A simple stellarator coil-design algorithm

S.R. Hudson & C. Zhu

Presented at U. Maryland, 13 Dec. 2017

1) The *Simplest Possible Algorithm*© *(SPA)* for designing stellarator coils is described.

2) The coil geometry has "maximum freedom", and
   the target function is "minimally constrained".

3) Fast, reliable and insightful numerical algorithms are enabled by exploiting 1st and 2nd derivatives with respect to the coil geometry *and* the "target surface".

1) P. Merkel, Nucl. Fus., **27** 867 (1987)
2) M. Landreman, Nucl. Fus., **57** 046003 (2017)
3) Caoxiang Zhu, Stuart R. Hudson *et al.,* Nucl. Fus., **58** 345 016008 (2018)
4) R.L. Dewar, S.R. Hudson & P.F. Price, Phys. Lett. A., **194** 49 (1994)

Vacuum fields in given domain described by Laplace

1. Given volume $\mathcal{V}$, with closed boundary $\mathcal{S} \equiv \partial\mathcal{V}$.
2. Vacuum fields satisfy $\nabla \times \mathbf{B} = 0$, suggests $\mathbf{B} = \nabla\Phi$.
3. Given $\mathbf{B} \cdot \mathbf{n}$ on $\mathcal{S}$.
4. Constraint of net flux $\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = 0$.
5. Toroidal flux $\Psi \equiv \oint_{\mathcal{L}} \mathbf{A} \cdot d\mathbf{l}$, (require one loop integral per "hole").
6. In $\mathcal{V}$, solution to $\nabla \cdot \nabla\Phi = 0$ is unique.

Mathematics becomes numerics: discretize currents and “regularize” functional.

1. Introduce $\mathbf{x}_i(l)$, $i = 1, \ldots, N_C$, closed current-carrying curves, and let $\bar{\mathbf{x}}(\theta, \zeta) \equiv \mathcal{S}$.
2. With finite degrees-of-freedom, cannot generally *exactly* recover arbitrary $B_n \equiv \mathbf{B} \cdot \mathbf{n}$ on $\mathcal{S}$.
3. Instead, minimize the **quadratic-flux functional** with a penalty on length,

$$\mathcal{F}[\mathbf{x}_i, \bar{\mathbf{x}}] \equiv \oint_{\mathcal{S}} \frac{1}{2} B_n^2 ds + \omega L, \quad \text{where } L[\mathbf{x}_i] = \sum_i \oint |\mathbf{x}_i'| \, dl. \tag{1}$$

4. That's all the physics. All that's left is:
   i. to use a suitable representation for the external currents, and
   ii. to construct the derivatives of $\mathcal{F}$.

5 Optimal coils for given surface are defined by $\dfrac{\delta \mathcal{F}}{\delta \mathbf{x}_i} = 0$.

The Biot-Savart law gives the magnetic field, variation in curves gives variation in magnetic field.

1. The magnetic field is from Biot-Savart,

$$\mathbf{B}_i(\bar{\mathbf{x}}) = I_i \oint_i \frac{\mathbf{x}_i' \times \mathbf{r}}{r^3}\, dl, \tag{1}$$

   where I_i is the current and $\mathbf{r}(\theta,\zeta,l) \equiv \bar{\mathbf{x}}(\theta,\zeta) - \mathbf{x}_i(l)$ and $\mathbf{x}_i' \equiv \dfrac{\partial \mathbf{x}_i}{\partial l}$.

2. For simplicity, set $I_i = 1$. (Trivial solutions avoided, and can ignore toroidal flux constraint.)

3. Variations in the curve induce variations in the field,

$$\delta\mathbf{B}(\bar{\mathbf{x}}) = \oint_i (\delta\mathbf{x}_i \times \mathbf{x}_i') \cdot \mathbf{R}_i\, dl, \tag{2}$$

   where $\mathbf{R} = \dfrac{3\,\mathbf{r}\,\mathbf{r}}{r^5} - \dfrac{\mathbf{I}}{r^3}$, and $\mathbf{I}$ is the "idemfactor", e.g. $\mathbf{I} = \mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k}$.

