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# Multiregion Relaxed MHD (MRxMHD) toroidal states with *flow*

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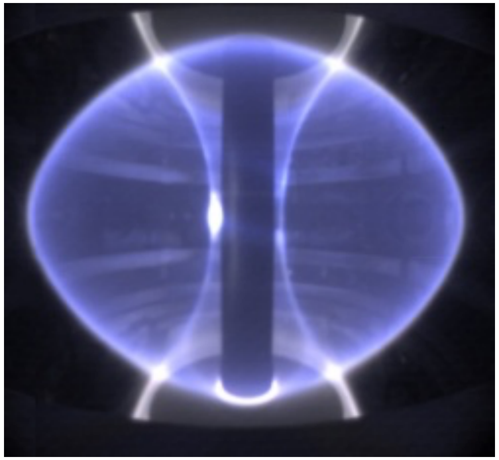
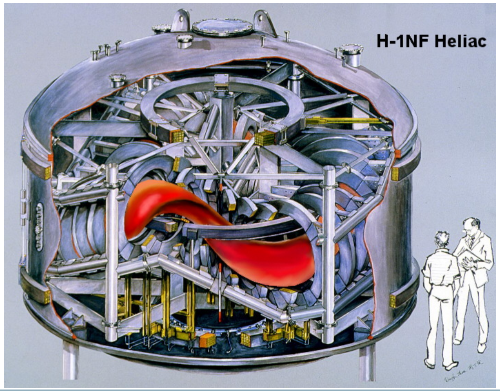
# Some abbreviations

**MRxMHD**: M stands for **Multi-region** (aka waterbag); Rx stands for (Taylor) **Relaxed**; MH for Magneto-Hydro; D stands for **Dynamics**

"2D" = *possessing a continuous symmetry, giving* integrable *magnetic field*
e.g axisymmetric Tokamaks (without Resonant Magnetic Perturbations = RMPs):



"3D" = *no continuous symmetry, allowing field-line chaos and islands*
e.g. non-axisymmetric Stellarators (without quasisymmetry?)



# Plan of talk

- Brief review of MRxMHD idea and realization in SPEC (Stepped-Pressure Equilibrium Code )
- New Dynamical MRxMHD: Force-free magnetic field ⇔ Euler fluid in each relaxation region
- Contrast 2 cases: Axisymmetric toroidal flow; Chaotic streamlines (extreme non-axisymmetry)
- Questions to address
- Preliminary numerical implementation
- Conclusion

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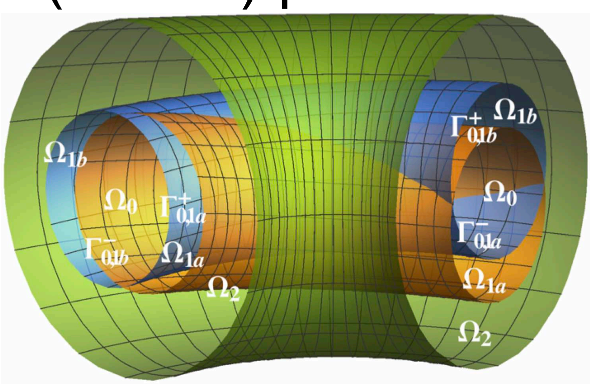
# Multiregion MHD concept

## *Aim: Simplest* ideal-like MHD model that works in 3D

- Confinement is maintained by thin transport barriers (interfaces) $\Gamma_{i,j}$ dividing the plasma into toroidal sub-regions $\Omega_i$

- Force balance across *interfaces* $\Gamma_{i,j}$ allows stepped pressure (& flow) profiles

