Title. . . .

Ideal MHD is used for. . . .

1. Equilibrium:
   i. Grad-Shafranov solvers; VMEC and NSTAB; . .
   ii. Reconstruction: e.g. EFIT, V3FIT, STELLOPT; . .
   iii. Linearly Perturbed Equilibria: e.g. IPEC, . .
2. Stability:
   i. kink,
   ii. ballooning,
   iii. peeling-ballooning,
3. Tearing mode analysis:
   i. calculation of Δ'

All of these codes presuppose an ideal MHD equilibrium with nested flux surfaces and smooth profiles

Use of RMPs to suppress ELMs requires 3D ideal MHD

“*MHD represents the simplest self-consistent model describing the macroscopic equilibrium and stability properties of a plasma.*”

“There is a general consensus that any configuration meriting consideration as a fusion reactor must satisfy the equilibrium and stability limits set by ideal MHD. If not, catastrophic termination of the plasma on a very short time scale .. is the usual consequence”

Ideal Magnetohydrodynamics, J.P. Friedberg

Ideal Force-Balance : ∇p=j×B
i. continuous solutions are physically unacceptable

$\nabla p = \mathbf{j} \times \mathbf{B}$ yields $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$. Note: $\mathbf{j}$ is current density. (1)

$\nabla \cdot \underbrace{(\sigma \mathbf{B} + \mathbf{j}_\perp)}_{\mathbf{j}} = 0$ yields $\mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$. (2)

$\mathbf{B} \cdot \nabla p(\psi) = 0$ yields $\mathbf{B} \cdot \nabla \psi = 0$. Use $\psi \equiv$ toroidal flux as surface label. (3)

Introduce angle, $\theta \equiv \bar{\theta} + \lambda(\bar{\theta}, \zeta)$, s.t. $\mathbf{B} \cdot \nabla \theta \equiv \mathbf{B} \cdot \nabla \bar{\theta}(1 + \lambda_\theta) = \iota\!\!-(\psi) \mathbf{B} \cdot \nabla \zeta$ (4)

Straight fieldline coordinates, (ψ, θ, ζ), allows $\sqrt{g}\mathbf{B} \cdot \nabla = \iota\!\!-\, \partial_\theta + \partial_\zeta$, $\mathbf{B} \cdot \nabla \zeta = \sqrt{g}^{-1}$. (5)

Fourier, $\sigma \equiv \sum_{m,n} \sigma_{m,n}(\psi) e^{i(m\theta - n\zeta)}$, Eqn(2) becomes $\boxed{(\iota\!\!-\, m - n)\sigma_{m,n} = i(\sqrt{g} \nabla \cdot \mathbf{j}_\perp)_{m,n}}$ (6)

Resonant parallel current density : $\sigma_{m,n} = \dfrac{p'}{x} + \delta(x)$, where $x \equiv \iota\!\!- - n/m$. (7)

Current $= \int_{\mathcal{S}} \mathbf{j} \cdot d\mathbf{s} = \int_\epsilon^\delta dx \int_0^{\pi/m} d\theta \, (\sigma \mathbf{B} \cdot \nabla \zeta \sqrt{g} + \mathbf{j}_\perp \cdot \nabla \zeta \sqrt{g}) \sim p'[\ln(\delta) - \ln(\epsilon)] \to \infty$ as $\epsilon \to 0$. (8)

{ Pressure gradients *near* rational surfaces produce infinite currents }

\+ { rational surfaces are dense } = no pressure allowed in solutions to ideal force-balance.

Ideal Force-Balance : ∇p=j×B
i. continuous solutions are physically unacceptable

$$\nabla p = \mathbf{j} \times \mathbf{B} \text{ yields } \mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2. \quad \text{Note: } \mathbf{j} \text{ is current density.} \tag{1}$$

$$\nabla \cdot \underbrace{(\sigma \mathbf{B} + \mathbf{j}_\perp)}_{\mathbf{j}} = 0 \text{ yields } \mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp. \tag{2}$$

