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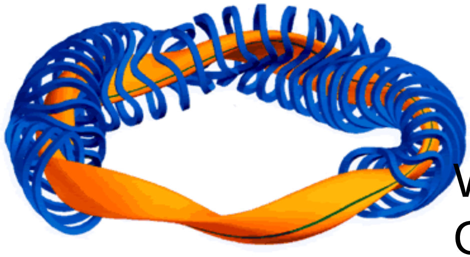
# Multiregion Relaxed MHD toroidal states with flow

Tokamak



W7X stellarator
Germany

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# Plan of talk

- Desiderata: Simple MHD model that can reduce to ideal MHD when appropriate but allows reconnection when not
- Dynamical MRxMHD: Force-free magnetic field⇔Euler fluid in each relaxation region
- Contrast 2 cases: Laminar flow (axisymmetry); Chaotic streamlines (non-axisymmetry)
- Preliminary numerical implementation

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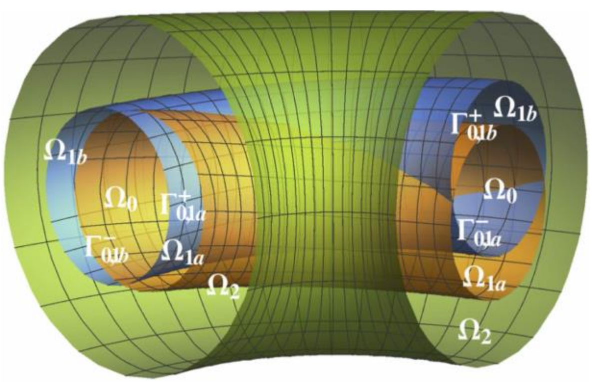
# Laminated MHD concept

## *Basic ideas*

- Confinement is maintained by thin transport barriers (interfaces) $\Gamma_{i,j}$ dividing the plasma into toroidal sub-regions $\Omega_i$
- Force balance across *interfaces* $\Gamma_{i,j}$ allows stepped pressure & flow profiles




Interfaces may be separ-atrices of magnetic islands / Kelvin's cat's eyes in fluid

- Within the *sub-regions* $\Omega_i$, use force-free magnetic fields, so no **j**X**B** force — field and fluid are *decoupled*
- Scope of applicability
  - "smart finite element" discretization for ideal-MHD 2D equilibrium calculations (incl. flow)
  - *new*, quasi-relaxed, MHD allowing islands & chaos in 3D equilibrium and dynamics

# MRxMHD theory

**MRxMHD**: M stands for **Multi-region** (aka waterbag) Rx stands for **Relaxed**; ..D stands for **Dynamics**

- ❑ Full ideal-MHD constraints *apply only within the interfaces* $\Gamma_{i,j}$
- ❑ Within the *sub-regions* $\Omega_i$ use generalized Taylor relaxation theory – relax nearly all the ideal-MHD constraints except for conservation of *macroscopic* magnetic helicity, entropy, and *microscopic* mass
- ❑ Cross-helicity not constrained, so fluid and magnetic field couple *only at interfaces*, which are current/vortex sheets
- ❑ Generalize minimum energy equilibrium approach by instead extremizing the MHD action (Hamilton's Principle)
- ❑ Details in Dewar et al 2015

# Euler-Lagrange Equations within $\Omega_i$

- Mass conservation (microscopic constraint)

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho \mathbf{v})$$

- $\delta p$ ➪ Isothermal equation of state

$$p = \tau_i \rho \quad (\text{N.B. } \tau_i = C_{\mathrm{s}i}^2)$$

- $\delta\mathbf{A}$ ➪ Beltrami equation

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_i \mathbf{B} \quad (\text{N.B. } \Rightarrow \mathbf{j}\times\mathbf{B} = 0)$$

- $\boldsymbol{\xi}$ ➪ Momentum equation (*Euler fluid*)

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\right) = -\boldsymbol{\nabla} p$$

# Steady axisymmetric toroidal flow

## Piecewise-constant *vorticity* $\boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$:

$$\boldsymbol{\omega}_i = 2\Omega_i \mathbf{e}_Z \Leftrightarrow \mathbf{v} = \Omega_i R \mathbf{e}_\phi \Longrightarrow \boldsymbol{\omega}_i \times \mathbf{v} = -2\Omega_i^2 R \mathbf{e}_R = -\nabla v_\phi^2$$

- Check Ideal Ohm's Law solvability condition for $\Phi$:

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \Omega_i R \left( \nabla \cdot \mathbf{B} - \frac{1}{R}\frac{\partial B_\phi}{\partial \phi} \right) \mathbf{e}_\phi = 0 \quad ✔☐✔☐$$

So rigid-body flow is compatible with ideal MHD for any axi-symmetric **B**, *including* MRxMHD's force-free Beltrami field!