4. Let me go through the algebra more slowly.

Variations in line integrals with respect to variations in the line: magnetic field

$$\mathbf{B} = \oint \frac{(\mathbf{x}' \times \mathbf{r})}{r^3}\, dl, \quad \text{where} \;\; \mathbf{r} \equiv \bar{\mathbf{x}} - \mathbf{x}, \;\; r \equiv \sqrt{\mathbf{r}\cdot\mathbf{r}}, \;\; \mathbf{x}' \equiv \partial_l \mathbf{x} \tag{1}$$

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$$\delta\mathbf{B} = \oint \frac{(\delta\mathbf{x}' \times \mathbf{r})}{r^3}\, dl \qquad - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3 \oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl$$

Variations in line integrals with respect to variations in the line: magnetic field

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\begin{aligned}
\mathbf{B} &= \oint \frac{(\mathbf{x}' \times \mathbf{r})}{r^3}\,dl, \quad \text{where } \mathbf{r} \equiv \bar{\mathbf{x}} - \mathbf{x}, \quad r \equiv \sqrt{\mathbf{r}\cdot\mathbf{r}}, \quad \mathbf{x}' \equiv \partial_l \mathbf{x} \qquad (1) \\
\delta\mathbf{B} &= \oint \frac{(\delta\mathbf{x}' \times \mathbf{r})}{r^3}\,dl \qquad - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\,dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\,dl \\
&= \oint \frac{(\delta\mathbf{x} \times \mathbf{x}')}{r^3}\,dl - 3\oint \frac{(\delta\mathbf{x} \times \mathbf{r})(\mathbf{r}\cdot\mathbf{x}')}{r^5}\,dl - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\,dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\,dl
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&= \oint \frac{(\delta\mathbf{x} \times \mathbf{x}')}{r^3}\, dl - 3\oint \frac{(\delta\mathbf{x} \times \mathbf{r})(\mathbf{r}\cdot\mathbf{x}')}{r^5}\, dl - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl \\
&= 2\oint \frac{(\delta\mathbf{x} \times \mathbf{x}')}{r^3}\, dl - 3\oint \frac{(\delta\mathbf{x} \times \mathbf{r})(\mathbf{r}\cdot\mathbf{x}')}{r^5}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl
\end{aligned}
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Correct, but not "transparent". Do tangential variations, $\delta\mathbf{x} \times \mathbf{x}' = 0$, change $\mathbf{B}$?

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$$
\delta\mathbf{B} = \oint \left[\frac{(\delta\mathbf{x} \times \mathbf{x}' \cdot \mathbf{r})\, 3\, \mathbf{r}}{r^5} - \frac{\delta\mathbf{x} \times \mathbf{x}'}{r^3} \right] dl \qquad (2)
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common factor

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$$= \oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \left(\frac{\mathbf{r}\, 3\, \mathbf{r}}{r^5} - \frac{\mathbf{I}}{r^3} \right) dl, \quad \text{where } \mathbf{v}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{v} = \mathbf{v}, \tag{3}$$

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$$= \oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \left(\frac{\mathbf{r}\, 3\, \mathbf{r}}{r^5} - \frac{\mathbf{I}}{r^3} \right) dl, \quad \text{where } \mathbf{v}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{v} = \mathbf{v}, \tag{3}$$

$$\delta\mathbf{B} = \oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \mathbf{R}\, dl \tag{4}$$

This is concise, and shows that tangential variations, $\delta\mathbf{x} \times \mathbf{x}' = 0$, do not alter the field.

Variations in line integrals with respect to variations in the line: length

$$L \equiv \oint (\mathbf{x}' \cdot \mathbf{x}')^{1/2}\, dl \qquad (1)$$

$$\delta L = \oint (\mathbf{x}' \cdot \mathbf{x}')^{-1/2} (\mathbf{x}' \cdot \delta\mathbf{x}')\, dl \qquad (2)$$

$$= \oint \delta\mathbf{x} \cdot \mathbf{x}' (\mathbf{x}' \cdot \mathbf{x}')^{-3/2} \mathbf{x}' \cdot \mathbf{x}''\, dl - \oint \delta\mathbf{x} \cdot \mathbf{x}'' (\mathbf{x}' \cdot \mathbf{x}')^{-1/2}\, dl \qquad (3)$$

Correct, but not transparent.

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Correct, but not transparent.

