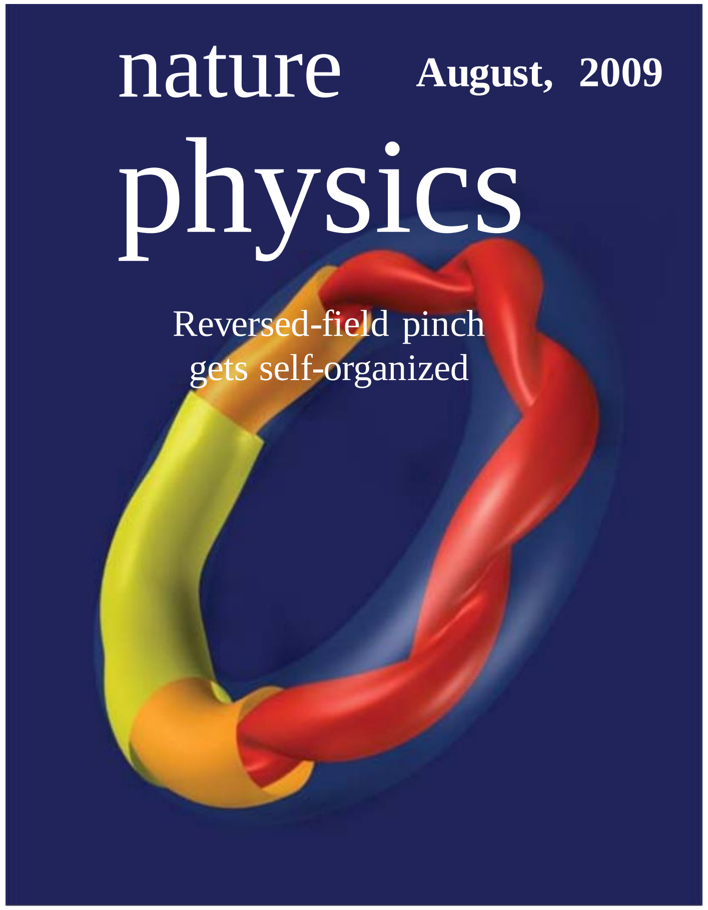
EXPERIMENTAL RESULTS
Overview of RFX-mod results
P. Martin et al., *Nuclear Fusion, 49 (2009) 104019*

Fig.6. Magnetic flux surfaces in the transition from a QSH state . . to a fully developed SHAx state . . The Poincaré plots are obtained considering only the axisymmetric field and dominant perturbation''



NUMERICAL CALCULATION USING STEPPED PRESSURE EQUILIBRIUM CODE
"Minimally Constrained Model of Self-Organized Helical States in Reversed-Field Pinches"
G. Dennis, S. Hudson, et al. PRL 111, 055003 (2013)]





Excellent Qualitative agreement between numerical calculation and experiment
→ this is first (and perhaps only?) equilibrium model able to explain internal helical state with two magnetic axes

Early and recent publications

Hole, Hudson & Dewar,	PoP,	2006	*(theoretical model)*
Hudson, Hole & Dewar,	PoP,	2007	*(theoretical model)*
Dewar, Hole et al.,	Entropy,	2008	*(theoretical model)*
Hudson, Dewar et al.,	PoP,	2012	*(SPEC)*
Dennis, Hudson et al.,	PoP,	2013	*(MRxMHD $\rightarrow$ ideal as $N_R \rightarrow \infty$)*
Dennis, Hudson et al.,	PRL,	2013	*(helical states in RFP = double Taylor state)*
Dennis, Hudson et al.,	PoP,	2014	*(MRxMHD+flow)*
Dennis, Hudson et al.,	PoP,	2014	*(MRxMHD+flow+pressure anisotrophy)*
Loizu, Hudson et al.,	PoP,	2015	*(first ever computation of 1/x & δ current-densities in ideal-MHD)*
Loizu, Hudson et al.,	PoP,	2015	*(well-defined, 3D MHD with discontinuous transform)*
Dewar, Yoshida et al.,	JPP,	2015	*(variational formulation of MRxMHD dynamics)*
Loizu, Hudson et al.,	PoP,	2016	*(pressure amplification of RMPs)*

Recent and upcoming invited talks

Hudson, Dewar, et al.,	2012	International Sherwood Fusion Theory Conference
Dennis, Hudson, et al.,	2013	International Sherwood Fusion Theory Conference
Dennis, Hudson, et. al.,	2013	International Stellarator Heliotron Workshop
Hole, Dewar, et al.,	2014	International Congress on Plasma Physics
Loizu, Hudson, et al.,	2015	International Sherwood Fusion Theory Conference
Loizu, Hudson, et al.,	2015	APS-DPP
Hudson, Loizu et al.,	2016	International Sherwood Fusion Theory Conference
Hudson, Loizu, et al.,	2016	Asia Pacific Plasma Theory Conference, 2016
Loizu, Hudson, et al.,	2016	Varenna Fusion Theory Conference