- Steady toroidal Euler flow momentum equation:

$$\boldsymbol{\omega} \times \mathbf{v} + \nabla \left( \frac{v_\phi^2}{2} + \ln \frac{\rho}{\rho_0} \right) = 0$$

- Gives *Bernoulli equation* $-\frac{v_\phi^2}{2} + \ln \frac{\rho}{\rho_0} = 0$: in agreement with, e.g., McClements & Hole 2010's *ideal MHD* result.

# "Relaxed" non-axisymmetric flow

## Static solution of Compressible Euler Fluid eqs in arb. Ω (Sato & Dewar arxiv):

- Dot momentum equation $\rho\mathbf{v}\cdot\nabla\mathbf{v} = -\tau\nabla\rho$ with $\mathbf{v}/\rho$ and integrate to give *Bernoulli equation* $v^2/2 + \tau\ln(\rho/\rho_0) = \mathrm{const}$ on *each* flow line.
- Only solution valid for *arbitrarily chaotic flow* within Ω is (with suitable choice of global constant $\rho_0$) $\rho = \rho_0\exp(-v^2/2\tau)$
- Gives nonlinear Beltrami equation:

$$\nabla\times\mathbf{v} = \alpha_0\exp(-v^2/2\tau)\mathbf{v}$$

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# Incompatibility of the two solutions

- Relaxed (Beltrami) fluid solution: vorticity *parallel* to flow
- Rigid toroidal flow solution: vorticity *perpendicular* to flow
- *Toroidal* kinetic energy terms in Bernoulli equations have *opposite* signs in the two solns.
- Unlike rigid toroidal flow, relaxed fluid does *not* in general satisfy ideal Ohm's Law solvability condition $\nabla \times (\mathbf{v} \times \mathbf{B}) = 0$.

# Open questions

- Are there steady 3-D MRxMHD solution that satisfy $\nabla \times (\mathbf{v} \times \mathbf{B}) = 0$?
- Can fluid relaxation be related to inverse cascade theory?
- Do adjacent non-axisymmetric layers with different mean velocities necessarily exchange angular momentum or is there a d’Alembert’s paradox? — Do non-axisymmetric steady flow equilibria *exist* in general, or only oscillatory solutions?

# Preliminary SPECF implementation I

Starting equation:

$\nabla \times \boldsymbol{u} = 0$ (Vorticity-free)
$\rho = \rho_0 e^{-u^2/2\tau}$ (Bernoulli relationship)
$\nabla \cdot \rho\boldsymbol{u} = 0$ (Continuity)

We get that

$$\nabla \cdot e^{-\frac{u^2}{2\tau}}(\nabla f + \boldsymbol{V}_0) = 0, \qquad *$$

where $f$ is a single valued per function and

$$\boldsymbol{u} = \nabla f + \boldsymbol{V}_0 = \nabla f + \psi_{tV}\nabla\theta + \psi_{pV}\nabla\xi.$$

This ensures that the toroidal and poloidal loop integral $\oint \boldsymbol{u}\cdot\boldsymbol{dl}$ equal to $\psi_{tV}$ and $\psi_{pV}$, the toroidal and poloidal vorticity flux, respectively.

Using the Chebyshev-Fourier reps in SPEC, writing $f$ into

$$f = \sum_{i,l} f_{e,i} T_{l,i}(s)\sin(m_i\theta - n_i\xi),$$

casting the equation * into matrix form

$$\underline{A(\boldsymbol{f}_{n-1})} \cdot \boldsymbol{f}_n = \underline{B(\boldsymbol{f}_{n-\mathbf{1}})} \cdot \begin{pmatrix}\psi_{tV}\\ \psi_{pV}\end{pmatrix},$$

in which matrix $\underline{A}$ and $\underline{B}$ depend on $\underline{f}$ due to the Boltzmann exponential factor, and are calculated iteratively by taking the last solution of $f$ until converged.