Use $(\delta\mathbf{x} \times \mathbf{x}') \cdot (\mathbf{x}' \times \mathbf{x}'') = (\delta\mathbf{x} \cdot \mathbf{x}') \cdot (\mathbf{x}' \cdot \mathbf{x}'') - (\delta\mathbf{x} \cdot \mathbf{x}'') \cdot (\mathbf{x}' \cdot \mathbf{x}')$.

$$\delta L = -\oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \boldsymbol{\kappa}, \quad \text{where} \quad \boldsymbol{\kappa} \equiv \frac{\mathbf{x}' \times \mathbf{x}''}{(\mathbf{x}' \cdot \mathbf{x}')^{3/2}} \tag{4}$$

The first variation with respect to variations in the curve is easy to calculate

1. The first variation of the penalized quadratic-flux, $\mathcal{F}[\mathbf{x}_i, \bar{\mathbf{x}}] \equiv \int_{\mathcal{S}} \frac{1}{2} B_n^2 ds + \omega L$, is

$$\delta\mathcal{F} = \oint_i \delta\mathbf{x}_i \cdot \frac{\delta F}{\delta \mathbf{x}_i}\, dl, \tag{1}$$

$$\text{where } \frac{\delta\mathcal{F}}{\delta\mathbf{x}_i} \equiv \mathbf{x}_i' \times \left(\oint_{\mathcal{S}} \mathbf{R}_{i,n}\, B_n\, ds + \omega\, \boldsymbol{\kappa}_i \right). \tag{2}$$

2. "Slow motion" descent algorithm is easy to implement,

$$\frac{\partial \mathbf{x}_i}{\partial \tau} = -\frac{\delta\mathcal{F}}{\delta\mathbf{x}_i}, \quad \frac{\partial\mathcal{F}}{\partial\tau} = -\oint_i \left(\frac{\delta\mathcal{F}}{\delta\mathbf{x}_i}\right)^2 dl \leq 0. \tag{3}$$

3. Coils cannot pass through surface (infinities); descent algorithm preserves linking

$$\text{Gauss linking integral} = \frac{1}{4\pi} \oint_i \oint_a \frac{\mathbf{x}_i - \mathbf{x}_a}{|\mathbf{x}_i - \mathbf{x}_a|^3} \cdot d\mathbf{x}_i \times d\mathbf{x}_a. \tag{4}$$

Second derivatives can be calculated, allows fast algorithms and sensitivity analysis.

1. Let $\mathbf{c} \equiv \{\mathbf{x}_{i,n}\}$, degrees-of-freedom that parameterize external currents.

   For example, $\mathbf{x}_i(l) = x_i(l)\mathbf{i} + y_i(l)\mathbf{j} + z_i(l)\mathbf{z}$ where

$$x_i(l) = \sum_n \left[x^c_{i,n} \cos(nl) + x^s_{i,n} \sin(nl) \right] \tag{1}$$

$$y_i(l) = \sum_n \left[y^c_{i,n} \cos(nl) + y^s_{i,n} \sin(nl) \right] \tag{2}$$

$$z_i(l) = \sum_n \left[z^c_{i,n} \cos(nl) + z^s_{i,n} \sin(nl) \right] \tag{3}$$

2. $\mathcal{F}(\mathbf{c} + \delta\mathbf{c}) \approx \mathcal{F}(\mathbf{c}) + \nabla_{\mathbf{c}}\mathcal{F} \cdot \delta\mathbf{c} + \frac{1}{2}\delta\mathbf{c}^T \cdot \nabla^2_{\mathbf{cc}}\mathcal{F} \cdot \delta\mathbf{c}$

3. Inverting Hessian allows Newton method.
   [C. Zhu, S.R. Hudson *et al.*, Plasma Phys. Control. Fusion, submitted (2018)]

4. Eigenvalues of Hessian describe sensitivity to coil placement errors.
   [C. Zhu, S.R. Hudson *et al.*, in preparation (2018)]

The quadratic-flux is an analytic function of the surface. So, what happens if the surface varies?