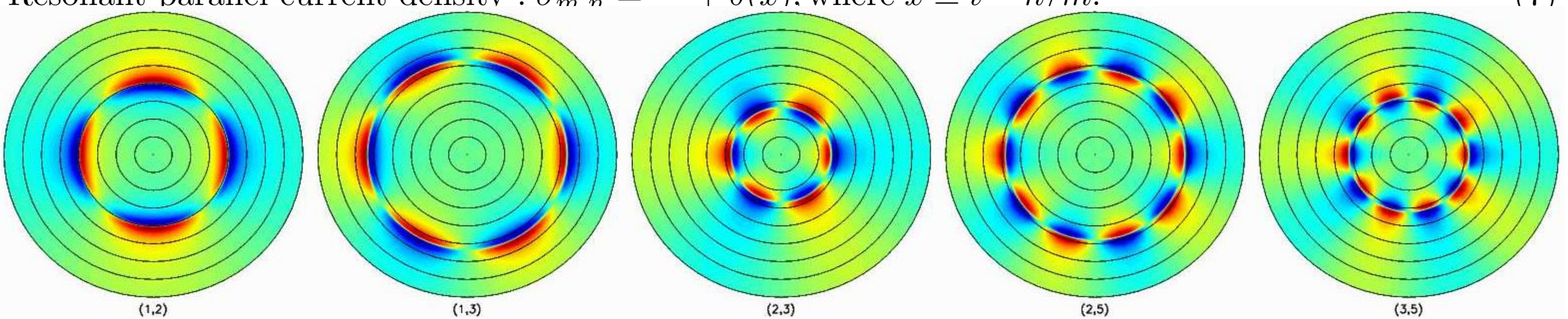
$$\mathbf{B} \cdot \nabla p(\psi) = 0 \text{ yields } \mathbf{B} \cdot \nabla \psi = 0. \text{ Use } \psi \equiv \text{ toroidal flux as surface label.} \tag{3}$$

$$\text{Introduce angle, } \theta \equiv \bar{\theta} + \lambda(\bar{\theta}, \zeta), \text{ s.t. } \mathbf{B} \cdot \nabla \theta \equiv \mathbf{B} \cdot \nabla \bar{\theta}(1 + \lambda_\theta) = \iota(\psi) \mathbf{B} \cdot \nabla \zeta \tag{4}$$

$$\text{Straight fieldline coordinates, } (\psi, \theta, \zeta), \text{ allows } \sqrt{g}\mathbf{B} \cdot \nabla = \iota\, \partial_\theta + \partial_\zeta, \quad \mathbf{B} \cdot \nabla \zeta = \sqrt{g}^{-1}. \tag{5}$$