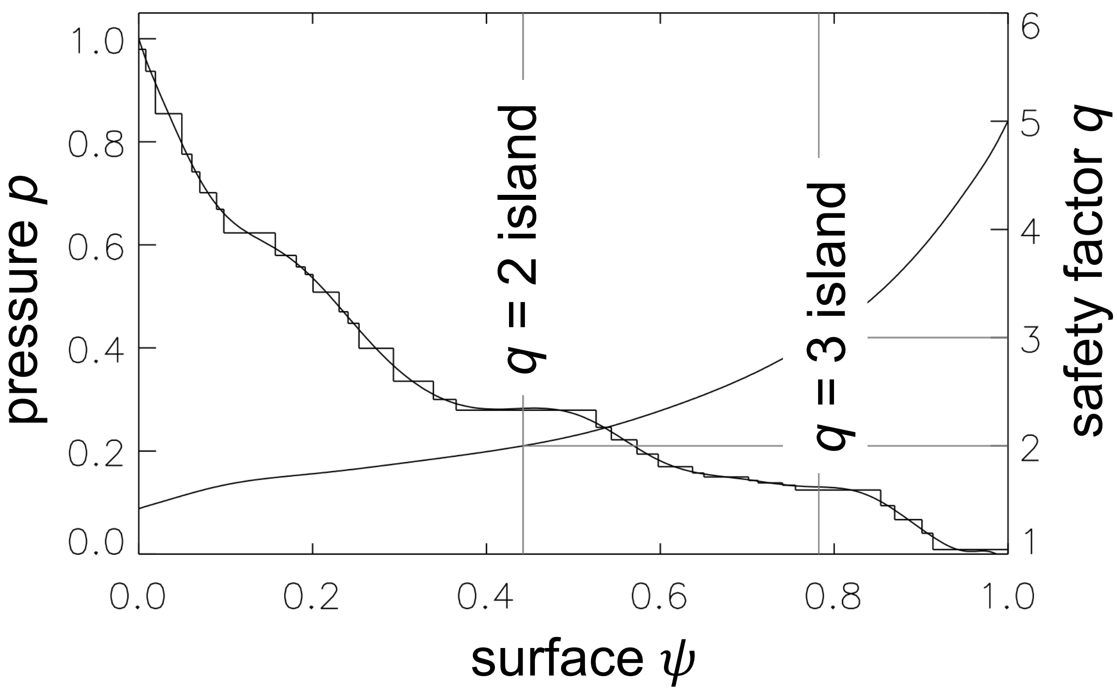



Flexible geometry: Interfaces not necessarily simple tori – may be separatrices of magnetic & fluid islands ("Kelvin's cat's eyes")

- Within the *sub-regions* $\Omega_i$, use force-free magnetic fields, so no **j**X**B** force — field and fluid are *decoupled*

- Dual goals
  - *Numerical*: well-posed "smart finite element" discretization for fast, convergent calculations
  - *Theoretical*: MHD model allowing islands, chaos & flow in 3D equilibria *and* dynamics

# MRxMHD implemented in SPEC




SPEC reconstruction of DIIID RMP equilibrium using many interfaces constructed using *highly irrational* *q* using a Farey tree algorithm: Hudson *et a*l PoP 2012

Stepped-Pressure (static) Equilibrium code (SPEC) has already demonstrated convergence, utility for data fitting, and physics interpret-ation of helical axis bifurcations in RFX.

Challenge now is to include *flow*.

# New MRxMHD theory

Generalize Taylor-type *minimum energy* equilibrium approach by instead *extremizing the MHD action* (Hamilton's Principle). Details in Dewar *et al*, J. Plasma Phys. 2015:

- Full ideal-MHD constraints *apply only within the interfaces* $\Gamma_{i,j}$
- Within the *sub-regions* $\Omega_i$ *relax* nearly all the ideal-MHD constraints except for conservation of *macroscopic* magnetic helicity, entropy, and *microscopic* mass
- Cross-helicity not constrained, so fluid and magnetic field couple *only at interfaces*, which are current/vortex sheets

# Result: Euler-Lagrange equations

(NB Mass conservation built in: $\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho \mathbf{v})$ )

- δ*p* ➪ Isothermal equation of state

$$p = \tau_i \rho \quad (\text{N.B. } \tau_i = C_{\mathrm{s}i}^2)$$

- δ**A** ➪ Beltrami equation (simple!)