1. The variation in F resulting from variations, $\delta \mathbf{x}_i$ and $\delta \bar{\mathbf{x}}$, in the geometry of the i-th coil and the surface is

$$\delta^2 F = \oint_i \delta \mathbf{x}_i \cdot \oint_{\mathcal{S}} \frac{\delta^2 F}{\delta \mathbf{x}_i \delta \bar{\mathbf{x}}} \cdot \delta \bar{\mathbf{x}} \; ds \, dl, \qquad (1)$$

where

$$\frac{\delta^2 F}{\delta \mathbf{x}_i \delta \bar{\mathbf{x}}} \equiv \mathbf{x}_i' \times (\mathbf{R}_S \cdot \nabla B_n + \mathbf{B}_S \cdot \nabla \mathbf{R}_n + B_n \mathbf{R} \cdot \mathbf{H}) \, \mathbf{n}, \qquad (2)$$

where

i. $\mathbf{B}_S \equiv \mathbf{B} - B_n \mathbf{n}$ is the projection of $\mathbf{B}$ in the tangent plane to $\bar{\mathbf{x}}$, and $\mathbf{R}_S \equiv \mathbf{R} - \mathbf{R}_n \mathbf{n}$.

ii. The mean curvature can be written $\mathbf{H} \equiv -\mathbf{n} \, (\nabla \cdot \mathbf{n})$.

3. The shape of the optimal coils must change with the surface to preserve $\nabla_{\mathbf{c}} \mathcal{F} = 0$,

$\nabla_{\mathbf{c}} \mathcal{F}(\mathbf{c} + \delta \mathbf{c}, \mathbf{s} + \delta \mathbf{s}) \approx \nabla^2_{\mathbf{cc}} \mathcal{F} \cdot \delta \mathbf{c} + \nabla^2_{\mathbf{cs}} \mathcal{F} \cdot \delta \mathbf{s} = 0$, and from this

$$\frac{\partial \mathbf{c}}{\partial \mathbf{s}} = - \left(\nabla^2_{\mathbf{cc}} \mathcal{F} \right)^{-1} \cdot \nabla^2_{\mathbf{cs}} \mathcal{F}. \qquad (3)$$

Variations of surface integrals with changes in the surface: surface area and mean curvature.

1. Parameterized surface, $\mathbf{x}(\theta, \zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal} \;\; \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \;\; \text{d(area)} \;\; ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

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   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \to \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce $\delta\mathbf{x}_\theta \equiv \partial_\theta \delta\mathbf{x}, \quad \delta\mathbf{x}_\zeta \equiv \partial_\zeta \delta\mathbf{x}$

$$\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| \quad = \quad \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2}\, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) \tag{2}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature.

1. Parameterized surface, $\mathbf{x}(\theta, \zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal} \;\; \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \;\; \text{d(area)} \;\; ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta, \zeta) \rightarrow \mathbf{x}(\theta, \zeta) + \delta \mathbf{x}(\theta, \zeta)$ induce

$$\begin{aligned}
\delta |\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2} [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2} \, 2 \, (\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta \mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta \mathbf{x}_\zeta) && (2) \\
&= \mathbf{n} \cdot (\delta \mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta \mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)
\end{aligned}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature.

1. Parameterized surface, $\mathbf{x}(\theta,\zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal } \ \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \ \ \text{d(area)} \ \ ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \to \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce

$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2}\, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n} \cdot (\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)
\end{aligned}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature.

1. Parameterized surface, $\mathbf{x}(\theta, \zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal } \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \quad \text{d(area)} \ \ ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta, \zeta) \to \mathbf{x}(\theta, \zeta) + \delta\mathbf{x}(\theta, \zeta)$ induce

$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2} \, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n} \cdot (\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)\\
\int \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta &= -\int \delta\mathbf{x} \cdot (\mathbf{x}_{\zeta\theta} \times \mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n}) \, d\theta d\zeta \\
&\quad + \int \delta\mathbf{x} \cdot (\mathbf{x}_{\theta\zeta} \times \mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n}) \, d\theta d\zeta && (5)
\end{aligned}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature.

1. Parameterized surface, $\mathbf{x}(\theta,\zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal } \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \quad \text{d(area)} \;\; ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \rightarrow \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce

$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2}\, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n} \cdot (\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)\\
\int \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta|\, d\theta d\zeta &= -\int \delta\mathbf{x} \cdot (\mathbf{x}_{\zeta\theta} \times \mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n})\, d\theta d\zeta \\
&\quad + \int \delta\mathbf{x} \cdot (\mathbf{x}_{\theta\zeta} \times \mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n})\, d\theta d\zeta && (5)\\
&= -\int \delta\mathbf{x} \cdot (\mathbf{x}_\zeta \partial_\theta - \mathbf{x}_\theta \partial_\zeta) \times \mathbf{n}\; d\theta d\zeta && (6)
\end{aligned}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature.