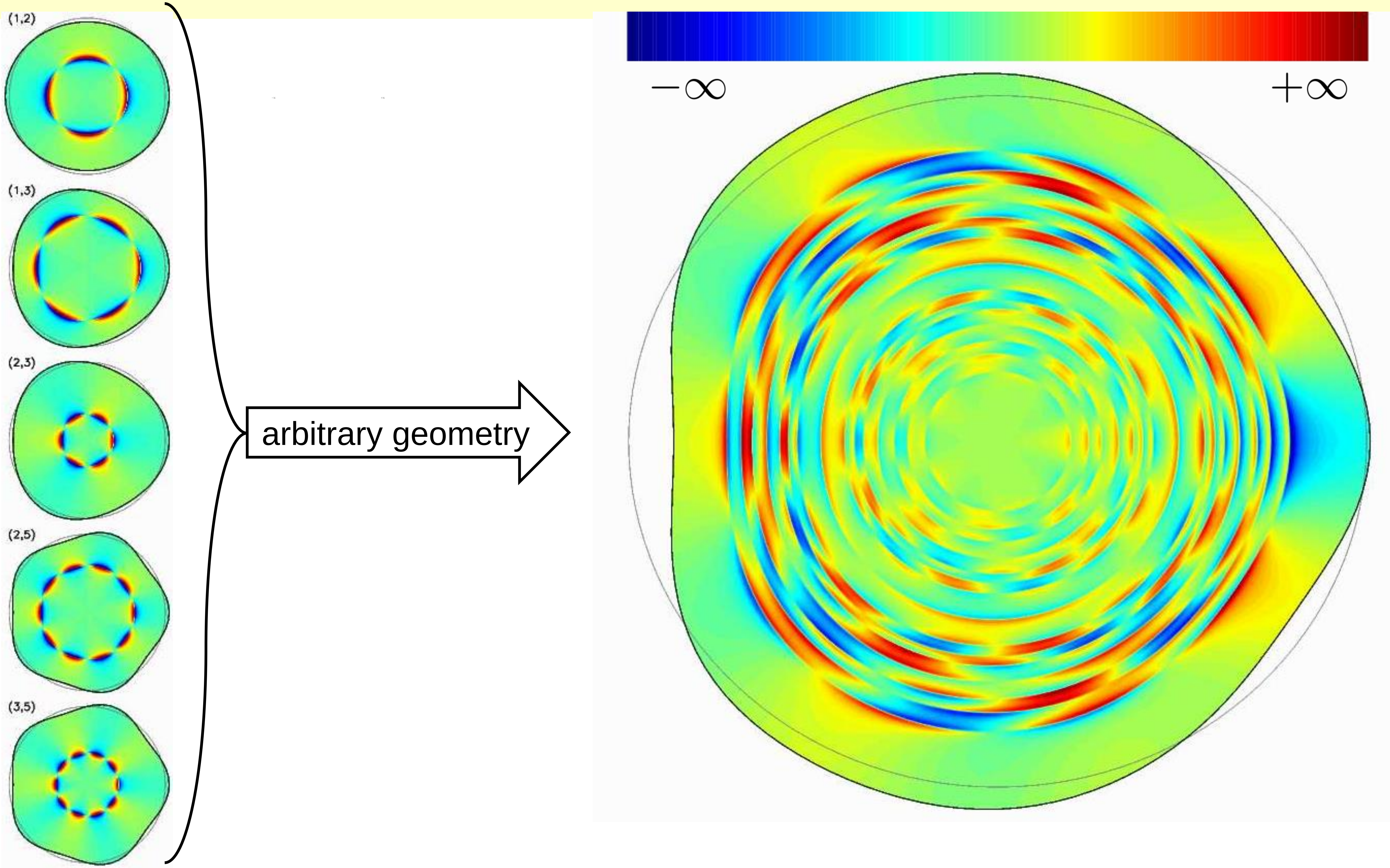
$$\text{Fourier, } \sigma \equiv \sum_{m,n} \sigma_{m,n}(\psi) e^{i(m\theta - n\zeta)}, \text{ Eqn(2) becomes } \boxed{(\iota m - n)\sigma_{m,n} = i(\sqrt{g} \nabla \cdot \mathbf{j}_\perp)_{m,n}} \tag{6}$$

$$\text{Resonant parallel current density : } \sigma_{m,n} = \frac{p'}{\ } + \delta(x), \text{ where } x \equiv \iota - n/m. \tag{7}$$



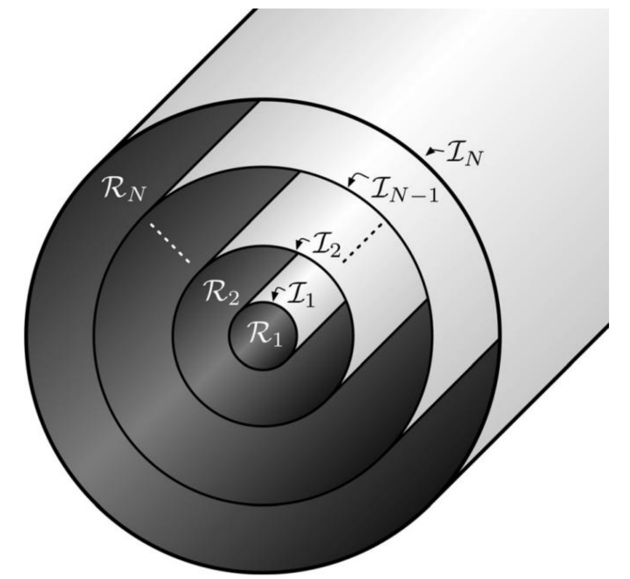
Ideal Force-Balance : ∇p=j×B

i. dense collection of alternating infinite currents



Multi-Region Relaxed MHD (MRxMHD)

equilibrium ≡ constrained, minimum-energy state
consistent with given pressure, transform, and boundary.