The boundary condition is

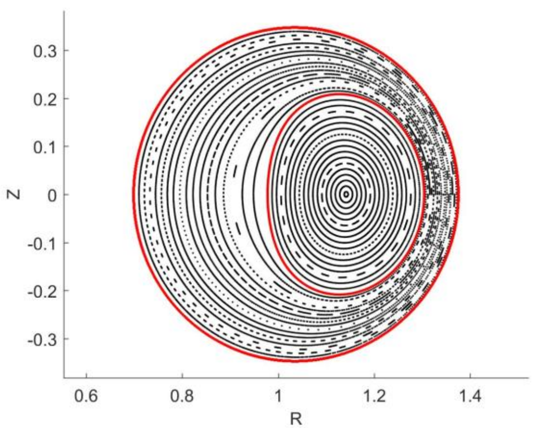
$$\boldsymbol{u}\cdot\boldsymbol{n} = u_s = 0.$$

Force Balance: $[[p_0 e^{-\frac{u^2}{2\tau}} + \frac{1}{2}B^2]] = 0$

# Preliminary SPECF implementation 2

Case1: Axisymmetric test case (this is some random geometry…)
If $\beta$ is low $\sim r^2/R^2$, the exponential factor to pressure does not influence force balance
So we need high $\beta \sim r/R$ (in our case $\beta \sim 1/3$)
Constrain rotational transform (Helicity is not avaliable currently)

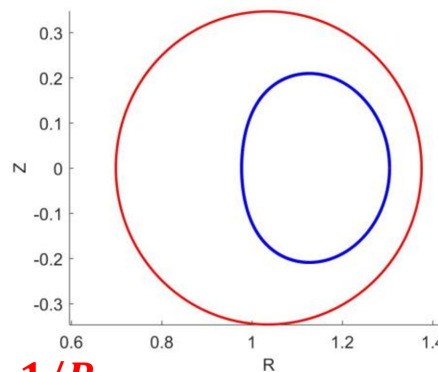
## No flow




I know I need to add many interfaces to match Michael's solution…

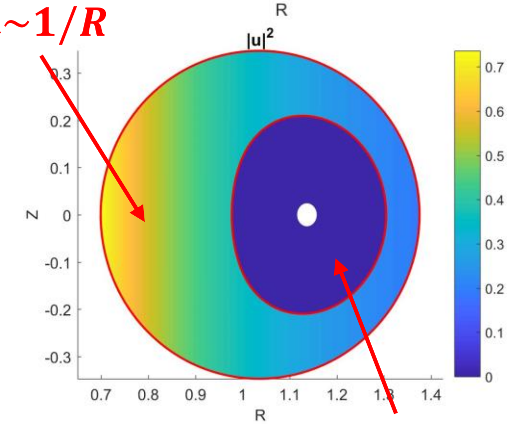
## Toroidal flow

Max Mach$^2$ ~ 0.7



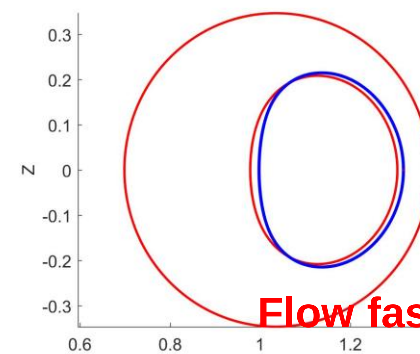

Force balance not much changed

$u \sim 1/R$



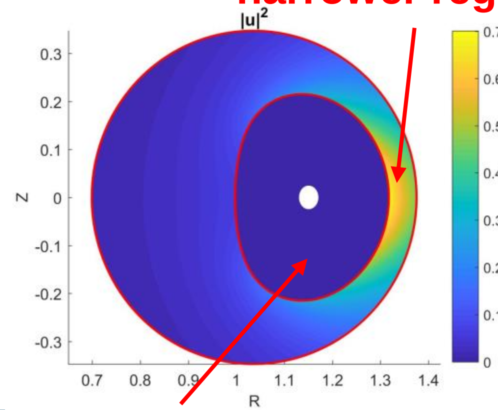

## Poloidal flow

Max Mach$^2$ ~ 0.7




Force balance changed

**Flow faster in narrower region**




**No flow at core because illly defined vorticity flux**

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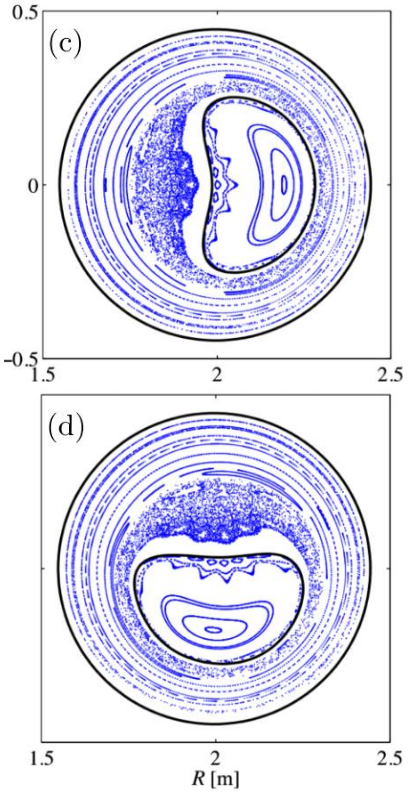
# Preliminary SPECF implementation 3