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_i \mathbf{B} \quad (\text{N.B. } \Rightarrow \mathbf{j}\times\mathbf{B} = 0)$$

- $\boldsymbol{\xi}$ in $\Omega_i$ ➪ Momentum equation (*Euler fluid*)

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\right) = -\boldsymbol{\nabla} p$$

**Problem:** Euler flow is *not* simple!

- $\boldsymbol{\xi}$ on $\Gamma_{i,j}$ ➪ Force balance $\left[\frac{B^2}{2\mu_0} + p\right]|_{i,j} = 0$

NB Centrifugal force balance does not appear explicitly — mediated by pressure

# 2D: Steady *axisymmetric* toroidal flow

Piecewise-constant vorticity $\boldsymbol{\omega} \equiv \nabla\times\mathbf{v}$:

$$\boldsymbol{\omega}_i = 2\Omega_i\mathbf{e}_Z \Leftrightarrow \mathbf{v} = \Omega_i R\mathbf{e}_\phi \Longrightarrow \boldsymbol{\omega}_i\times\mathbf{v} = -2\Omega_i^2 R\mathbf{e}_R = -\nabla v_\phi^2$$

- Steady toroidal Euler flow momentum equation:

$$\boldsymbol{\omega}\times\mathbf{v} + \nabla\left(\frac{v_\phi^2}{2} + \ln\frac{\rho}{\rho_0}\right) = 0$$

- Gives *Bernoulli equation* $-\frac{v_\phi^2}{2} + \ln\frac{\rho}{\rho_0} = 0$: in agreement with, e.g., McClements & Hole's 2010 *ideal MHD* result ✔

- Ideal Ohm's Law solvability condition for $\Phi$:

$$\nabla\times(\mathbf{v}\times\mathbf{B}) = \Omega_i R\left(\nabla\cdot\mathbf{B} - \frac{1}{R}\frac{\partial B_\phi}{\partial\phi}\right)\mathbf{e}_\phi = 0 \quad ✔✔$$

So rigid-body flow is *compatible with ideal MHD* for any axi-symmetric **B**, *including* MRxMHD's force-free Beltrami field

# 3D. "Relaxed" *non*-axisymmetric flow

## Static solution of Compressible Euler Fluid eqs. in arb. Ω (Sato & Dewar arxiv):

- Dot momentum equation $\rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\tau\boldsymbol{\nabla}\rho$ with $\mathbf{v}/\rho$ and integrate to give *Bernoulli equation* $v^2/2 + \tau\ln(\rho/\rho_0) = \mathrm{const}$ on *each* flow line.
- Only solution valid for *arbitrarily chaotic flow* within Ω is (with suitable choice of global constant $\rho_0$ ) $\rho = \rho_0 \exp(-v^2/2\tau)$
- Gives nonlinear Beltrami equation:

$$\boldsymbol{\nabla}\times\mathbf{v} = \alpha_0 \exp(-v^2/2\tau)\mathbf{v}$$

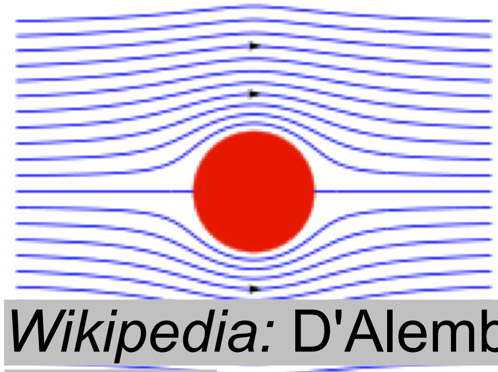
# Problem: Incompatibility of solutions

*2D and 3D examples both satisfy Euler flow equations, but are very different:*

- Relaxed (3D Beltrami) fluid solution: vorticity *parallel* to flow
- Rigid toroidal flow 2D solution: vorticity *perpendicular* to flow
- *Toroidal* kinetic energy terms in Bernoulli equations have *opposite* signs in the two solns.
- Unlike rigid toroidal flow, relaxed fluid does *not* in general satisfy ideal Ohm's Law solvability condition $\nabla\times(\mathbf{v}\times\mathbf{B}) = 0$.

# Questions re 3-D flow equilibria

- Are there steady 3-D MRxMHD solutions that satisfy $\nabla\times(\mathbf{v}\times\mathbf{B}) = 0$?
- Is there a better fluid relaxation theory based on quasi-2D (“shallow water”) inverse cascade theory?
- Does d’Alembert’s paradox (no drag) apply so non-trivial steady stepped flows exist ?



*Wikipedia:* D'Alembert's paradox

- Or, do only *time-dependent* (e.g. oscillatory) solutions exist?