1. Parameterized surface, $\mathbf{x}(\theta,\zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal} \;\; \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \;\; \text{d(area)} \;\; ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \to \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce

$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2} \, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n} \cdot (\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)\\
\int \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta &= -\int \delta\mathbf{x} \cdot (\mathbf{x}_{\zeta\theta} \times \mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n}) \, d\theta d\zeta \\
&\quad + \int \delta\mathbf{x} \cdot (\mathbf{x}_{\theta\zeta} \times \mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n}) \, d\theta d\zeta && (5)\\
&= -\int \delta\mathbf{x} \cdot (\mathbf{x}_\zeta \partial_\theta - \mathbf{x}_\theta \partial_\zeta) \times \mathbf{n} \; d\theta d\zeta && (6)\\
&= \int \delta\mathbf{x} \cdot \mathbf{H} \, ds, \;\; \text{where} \;\; \mathbf{H} \equiv -\mathbf{n}\,(\nabla \cdot \mathbf{n}) = \text{mean curvature.} && (7)
\end{aligned}$$

Can the surface be varied to simplify the coils under the constraint of conserved plasma properties?

1. Introduce a measure of coil complexity, $\mathcal{C}(\mathbf{c})$, that we wish to minimize, e.g. integrated torsion,

$$\mathcal{C} \equiv \oint \frac{\mathbf{x}' \cdot \mathbf{x}'' \times \mathbf{x}'''}{|\mathbf{x}' \times \mathbf{x}''|^2} dl \tag{1}$$

   which quantifies "non-planar-ness" of the coils.

2. Introduce a plasma property, $\mathcal{P}(\bar{\mathbf{x}})$, that we wish to constrain, e.g. rotational-transform on axis.

3. Can minimize coil complexity subject to constrained plasma properties, i.e. extremize

$$\mathcal{G}(\bar{\mathbf{x}}) \equiv \mathcal{C}(\mathbf{x}_i(\bar{\mathbf{x}})) + \lambda \left[\mathcal{P}(\bar{\mathbf{x}}) - \mathcal{P}_0\right]. \tag{2}$$

4. Rotational-transform is an easy first test: rotational-transform can be produced [Mercier (1964)]

   i. by shaping the boundary (i.e., rotating elliptical boundary), or by

   ii. by shaping the magnetic axis (through torsion), or

   iii. by both.

5. Solutions satisfy $\dfrac{\partial \mathbf{x}_i}{\partial \bar{\mathbf{x}}} \cdot \dfrac{\partial \mathcal{C}}{\partial \mathbf{x}_i} + \lambda \dfrac{\partial \mathcal{P}}{\partial \bar{\mathbf{x}}} = 0.$

A circular cross-section on an axis with torsion gives simpler coils than a rotating ellipse on a circular magnetic axis.

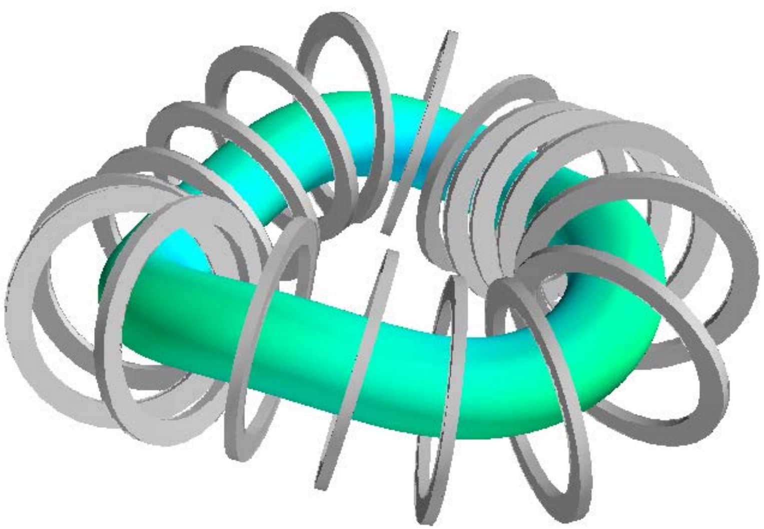
1. "Simple" in this case means more planar. Have not yet measured coil-coil spacing, for example.

