Differentiating the coil geometry with respect to the plasma boundary

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1) The *Simplest Possible Algorithm*© *(SPA)* for designing stellarator coils is described.

2) The coil geometry has "maximum freedom", and the target function is "minimally constrained".

3) Fast, reliable and insightful numerical algorithms are enabled by exploiting 1st and 2nd derivatives with respect to the coil geometry *and* the "target surface".

1) P. Merkel, Nucl. Fus., **27** 867 (1987)
2) R.L. Dewar, S.R. Hudson & P.F. Price, Phys. Lett. A., **194** 49 (1994)
3) M. Landreman, Nucl. Fusion, **57** 046003 (2017)

Vacuum fields in given domain uniquely defined by supplied boundary conditions

1. Given volume $\mathcal{V}$, with closed boundary, $\mathcal{S} \equiv \partial\mathcal{V}$.
2. Vacuum fields satisfy $\nabla \times \mathbf{B} = 0$, which suggests $\mathbf{B} = \nabla\Phi$.
3. Given a suitable boundary condition, e.g. $\mathbf{B} \cdot \mathbf{n}$ on $\mathcal{S}$.
4. Divergence-free fields, $\nabla \cdot \mathbf{B} = 0$, implies constraint of net flux $\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = 0$.
5. Toroidal flux $\Psi \equiv \oint_{\mathcal{L}} \mathbf{A} \cdot d\mathbf{l}$, (require one loop integral per "hole").
6. In $\mathcal{V}$, solution to $\nabla \cdot \nabla\Phi = 0$ is unique.

Task is to design coils that provide required $\mathbf{B} \cdot \mathbf{n}$ on given surface.

Minima of "regularized" functional give required set of external current-carrying coils

1. Introduce $\mathbf{x}_i(l),\ i = 1, \ldots, N_C,$ to represent closed current-carrying curves.

2. Let $\bar{\mathbf{x}}(\theta, \zeta) \equiv \mathcal{S}$ represent plasma boundary.

3. With finite degrees-of-freedom, cannot *exactly* recover arbitrary $B_n \equiv \mathbf{B} \cdot \mathbf{n}$ on $\mathcal{S}$.

4. Instead, minimize **quadratic-flux functional** with penalty on length,

$$\mathcal{F}[\mathbf{x}_i, \bar{\mathbf{x}}] \equiv \oint_{\mathcal{S}} \frac{1}{2} B_n^2 ds + \omega L, \quad \text{where } L[\mathbf{x}_i] = \sum_i \oint |\mathbf{x}_i'|\, dl. \tag{1}$$

5. Numerically need to find minima, perform sensitivity studies, and advantageous to construct derivatives.

6. Optimal coils for given surface are defined by $\left.\dfrac{\delta \mathcal{F}}{\delta \mathbf{x}_i}\right|_{\bar{\mathbf{x}}} = 0.$

7. Simple to include (i) an additional factor $\displaystyle\oint_{\mathcal{S}} \frac{1}{2} w_{m,n} |B^n_{m,n}| ds$ to reflect that some "error fields" are more important to control than others; and (ii) additional "engineering" penalties, such as coil-coil distance.

Variations in line integrals with respect to variations in the line: length

$$L \equiv \oint (\mathbf{x}' \cdot \mathbf{x}')^{1/2} \, dl \tag{1}$$

$$\delta L = \oint (\mathbf{x}' \cdot \mathbf{x}')^{-1/2} (\mathbf{x}' \cdot \delta\mathbf{x}') \, dl \tag{2}$$

$$= \oint \delta\mathbf{x} \cdot \mathbf{x}' (\mathbf{x}' \cdot \mathbf{x}')^{-3/2} \mathbf{x}' \cdot \mathbf{x}'' \, dl - \oint \delta\mathbf{x} \cdot \mathbf{x}'' (\mathbf{x}' \cdot \mathbf{x}')^{-1/2} \, dl \tag{3}$$

Correct, but not transparent. Do tangential variations, $\delta\mathbf{x} \times \mathbf{x}' = 0$, change length?