# Preliminary SPECF implementation 4

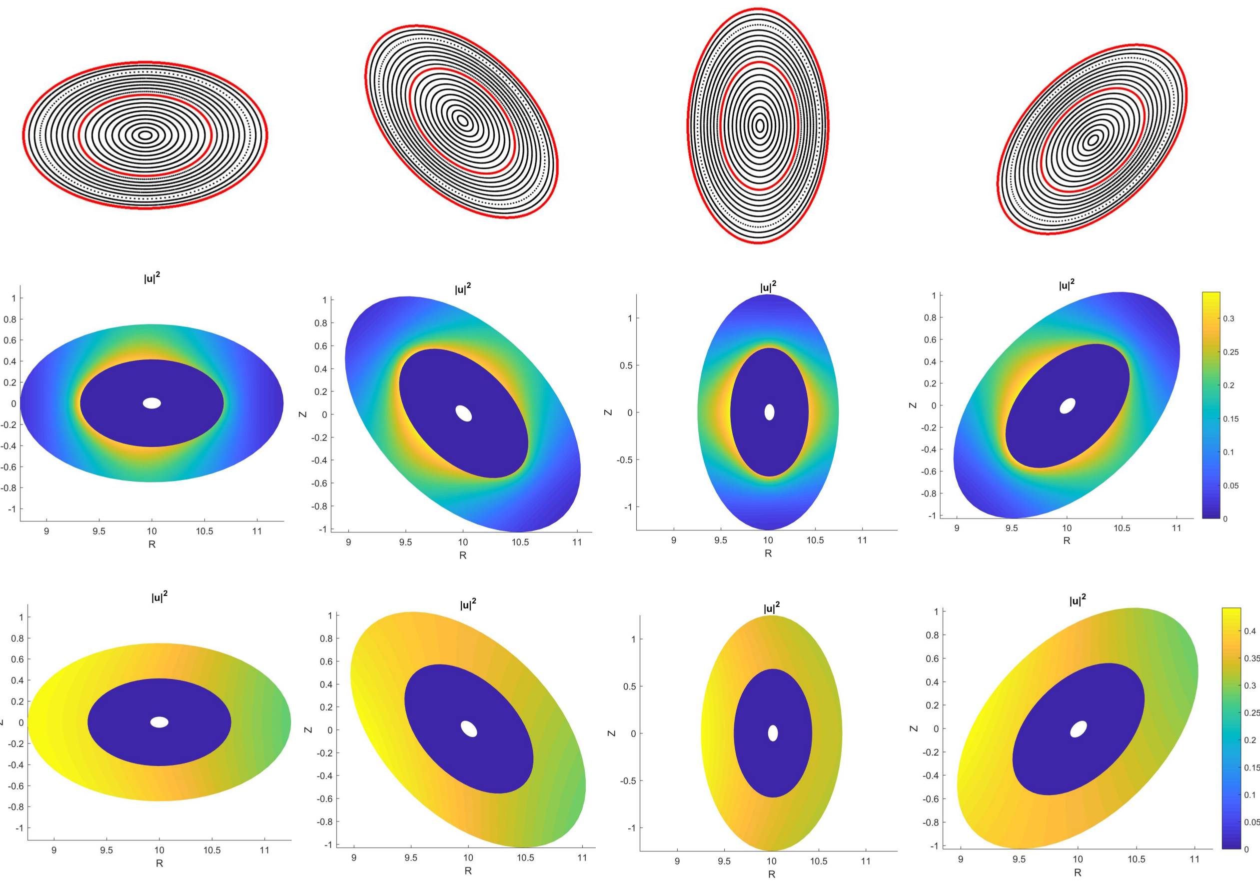
Case 3: *3D* Stellarator test case (toroidal periodicity = 5, zero $\beta$)

No viscosity is used — We find *steady* converged solutions using free-slip boundary conditions with flow discontinuity at interface: it appears **there is a d'Alembert's paradox operating in our case**

Field Poincaré

Poloidal flow only $|\boldsymbol{u}|^2$ plot

Toroidal flow only $|\boldsymbol{u}|^2$ plot



# Conclusion

- Have shown that MRxMHD is compatible with ideal MHD for *axisymmetric* toroidal flow equilibria
- Have found that the most general *non*-axisymmetric “relaxed Euler flow” equilibria cannot reduce to the axisymmetric toroidal flow equilibria
- Have implemented a preliminary version of the SPEC code with flow (SPECF)
- Have enunciated some questions that need to be addressed (another is physics of the interfaces — next talk)

# THE END

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# Abstract

## Multiregion Relaxed MHD toroidal states with flow

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The action-based formulation[1] of Multiregion Relaxed MHD (MRxMHD) encompasses both steady-flow statics, and dynamics on a slower timescale than Taylor relaxation. We consider the case of a toroidal plasma laminated into multiple nested annular toroidal relaxation regions, separated by interfaces supporting current sheets. Unlike ideal MHD, Taylor relaxation allows reconnection at resonant surfaces to occur within these regions. However, the physical applicability of the model depends on the interfaces between them being ideal, i.e. *stable* against reconnection for times much longer than the relaxation timescale.