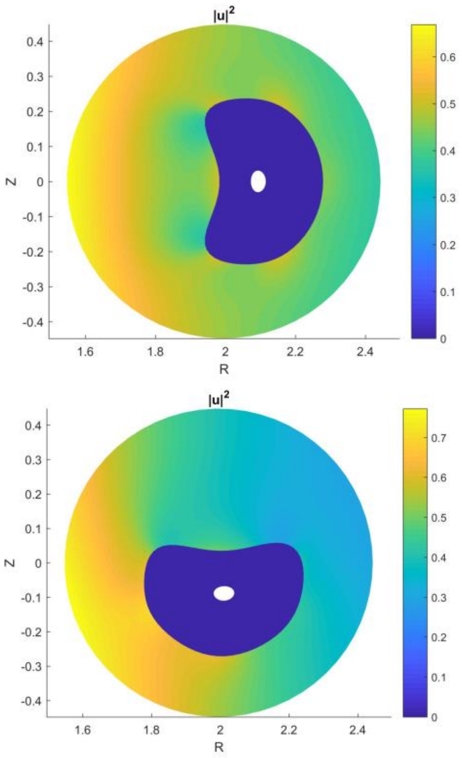
Case 2: Reverse field pinches, single helical axis (SHAx)
Very small beta, flow plays no role in force balance