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Use $(\delta\mathbf{x} \times \mathbf{x}') \cdot (\mathbf{x}' \times \mathbf{x}'') = (\delta\mathbf{x} \cdot \mathbf{x}') \cdot (\mathbf{x}' \cdot \mathbf{x}'') - (\delta\mathbf{x} \cdot \mathbf{x}'') \cdot (\mathbf{x}' \cdot \mathbf{x}')$.

$$\delta L = -\oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \boldsymbol{\kappa}, \quad \text{where} \quad \underbrace{\boldsymbol{\kappa} \equiv \frac{\mathbf{x}' \times \mathbf{x}''}{(\mathbf{x}' \cdot \mathbf{x}')^{3/2}}}_{\text{curvature}} \tag{4}$$

The Biot-Savart law gives the magnetic field, variation in curves gives variation in magnetic field

1. The magnetic field is from Biot-Savart,

$$\mathbf{B}_i(\bar{\mathbf{x}}) = I_i \oint_i \frac{\mathbf{x}_i' \times \mathbf{r}}{r^3}\, dl, \tag{1}$$

   where I_i is the current and $\mathbf{r}(\theta, \zeta, l) \equiv \bar{\mathbf{x}}(\theta, \zeta) - \mathbf{x}_i(l)$ and $\mathbf{x}_i' \equiv \dfrac{\partial \mathbf{x}_i}{\partial l}$.

2. For simplicity, set $I_i = 1$. (Trivial solutions avoided, ignore toroidal flux constraint.)

3. Variations in the curve induce variations in the field,

$$\delta\mathbf{B}(\bar{\mathbf{x}}) = \oint_i (\delta\mathbf{x}_i \times \mathbf{x}_i') \cdot \mathbf{R}_i\, dl, \tag{2}$$

   where $\mathbf{R} = \dfrac{3\,\mathbf{r}\,\mathbf{r}}{r^5} - \dfrac{\mathbf{I}}{r^3}$, and $\mathbf{I}$ is the "idemfactor", e.g. $\mathbf{I} = \mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k}$.

4. Let me go through the algebra more slowly.

Variations in line integrals with respect to variations in the line: magnetic field

$$\mathbf{B} = \oint \frac{(\mathbf{x}' \times \mathbf{r})}{r^3}\, dl, \quad \text{where} \;\; \mathbf{r} \equiv \bar{\mathbf{x}} - \mathbf{x}, \;\; r \equiv \sqrt{\mathbf{r}\cdot\mathbf{r}}, \;\; \mathbf{x}' \equiv \partial_l \mathbf{x} \tag{1}$$

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$$\delta\mathbf{B} = \oint \frac{(\delta\mathbf{x}' \times \mathbf{r})}{r^3}\, dl - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3 \oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl$$

Variations in line integrals with respect to variations in the line: magnetic field

$$
\begin{aligned}
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\delta\mathbf{B} &= \oint \frac{(\delta\mathbf{x}' \times \mathbf{r})}{r^3}\, dl \qquad - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl \\
&= \oint \frac{(\delta\mathbf{x} \times \mathbf{x}')}{r^3}\, dl - 3\oint \frac{(\delta\mathbf{x} \times \mathbf{r})(\mathbf{r}\cdot\mathbf{x}')}{r^5}\, dl - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl
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&= 2\oint \frac{(\delta\mathbf{x} \times \mathbf{x}')}{r^3}\, dl - 3\oint \frac{(\delta\mathbf{x} \times \mathbf{r})(\mathbf{r}\cdot\mathbf{x}')}{r^5}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl
\end{aligned} \tag{2}$$

Correct, but not "transparent". Do tangential variations, $\delta\mathbf{x} \times \mathbf{x}' = 0$, change $\mathbf{B}$?