It has been postulated[2] that plasma flow may stabilize such current sheets even if they occur on surfaces that resonate with boundary perturbations in 3D geometries such as stellarators, or tokamaks with resonant magnetic perturbation (RMP) coils. This motivates the extension, now under development, of the 3D-MRxMHD-based *equilibrium* code SPEC[3] to allow plasma flow with reasonably general flow profiles. However, it is not clear[4] that stationary 3D states with other than rigid-rotation flow exist, motivating development of a 3D MRxMHD *initial value* code to model oscillatory states and nonlinear instabilities.

The formulation of Ref. 1 describes the plasma in each region as an ideal Euler fluid, which is too general for practical purposes as it allows all the turbulent complexity of such a fluid. This motivates developing a Taylor-like relaxation model[5] for fluids, based on minimizing total energy with constant mass, entropy and fluid helicity (or, equivalently, minimizing fluid helicity at constant mass, entropy and energy). This leads to a compressible Beltrami equation, $\nabla \times \mathbf{v} = \alpha_0 \exp\left(-v^2/2\tau\right)\mathbf{v}$, where $\alpha_0$ and $\tau$ are constant in each region, $\tau$ being the square of the isothermal sound speed in that region. The simplest case is $\alpha_0 = 0$, i.e. the flow has *zero vorticity*, but, because our relaxation regions are not simply connected, non-trivial rotation profiles can still be treated.

**References:**

1. R.L. Dewar, *et al.*, J. Plasma Phys., **81**, 515810604-1–22, (2015).
2. R.L. Dewar, S.R. Hudson *et al.*, Phys. Plasmas, **24**, 042507-1–18, (2017).
3. S.R. Hudson, R.L. Dewar *et al.*, Phys. Plasmas **19**, 112502-1–18, (2012).
4. G.R. Dennis, S.R. Hudson, R.L. Dewar and M.J. Hole, Phys. Plasmas **19**, 042501-1–9, (2014).
5. N. Sato and R.L. Dewar, *Relaxation of Compressible Euler Flow in a Toroidal Domain* https://arxiv.org/pdf/1708.06193.pdf.

# Preliminary SPECF implementation I

For simplicity assume *zero vorticity*:
$\nabla\times\boldsymbol{u} = 0$ (irrotational—potential flow)
$\rho = \rho_0 e^{-u^2/2\tau}$ (Bernoulli relationship)
$\nabla\cdot\rho\boldsymbol{u} = 0$ (Continuity)

We get that

$$\nabla\cdot e^{-\frac{u^2}{2\tau}}(\nabla f + \boldsymbol{V}_0) = 0, \qquad *$$

where $f$ is a single valued periodic function and

$$\boldsymbol{u} = \nabla f + \boldsymbol{V}_0 = \nabla f + \psi_{tV}\nabla\theta + \psi_{pV}\nabla\xi.$$

This ensures that the toroidal and poloidal loop integrals $\oint \boldsymbol{u}\cdot\boldsymbol{dl}$ equal $\psi_{tV}$ and $\psi_{pV}$, the toroidal and poloidal vorticity flux, respectively.

Using the Chebyshev-Fourier reps in SPEC, writing $f$ into

$$f = \sum_{i,l} f_{e,i} T_{l,i}(s)\sin(m_i\theta - n_i\xi),$$

casting the equation * into matrix form

$$\underline{A(\boldsymbol{f}_{n-1})}\cdot\boldsymbol{f}_n = \underline{B(\boldsymbol{f}_{n-\mathbf{1}})}\cdot\begin{pmatrix}\psi_{tV}\\ \psi_{pV}\end{pmatrix},$$

in which matrix $\underline{A}$ and $\underline{B}$ depend on $\underline{f}$ due to the Boltzmann exponential factor, and are calculated iteratively by taking the last solution of $f$ until converged.