1. **Stepped Pressure Equilibrium Code** (SPEC) extremizes

$$\mathcal{F} \equiv \sum_{i=1}^{N_R} \left\{ \int_{\mathcal{R}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv - \frac{\mu_i}{2} \left(\int_{\mathcal{R}_i} \mathbf{A} \cdot \mathbf{B} \, dv - H_i \right) \right\},$$

$$\underbrace{\delta \mathbf{B} \text{ is arbitrary in } \mathcal{R}_i}_{\text{i.e. islands in } \mathcal{R}_i}, \qquad \text{and} \qquad \underbrace{\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) \text{ on } \mathcal{I}_i}_{\text{topological constraint}}.$$

2. The equilibrium state: $\underbrace{\nabla \times \mathbf{B} = \mu_i \mathbf{B}}_{\text{Taylor relaxed}}$ in each $\mathcal{R}_i$, and $\underbrace{[[p + B^2/2]] = 0}_{\text{“ideal”}}$ across each $\mathcal{I}_i$.

3. **Unification of Taylor relaxation and ideal MHD**

If	$1 = N_R$	globally-relaxed, Taylor state,	$\nabla \times \mathbf{B} = \mu \mathbf{B}$
If	$1 < N_R < \infty$	partial chaos, stepped-pressure	
If	$N_R = \infty$	globally-ideal, ideal-MHD,	$\nabla p = \mathbf{j} \times \mathbf{B}$

Title. . . .

Early and recent publications

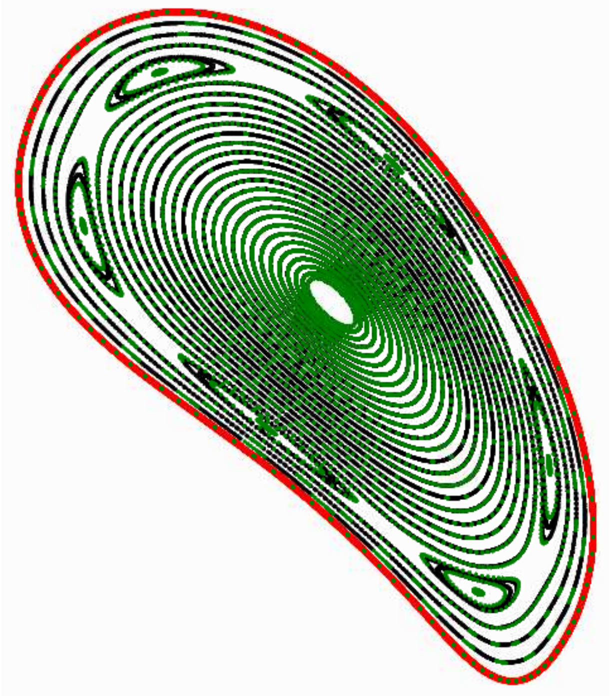
1)	Hole, Hudson & Dewar,	PoP 2006	(theoretical model)
2)	Hudson, Hole & Dewar,	PoP 2007	
3)	Dewar, Hole et al.	Entropy 2008	
4)	Hudson, Dewar et al.,	PoP 2012	*(SPEC)*
5)	Dennis, Hudson et al.	PoP 2013	*(MRxMHD $\rightarrow$ ideal as $N_R \rightarrow \infty$)*
6)	Dennis, Hudson et al.,	PRL 2013	*(helical states in RFP = double Taylor state)*
7)	Dennis, Hudson et al.,	PoP 2014	*(MRxMHD+flow)*
8)	Dennis, Hudson et al.,	PoP 2014	*(MRxMHD+flow+pressure anisotrophy)*
9)	Loizu, Hudson et al.	PoP 2015	*(first ever computation of 1/x & δ current-densities in ideal-MHD)*
10)	Loizu, Hudson et al.	PoP, 2015	*(well-defined, 3D ideal MHD with discontinuous transform, RMP penetration)*
11)	Dewar, Yoshida et al.	JPP, 2015	*(variational formulation of MRxMHD dynamics)*
12)	Loizu, Hudson et al.,	PoP, 2016	*(pressure amplification of RMPs)*

Recent and upcoming invited talks

1)	Hudson, Dewar, et al.	2012	International Sherwood Fusion Theory Conference
2)	Dennis, Hudson, et al.	2013	International Sherwood Fusion Theory Conference
3)	Dennis, Hudson, et. al	2013	International Stellarator Heliotron Workshop
4)	Hole, Dewar, et al.	2014	International Congress on Plasma Physics
5)	Loizu, Hudson, et al.	2015	International Sherwood Fusion Theory Conference
6)	Loizu, Hudson, et al.	2015	APS-DPP
7)	Hudson, Loizu, et al.,	2016	Asia Pacific Plasma Theory Conference, 2016
8)	Loizu, Hudson, et al.	2016	Varenna Fusion Theory Conference

Ongoing Efforts

1) Linearly perturbed calculations ✓ (already published)
2) Vacuum benchmark, W7-X, OP1.1, ✓ SPEC vs. Biot-Savart shown →
3) Non-stellarator symmetric capability ✓ (completed, not-yet published)
4) Free-boundary capability ☐ (almost finished)



W7-X vacuum verification
excellent agreement!