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$$\delta\mathbf{B} = \oint \left[\frac{(\delta\mathbf{x} \times \mathbf{x}' \cdot \mathbf{r})\, 3\, \mathbf{r}}{r^5} - \frac{\delta\mathbf{x} \times \mathbf{x}'}{r^3} \right] dl \tag{3}$$

Variations in line integrals with respect to variations in the line: magnetic field

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common factor

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$$\delta\mathbf{B} = \oint \left[\frac{(\delta\mathbf{x} \times \mathbf{x}' \cdot \mathbf{r})\, 3\,\mathbf{r}}{r^5} - \frac{\delta\mathbf{x} \times \mathbf{x}'}{r^3}\right] dl \tag{3}$$

$$= \oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \left(\frac{\mathbf{r}\, 3\, \mathbf{r}}{r^5} - \frac{\mathbf{I}}{r^3}\right) dl, \quad \text{where } \mathbf{v}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{v} = \mathbf{v}, \tag{4}$$

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&= \oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \left(\frac{\mathbf{r}\, 3\,\mathbf{r}}{r^5} - \frac{\mathbf{I}}{r^3} \right) dl, \quad \text{where } \mathbf{v}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{v} = \mathbf{v}, && (4) \\
\delta\mathbf{B} &= \oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \mathbf{R}\, dl && (5)
\end{aligned}
$$

This is concise, and shows that tangential variations, $\delta\mathbf{x} \times \mathbf{x}' = 0$, do not alter the field.

The first variation with respect to variations in the curve is easy to calculate

1. The first variation of the penalized quadratic-flux, $\mathcal{F}[\mathbf{x}_i, \bar{\mathbf{x}}] \equiv \int_{\mathcal{S}} \frac{1}{2} B_n^2 ds + \omega L$, is

$$\delta \mathcal{F} = \oint_i \delta \mathbf{x}_i \cdot \left. \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right|_{\bar{\mathbf{x}}} dl, \text{ where } \left. \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right|_{\bar{\mathbf{x}}} \equiv \mathbf{x}_i' \times \left(\oint_{\mathcal{S}} \mathbf{R}_{i,n} \, B_n \, ds + \omega \, \boldsymbol{\kappa}_i \right). \tag{1}$$

2. "Slow motion" steepest-descent algorithm is easy to implement,

$$\frac{\partial \mathbf{x}_i}{\partial \tau} = - \left. \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right|_{\bar{\mathbf{x}}}, \quad \frac{\partial \mathcal{F}}{\partial \tau} = - \oint_i \left(\frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right)^2 dl \leq 0. \tag{2}$$

3. Coils cannot continuously pass through surface, as this would produce infinities; so the descent algorithm preserves

   the Gauss linking integral $= \dfrac{1}{4\pi} \oint_i \oint_a \dfrac{\mathbf{x}_i - \mathbf{x}_a}{|\mathbf{x}_i - \mathbf{x}_a|^3} \cdot d\mathbf{x}_i \times d\mathbf{x}_a$,

   and thereby avoids the trivial solution that the coils are removed to infinity.

Flexible Optimized Coils Using Space (FOCUS) curves

Caoxiang Zhu, Stuart R. Hudson *et al., "New method to design stellarator coils without the winding surface",* Nucl. Fusion **58**, 016008 (2017)

Second derivatives can be calculated, allows fast algorithms and sensitivity analysis

1. Let $\mathbf{c} \equiv \{\mathbf{x}_{i,n}\}$, degrees-of-freedom that parameterize external currents.

   For example, $\mathbf{x}_i(l) = x_i(l)\mathbf{i} + y_i(l)\mathbf{j} + z_i(l)\mathbf{z}$ where

$$x_i(l) = \sum_n \left[x_{i,n}^c \cos(nl) + x_{i,n}^s \sin(nl)\right] \tag{1}$$

$$y_i(l) = \sum_n \left[y_{i,n}^c \cos(nl) + y_{i,n}^s \sin(nl)\right] \tag{2}$$

$$z_i(l) = \sum_n \left[z_{i,n}^c \cos(nl) + z_{i,n}^s \sin(nl)\right] \tag{3}$$

2. $\mathcal{F}(\mathbf{c} + \delta\mathbf{c}) \approx \mathcal{F}(\mathbf{c}) + \nabla_{\mathbf{c}}\mathcal{F} \cdot \delta\mathbf{c} + \frac{1}{2}\delta\mathbf{c}^T \cdot \nabla^2_{\mathbf{cc}}\mathcal{F} \cdot \delta\mathbf{c}$

3. Inverting Hessian allows Newton method.
   [C. Zhu, S.R. Hudson *et al.*, Plasma Phys. Control. Fusion, in press (2018)]