The boundary condition is

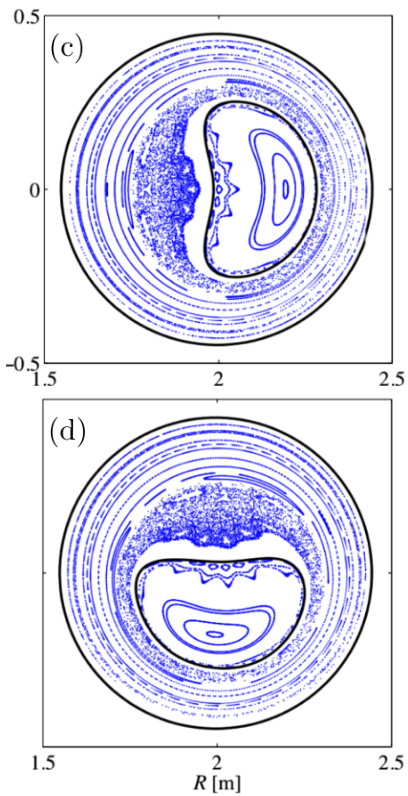
$$\boldsymbol{u}\cdot\boldsymbol{n} = u_s = 0.$$

Flow enters force balance only through pressure: $[[p_0 e^{-\frac{u^2}{2\tau}} + \frac{1}{2}B^2]] = 0$

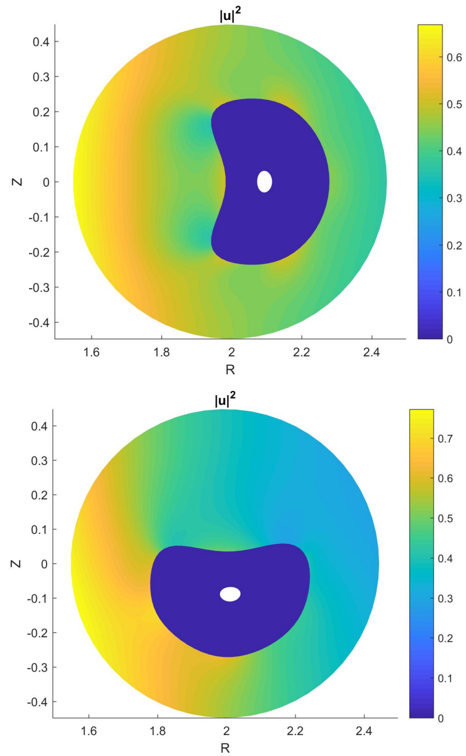
# Preliminary SPECF implementation 3

Case 2: Reversed field pinches, single helical axis (SHAx)
Very small beta: pressure, and hence flow, plays negligible role in force balance