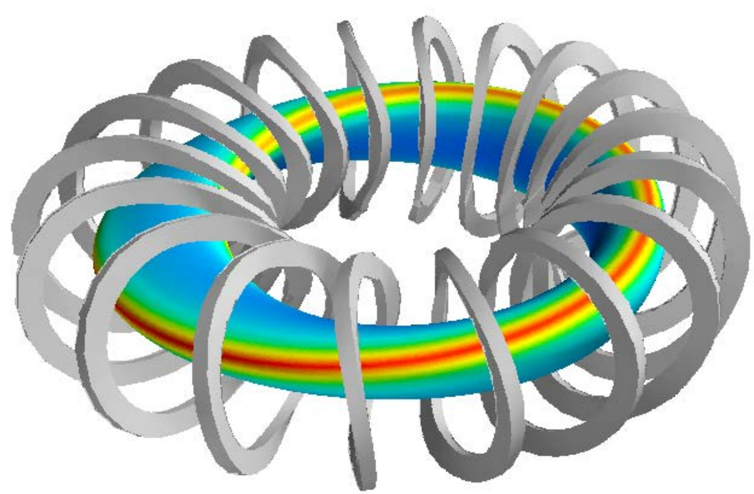
2. The following have

   i. the same rotational-transform on axis, $\iota\!\!\!-\approx 0.101$, and good flux surfaces,

   ii. total volume $= 0.7986m^3$, 18 coils,

   iii. average int. torsion of coils is
   $0.005m^{-1}$ for circular cross-section high-torsion axis, and
   $0.800m^{-1}$ for rotating elliptical cross-section circular axis.





Variations of surface integrals with changes in the surface: surface area and mean curvature.

$$
\begin{aligned}
\text{area} &= \int_{\mathcal{S}} ds, \ \text{where } ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta, \ \text{ and } \ \mathbf{x}_\theta \equiv \partial_\theta \mathbf{x} & (1)\\
|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= [(\mathbf{x}_\theta \times \mathbf{x}_\zeta) \cdot (\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2} & (2)\\
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \mathbf{n} \cdot (\delta \mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta \mathbf{x}_\zeta \times \mathbf{x}_\theta), \ \text{ where } \ \mathbf{n} = (\mathbf{x}_\theta \times \mathbf{x}_\zeta)/|\mathbf{x}_\theta \times \mathbf{x}_\zeta| & (3)\\
\delta(\text{area}) &= \int_{\mathcal{S}} \partial_\theta \delta \mathbf{x} \cdot \mathbf{x}_\zeta \times \mathbf{n} \, d\theta d\zeta - \int_{\mathcal{S}} \partial_\zeta \delta \mathbf{x} \cdot \mathbf{x}_\theta \times \mathbf{n} \, d\theta d\zeta & (4)\\
&= - \int_{\mathcal{S}} \delta \mathbf{x} \cdot (\mathbf{x}_{\zeta\theta} \times \mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n}) \, d\theta d\zeta & (5)\\
&\quad + \int_{\mathcal{S}} \delta \mathbf{x} \cdot (\mathbf{x}_{\theta\zeta} \times \mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n}) \, d\theta d\zeta & (6)\\
&= - \int_{\mathcal{S}} \delta \mathbf{x} \cdot (\mathbf{x}_\zeta \partial_\theta - \mathbf{x}_\theta \partial_\zeta) \times \mathbf{n} \, d\theta d\zeta & (7)\\
&= - \int_{\mathcal{S}} \delta \mathbf{x} \cdot (\mathbf{n} \times \nabla) \times \mathbf{n} \, ds, \ \text{ where } \ \mathbf{n} = \nabla s / |\nabla s| \ \text{ and } \ \nabla \equiv \nabla s \, \partial_s + \nabla \theta \, \partial_\theta + \nabla \zeta \, \partial_\zeta, & (8)\\
&= - \int_{\mathcal{S}} \delta \mathbf{x} \cdot \mathbf{n} \, (\nabla \cdot \mathbf{n}) \, ds & (9)\\
&= - \int_{\mathcal{S}} \delta \mathbf{x} \cdot \mathbf{H} \, ds, \ \text{ mean curvature } \ \mathbf{H} \equiv \mathbf{n} \, (\nabla \cdot \mathbf{n}) & (10)
\end{aligned}
$$

1) R.L. Dewar, S.R. Hudson & P.F. Price, Phys. Lett. A., **194** 49 (1994)