4. Eigenvalues of Hessian describe sensitivity to coil placement errors.
   [C. Zhu, S.R. Hudson *et al.*, Plasma Phys. Control. Fusion, in press (2018)]

5. A piecewise-linear representation is under construction.

Variations of surface integrals with changes in the surface: surface area and mean curvature

1. Parametrized surface, $\mathbf{x}(\theta, \zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal} \;\; \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \;\; \text{d(area)} \;\; ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

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   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \to \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce $\delta\mathbf{x}_\theta \equiv \partial_\theta \delta\mathbf{x}, \quad \delta\mathbf{x}_\zeta \equiv \partial_\zeta \delta\mathbf{x}$

$$\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| \ \ = \ \ \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2} \, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) \tag{2}$$

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   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta, \zeta) \rightarrow \mathbf{x}(\theta, \zeta) + \delta \mathbf{x}(\theta, \zeta)$ induce

$$\begin{aligned} \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2} \, 2 \, (\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta \mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta \mathbf{x}_\zeta) && (2) \\ &= \mathbf{n} \cdot (\delta \mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta \mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3) \end{aligned}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature

1. Parametrized surface, $\mathbf{x}(\theta,\zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal} \;\; \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \quad \text{d(area)} \;\; ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \rightarrow \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce

$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2}\, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n} \cdot (\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)
\end{aligned}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature

1. Parametrized surface, $\mathbf{x}(\theta,\zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal } \; \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \quad \mathrm{d(area)} \;\; ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \to \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce

$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2} \, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n} \cdot (\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)\\
\int \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta &= -\int \delta\mathbf{x} \cdot (\mathbf{x}_{\zeta\theta} \times \mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n}) \, d\theta d\zeta \\
&\quad + \int \delta\mathbf{x} \cdot (\mathbf{x}_{\theta\zeta} \times \mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n}) \, d\theta d\zeta && (5)
\end{aligned}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature

1. Parametrized surface, $\mathbf{x}(\theta,\zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal} \;\; \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \;\; \mathrm{d(area)} \;\; ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \to \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce

$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2}\, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n}\cdot(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta\cdot(\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta\cdot(\mathbf{x}_\theta \times \mathbf{n}) && (4)\\
\int \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta|\, d\theta d\zeta &= -\int \delta\mathbf{x}\cdot(\mathbf{x}_{\zeta\theta} \times \mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n})\, d\theta d\zeta \\
&\quad + \int \delta\mathbf{x}\cdot(\mathbf{x}_{\theta\zeta} \times \mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n})\, d\theta d\zeta && (5)\\
&= -\int \delta\mathbf{x}\cdot(\mathbf{x}_\zeta\partial_\theta - \mathbf{x}_\theta\partial_\zeta) \times \mathbf{n}\; d\theta d\zeta && (6)
\end{aligned}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature

1. Parametrized surface, $\mathbf{x}(\theta,\zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal} \;\; \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \;\; \text{d(area)} \;\; ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta|\, d\theta d\zeta. \tag{1}$$

   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \to \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce

$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2}\, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n}\cdot(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)\\
\int \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta|\, d\theta d\zeta &= -\int \delta\mathbf{x}\cdot(\mathbf{x}_{\zeta\theta} \times \mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n})\, d\theta d\zeta \\
&\quad + \int \delta\mathbf{x}\cdot(\mathbf{x}_{\theta\zeta} \times \mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n})\, d\theta d\zeta && (5)\\
&= -\int \delta\mathbf{x}\cdot(\mathbf{x}_\zeta \partial_\theta - \mathbf{x}_\theta \partial_\zeta) \times \mathbf{n}\; d\theta d\zeta && (6)\\
&= -\int \delta\mathbf{x}\cdot\mathbf{n}\,(\nabla\cdot\mathbf{n}) ds, \quad \text{and only normal variations matter.} && (7)
\end{aligned}$$

The quadratic-flux is an analytic function of the surface. So, what happens if the surface varies?