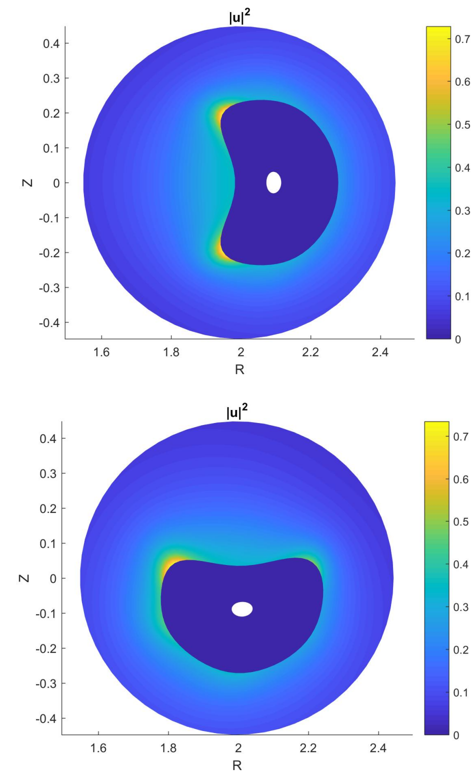
### No flow




### Toroidal flow
Max Mach$^2$ ~ 0.7



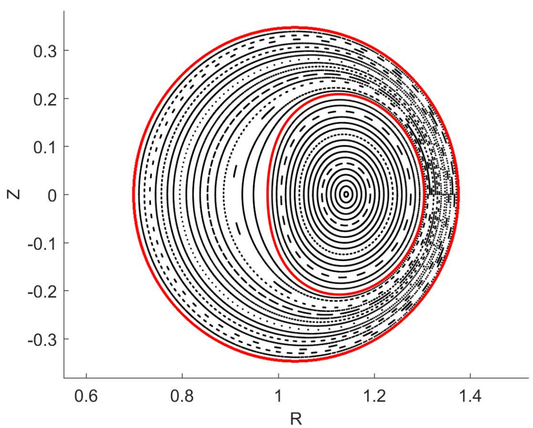

### Poloidal flow
Max Mach$^2$ ~ 0.7



# Preliminary SPECF implementation 2

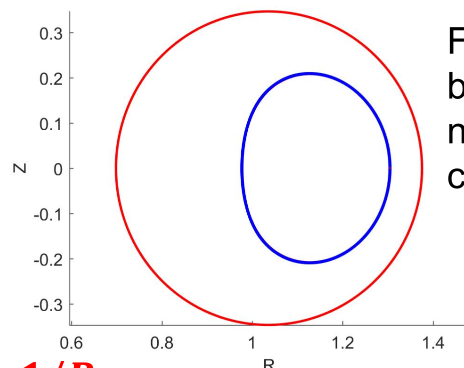
Case1: Axisymmetric test case, $\beta \sim 1/3$.
Constrain rotational transform (Helicity constraint is not implemented currently)