Milestones

1)	Publish vacuum verification	2016	
2)	Complete free-boundary	2016	
3)	W7-X calculations	2016 / 2017	
4)	Linear stability calculations	2017	
5)	Include flow	2017	*(collaboration with Dr. Hamdi)*

Active Collaborators

1)	S. R. Hudson, A. Bhattacharjee	Princeton Plasma Physics Laboratory	
2)	R.L. Dewar, M. J. Hole	Australian National University	*(awarded 3 Australian Research Council Grants)*
3)	J. Loizu, P. Helander, C. Nuehrenberg	Max-Planck Institute for Plasma Physics	
4)	Z. Yoshida	University of Tokyo	*(indirect collaboration via Prof.Dewar)*
5)	H. AbdelHamid	University of Tokyo	*(presently visiting PPPL)*

SPEC vc M3D-C_1

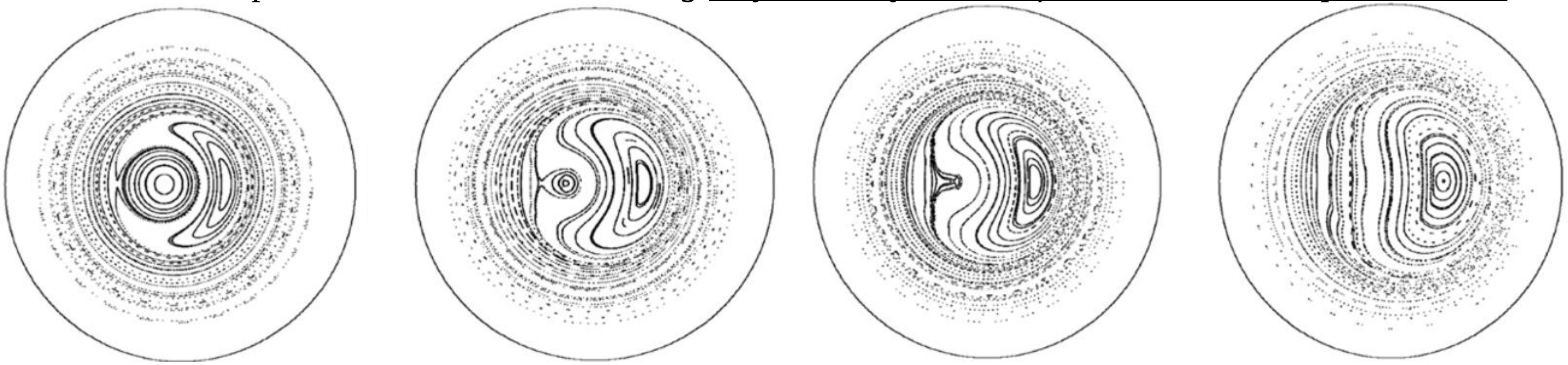
1) How can an equilibrium code be verified against an initial-value code?