1. The variation in $\mathcal{F}$ resulting from variations, $\delta\mathbf{x}_i$ and $\delta\bar{\mathbf{x}}$, in the geometry of the i-th coil and the surface is

$$\begin{aligned} \delta^2\mathcal{F} &\equiv \oint_i \delta\mathbf{x}_i \cdot \oint_{\mathcal{S}} \frac{\delta^2 F}{\delta\mathbf{x}_i \delta\bar{\mathbf{x}}} \cdot \delta\bar{\mathbf{x}} \;\; ds\, dl, && (1) \\ \frac{\delta^2\mathcal{F}}{\delta\mathbf{x}_i \delta\bar{\mathbf{x}}} &= \mathbf{x}_i' \times (\mathbf{R}_S \cdot \nabla B_n + \mathbf{B}_S \cdot \nabla \mathbf{R}_n + B_n\, \mathbf{R} \cdot \mathbf{H})\, \mathbf{n}, && (2) \end{aligned}$$

where

i. $\mathbf{B}_S \equiv \mathbf{B} - B_n\mathbf{n}$ is projection of $\mathbf{B}$ in the tangent plane to $\bar{\mathbf{x}}$, and $\mathbf{R}_S \equiv \mathbf{R} - \mathbf{R}_n\mathbf{n}$,

ii. the mean curvature can be written $\mathbf{H} \equiv -\mathbf{n}\,(\nabla \cdot \mathbf{n})$,

iii. the calculus of variations of the quadratic-flux w.r.t. surface variations was presented by Dewar *et al.* [Phys. Lett. A **194**, 49 (1994)].

3. The shape of the optimal coils must change with the surface to preserve $\nabla_{\mathbf{c}}\mathcal{F} = 0$,

$\nabla_{\mathbf{c}}\mathcal{F}(\mathbf{c} + \delta\mathbf{c}, \mathbf{s} + \delta\mathbf{s}) \approx \nabla^2_{\mathbf{cc}}\mathcal{F} \cdot \delta\mathbf{c} + \nabla^2_{\mathbf{cs}}\mathcal{F} \cdot \delta\mathbf{s} = 0$, and from this

$$\frac{\partial\mathbf{c}}{\partial\mathbf{s}} = -\left(\nabla^2_{\mathbf{cc}}\mathcal{F}\right)^{-1} \cdot \nabla^2_{\mathbf{cs}}\mathcal{F}. \qquad (3)$$

Part Two:
Can the surface be varied to simplify the coils under the constraint of conserved plasma properties?

1. Introduce a measure of coil complexity, $\mathcal{C}(\mathbf{c})$, that we wish to minimize,

   e.g. integrated torsion, $\mathcal{C} \equiv \oint \frac{\mathbf{x}' \cdot \mathbf{x}'' \times \mathbf{x}'''}{|\mathbf{x}' \times \mathbf{x}''|^2} dl$

   which quantifies the "non-planar-ness" of the coils.

2. Introduce a plasma property, $\mathcal{P}(\bar{\mathbf{x}})$, that we wish to constrain.

3. Can minimize coil complexity subject to constrained plasma properties, i.e. extremize

$$\mathcal{G}(\bar{\mathbf{x}}) \equiv \mathcal{C}(\mathbf{x}_i(\bar{\mathbf{x}})) + \lambda \left[\mathcal{P}(\bar{\mathbf{x}}) - \mathcal{P}_0\right], \tag{1}$$

   where λ is a Lagrange multiplier.

5. Solutions satisfy $\dfrac{\partial \mathbf{x}_i}{\partial \bar{\mathbf{x}}} \cdot \dfrac{\partial \mathcal{C}}{\partial \mathbf{x}_i} + \lambda \dfrac{\partial \mathcal{P}}{\partial \bar{\mathbf{x}}} = 0.$

Example: rotational-transform on axis depends on "ellipticity" and torsion of axis.