## No flow



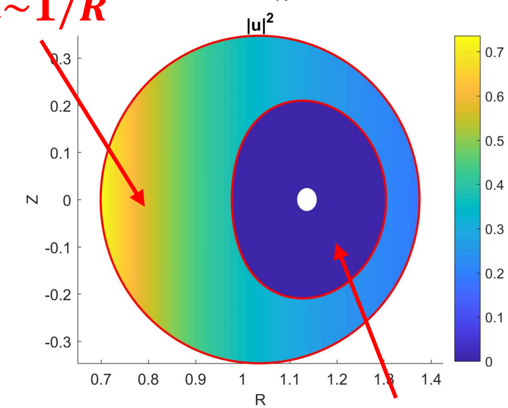

## Toroidal flow

Max Mach$^2$ ~ 0.7



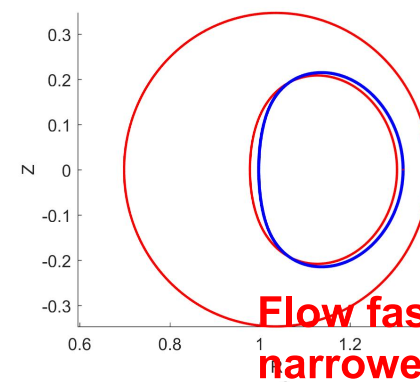

Force balance not much changed

$u \sim 1/R$



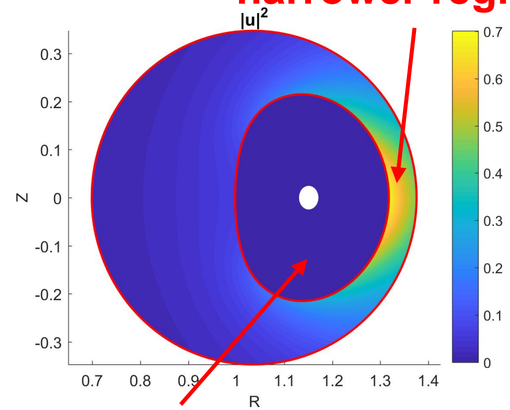

## Poloidal flow

Max Mach$^2$ ~ 0.7



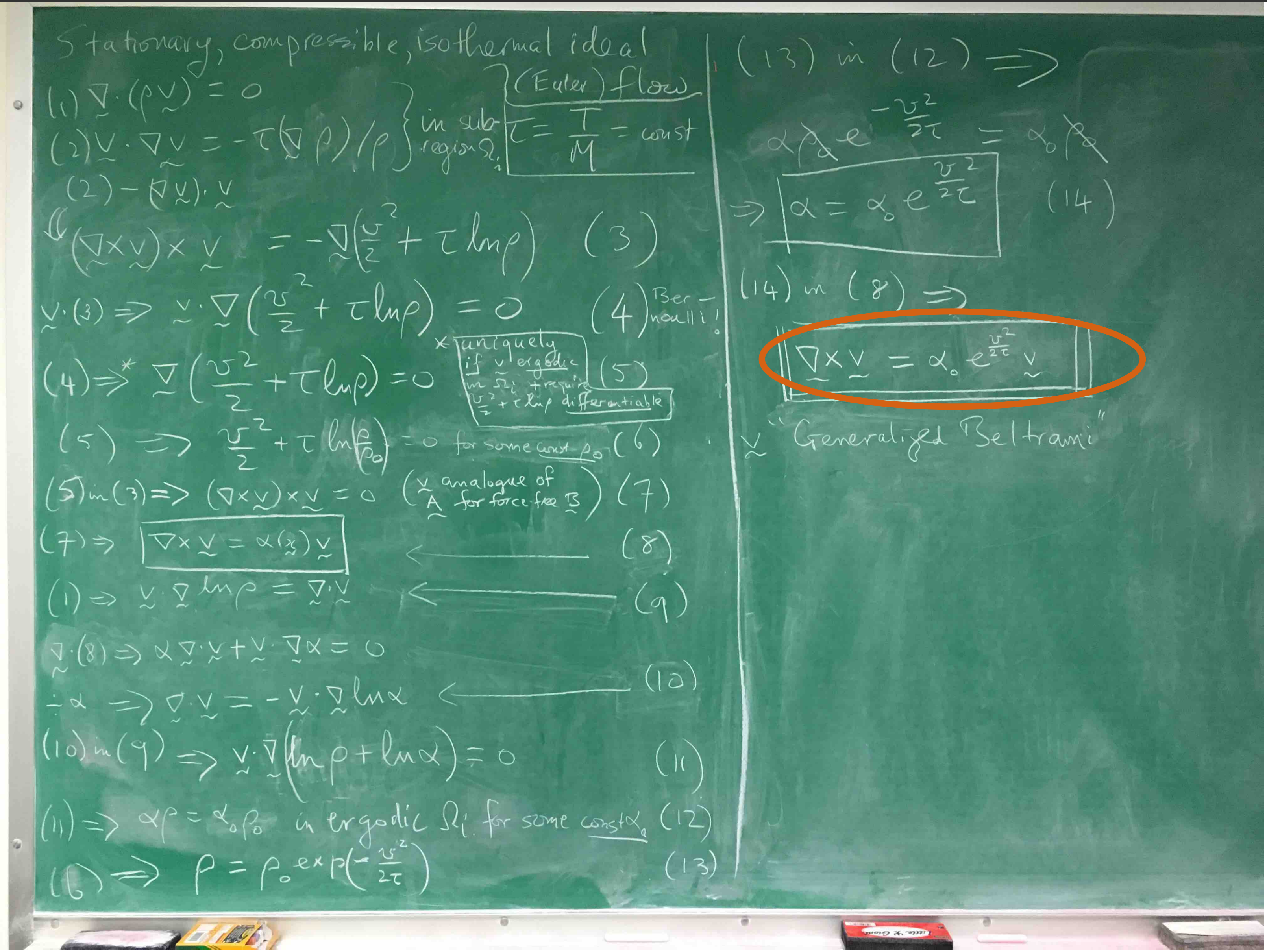

Force balance changed

**Flow faster in narrower region**




**No flow at core because ill-defined vorticity flux**

# Brainstorm: 3D Beltrami flow



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# Derived from MRxMHD Lagrangian

## *kinetic energy – MHD potential energy +* Lagrange multiplier constraint terms:

- MHD Lagrangian density in region $i$

$$\mathcal{L}^{\mathrm{MHD}} = \rho\frac{v^2}{2} - \frac{p}{\gamma - 1} - \frac{B^2}{2\mu_0}$$

- Constrained Lagrangian in region $i$

$$L_i = \int_{\Omega_i} \mathcal{L}^{\mathrm{MHD}} dV + \tau_i(S_i - S_{i0}) + \mu_i\left(K_i - K_{i0}\right)$$

- Helicity and entropy *macroscopic* invariants

$$K_i \equiv \int_{\Omega_i} \frac{\mathbf{A}\cdot\mathbf{B}}{2\mu_0} dV \qquad S_i \equiv \int_{\Omega_i} \frac{\rho}{\gamma - 1} \ln\left(\kappa\frac{p}{\rho^\gamma}\right) dV$$