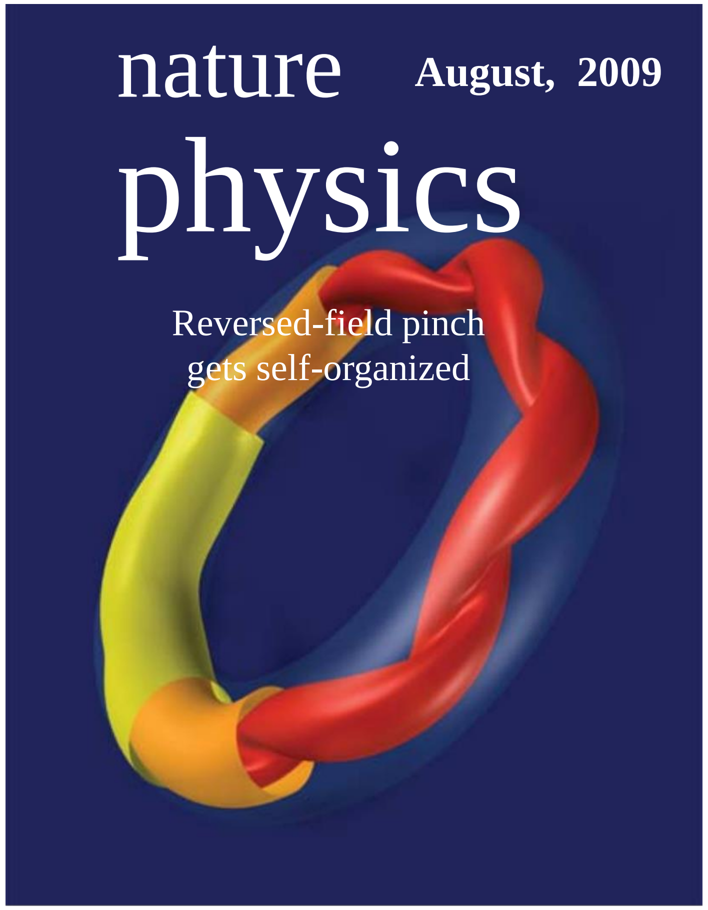
2) Can self-organized stationary states in tokamaks [Jardin et al. PRL] be explained as constrained minimum energy states? *(preliminary discussions with A. Bhattacharjee, H. Abdelhamid)*

MRxMHD explains self-organization of Reversed Field Pinch into internal helical state

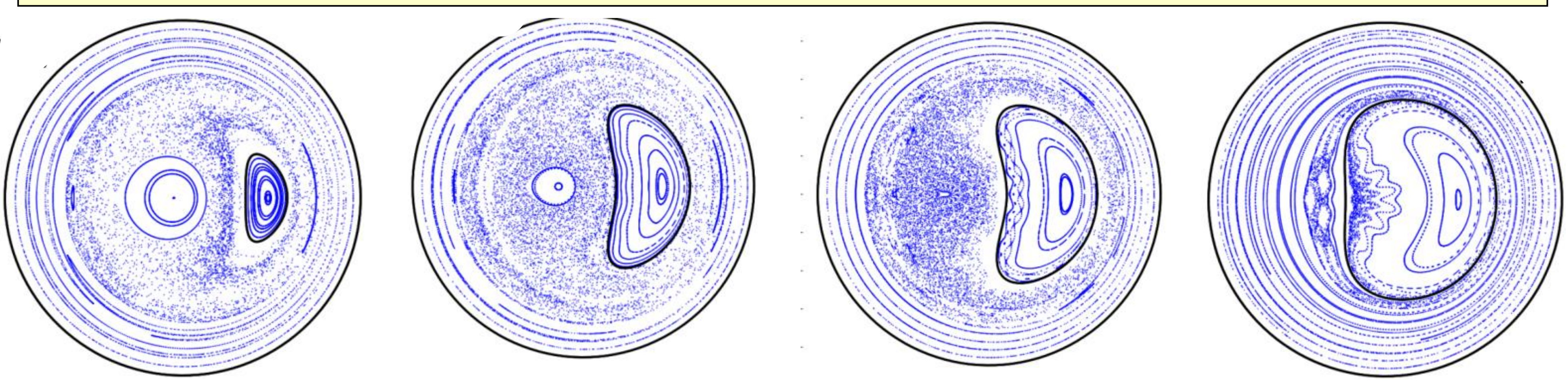
EXPERIMENTAL RESULTS
Overview of RFX-mod results
P. Martin et al., *Nuclear Fusion, 49 (2009) 104019*

Fig.6. Magnetic flux surfaces in the transition from a QSH state . . to a fully developed SHAx state . . The Poincaré plots are obtained considering only the axisymmetric field and dominant perturbation"





NUMERICAL CALCULATION USING STEPPED PRESSURE EQUILIBRIUM CODE
Taylor relaxation and reversed field pinches
G. Dennis, R. Dewar, S. Hudson, M. Hole, 2012 20th Australian Institute of Physics Congress



Excellent Qualitative agreement between numerical calculation and experiment
→ this is first (and perhaps only?) equilibrium model able to explain internal helical state with two magnetic axes