1. Rotational-transform on axis, $\iota\!\!\!-_a$,
   can be produced in vacuum [Mercier (1964)]

    i. by shaping the boundary
       (i.e., rotating ellipse),

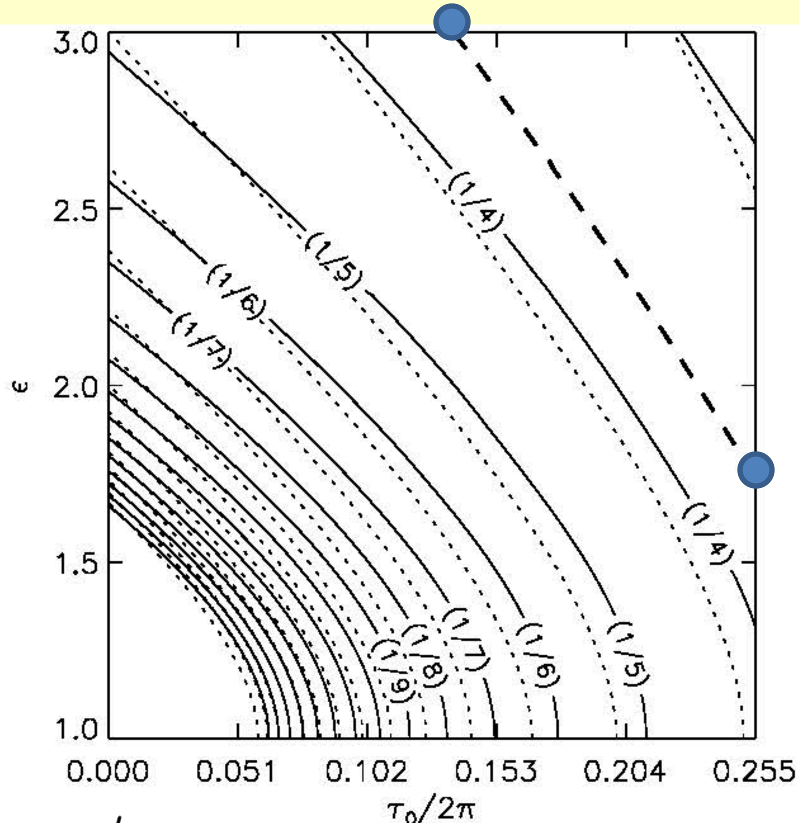
    ii. by shaping the magnetic axis
        (through torsion),

    iii. or by both;

$$\iota\!\!\!-_a = \frac{(\epsilon-1)^2}{\epsilon^2+1}\frac{N}{2} + \frac{2\epsilon}{\epsilon^2+1}\bar{\tau}. \qquad (1)$$

2. There is freedom to change boundary at $\iota\!\!\!-_a = const.$,
   which can be used to simplify the coils.

3. A two-parameter family of surfaces parametrized by ellipticity, ϵ, and integrated axis torsion, $\bar{\tau}$, with $R = 1.0$ and $a = 0.2$, is constructed; coils constructed using FOCUS, $\iota\!\!\!-_a$ measured numerically.

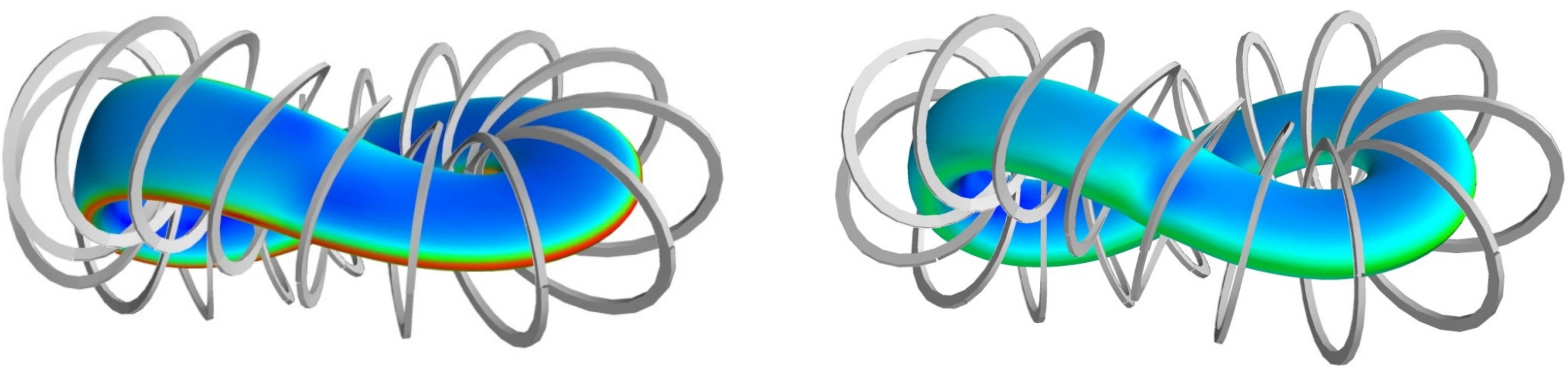


contours of $\iota\!\!\!-_a$:
dotted from Eqn(1);
solid from coil field;
dashed $\iota\!\!\!-_a = \dfrac{1+\gamma}{3+4\gamma} \approx 0.276$

A circular cross-section with axis torsion gives simpler coils than a rotating ellipse with circular magnetic axis

1. "Simple" in this case means more planar.
2. The following have
   i. the same rotational-transform on axis, $\iota_a \approx 0.276$, and good flux surfaces,
   ii. total volume $= 0.799m^3$, 18 coils, $N_{FP} \equiv$ field-periods $= 1$,
   iii. average length and complexity of the coils is

$$\langle L \rangle = 3.07m \text{ and } \langle \mathcal{C} \rangle = 0.66m^{-1}, \text{ and } \langle L \rangle = 2.88m \text{ and } \langle \mathcal{C} \rangle = 0.12m^{-1}.$$



3. Color indicates mean curvature.

Another example:
one purely elliptical, the other purely torsion

1. The following have

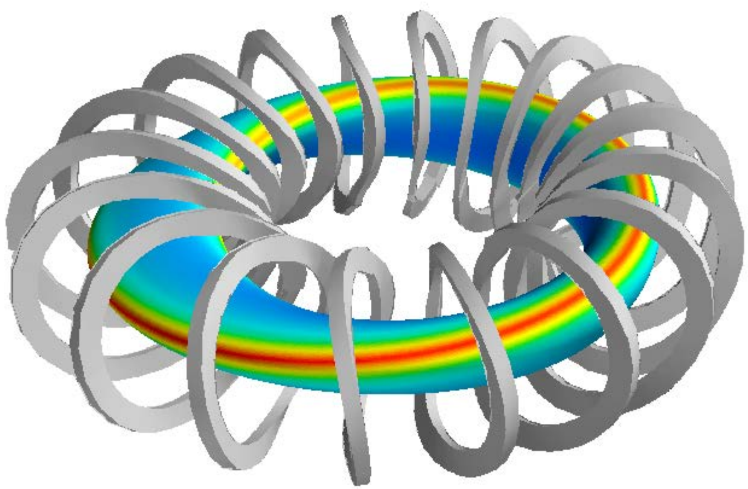
   i. the same rotational-transform on axis, $\iota \approx 0.101$, and good flux surfaces,

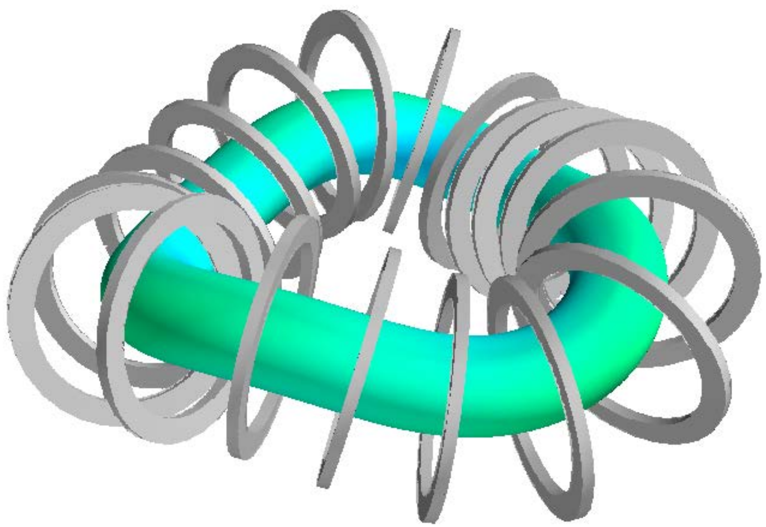
   ii. total volume $= 0.7986m^3$, 18 coils, $N_{FP} = 1$,

The average complexity of the coils is:

$\langle \mathcal{C} \rangle = 0.800m^{-1}$,

$\langle \mathcal{C} \rangle = 0.005m^{-1}$.





Summary

1) The *Simplest Possible Algorithm*© *(SPA)* for designing stellarator coils is described.

2) The coil geometry has "maximum freedom", and the target function is "minimally constrained". (Additional constraints can be added.)

3) Fast, reliable and insightful numerical algorithms are enabled by exploiting 1^{st} and 2^{nd} derivatives with respect to the coil geometry *and* the "target surface".

Some relevant papers

1) P. Merkel, Nucl. Fusion **27**, 867 (1987)
2) R.L. Dewar, S.R. Hudson & P.F. Price, Phys. Lett. A **194**, 49 (1994)
3) M. Landreman, Nucl. Fusion **57**, 046003 (2017)

4) Caoxiang Zhu, Stuart R. Hudson *et al., "New method to design stellarator coils without the winding surface",* Nucl. Fusion **58**, 016008 (2017)
5) Caoxiang Zhu, Stuart R. Hudson *et al.*, *"Designing stellarator coils using a Newton method",* Plasma Phys. Control. Fusion, in press (2018)
6) Caoxiang Zhu, Stuart R. Hudson *et al.*, "*Hessian matrix approach for determining error field sensitivity to coil deviations",* Plasma Phys. Control. Fusion, in press (2018)

Variations of surface integrals with changes in the surface: surface area and mean curvature.

$$
\begin{aligned}
\text{area} &= \int_{\mathcal{S}} ds, \ \text{where } ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta, \ \text{ and } \ \mathbf{x}_\theta \equiv \partial_\theta \mathbf{x} && (1)\\
|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= [(\mathbf{x}_\theta \times \mathbf{x}_\zeta)\cdot(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2} && (2)\\
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \mathbf{n}\cdot(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta), \ \text{ where } \ \mathbf{n} = (\mathbf{x}_\theta \times \mathbf{x}_\zeta)/|\mathbf{x}_\theta \times \mathbf{x}_\zeta| && (3)\\
\delta(\text{area}) &= \int_{\mathcal{S}} \partial_\theta \delta\mathbf{x}\cdot\mathbf{x}_\zeta \times \mathbf{n}\, d\theta d\zeta - \int_{\mathcal{S}} \partial_\zeta \delta\mathbf{x}\cdot\mathbf{x}_\theta \times \mathbf{n}\, d\theta d\zeta && (4)\\
&= -\int_{\mathcal{S}} \delta\mathbf{x}\cdot(\mathbf{x}_{\zeta\theta}\times\mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n})\, d\theta d\zeta && (5)\\
&\quad + \int_{\mathcal{S}} \delta\mathbf{x}\cdot(\mathbf{x}_{\theta\zeta}\times\mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n})\, d\theta d\zeta && (6)\\
&= -\int_{\mathcal{S}} \delta\mathbf{x}\cdot(\mathbf{x}_\zeta \partial_\theta - \mathbf{x}_\theta \partial_\zeta)\times\mathbf{n}\, d\theta d\zeta && (7)\\
&= -\int_{\mathcal{S}} \delta\mathbf{x}\cdot(\mathbf{n}\times\nabla)\times\mathbf{n}\, ds, \ \text{ where } \ \mathbf{n} = \nabla s/|\nabla s| \ \text{ and } \ \nabla \equiv \nabla s\,\partial_s + \nabla\theta\,\partial_\theta + \nabla\zeta\,\partial_\zeta, && (8)\\
&= -\int_{\mathcal{S}} \delta\mathbf{x}\cdot\mathbf{n}\,(\nabla\cdot\mathbf{n})\, ds && (9)\\
&= -\int_{\mathcal{S}} \delta\mathbf{x}\cdot\mathbf{H}\, ds, \ \text{ mean curvature } \ \mathbf{H} \equiv \mathbf{n}\,(\nabla\cdot\mathbf{n}) && (10)
\end{aligned}
$$

1) R.L. Dewar, S.R. Hudson & P.F. Price, Phys. Lett. A., **194** 49 (1994)