## No flow



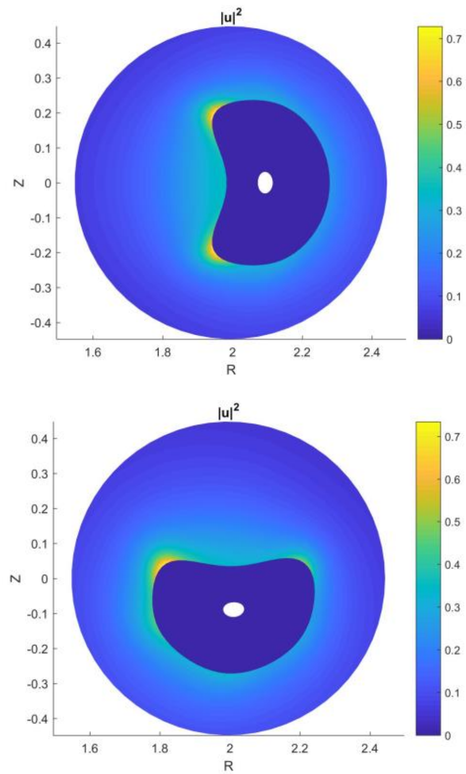

## Toroidal flow

Max Mach$^2$ ~ 0.7




## Poloidal flow

Max Mach$^2$ ~ 0.7



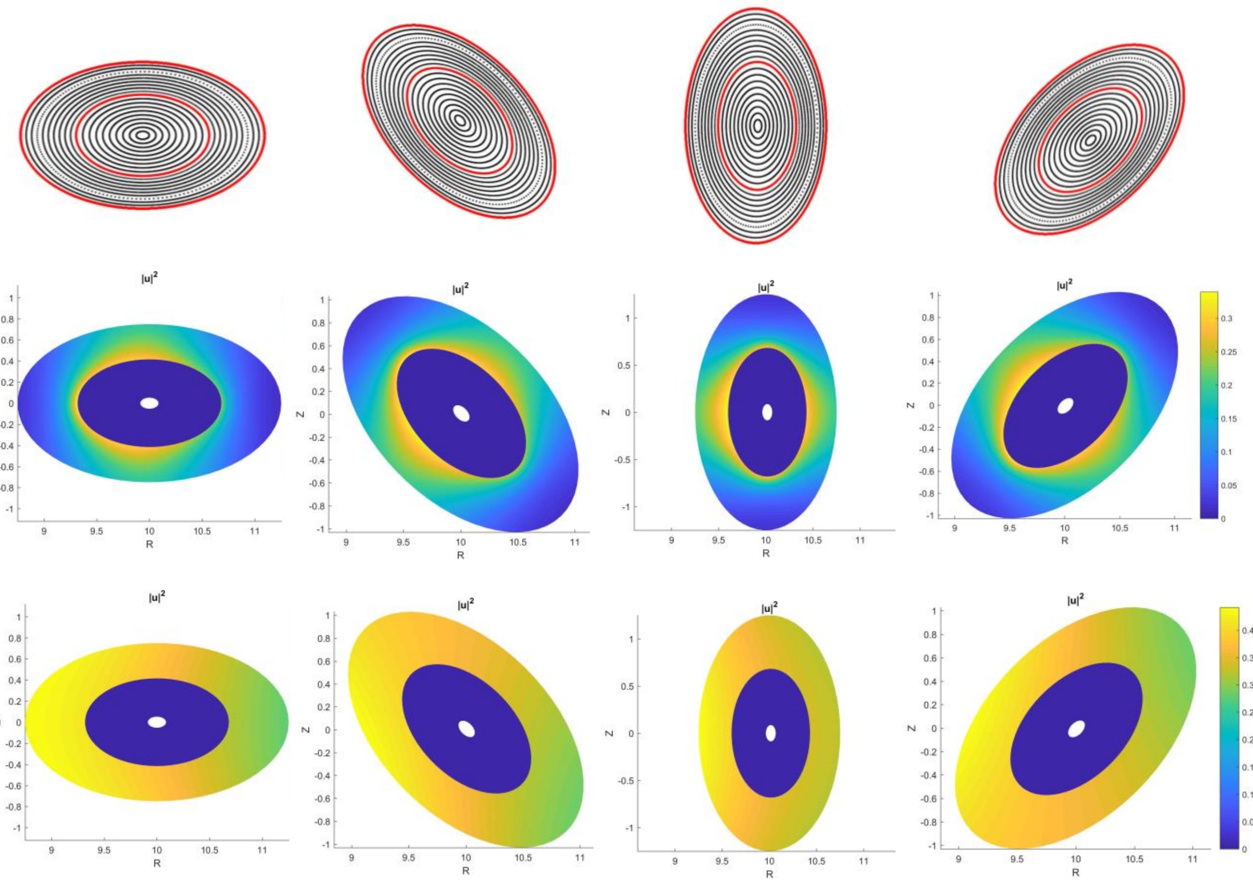
# Preliminary SPECF implementation 4

Case 3: Stellerator test case (toroidal periodicity = 5)
Note: zero $\beta$ case
No viscosity is used. <u>**There is d‘Alembert's paradox in our case**</u>.
One need to consider interface as a boundary layer (not finished).
But regardless of the lack of boundary layer, we get solutions.

Field Poincaré

Poloidal flow only $|\boldsymbol{u}|^2$ plot

Toroidal flow only $|\boldsymbol{u}|^2$ plot



# Conclusion

- Have shown that MRxMHD is compatible with ideal MHD for *axisymmetric* toroidal flow equilibria
- Have found that the most general *non*-axisymmetric “relaxed Euler flow” equilibria cannot reduce to the axisymmetric toroidal flow equilibria
- Have implemented a preliminary version of the SPEC code with flow (SPECF)
- Have enunciated some open questions that need to be addressed

# Abstract

## Multiregion Relaxed MHD toroidal states with flow

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The action-based formulation[1] of Multiregion Relaxed MHD (MRxMHD) encompasses both steady-flow statics, and dynamics on a slower timescale than Taylor relaxation. We consider the case of a toroidal plasma laminated into multiple nested annular toroidal relaxation regions, separated by interfaces supporting current sheets. Unlike ideal MHD, Taylor relaxation allows reconnection at resonant surfaces to occur within these regions. However, the physical applicability of the model depends on the interfaces between them being ideal, i.e. *stable* against reconnection for times much longer than the relaxation timescale.

It has been postulated[2] that plasma flow may stabilize such current sheets even if they occur on surfaces that resonate with boundary perturbations in 3D geometries such as stellarators, or tokamaks with resonant magnetic perturbation (RMP) coils. This motivates the extension, now under development, of the 3D-MRxMHD-based *equilibrium* code SPEC[3] to allow plasma flow with reasonably general flow profiles. However, it is not clear[4] that stationary 3D states with other than rigid-rotation flow exist, motivating development of a 3D MRxMHD *initial value* code to model oscillatory states and nonlinear instabilities.

The formulation of Ref. 1 describes the plasma in each region as an ideal Euler fluid, which is too general for practical purposes as it allows all the turbulent complexity of such a fluid. This motivates developing a Taylor-like relaxation model[5] for fluids, based on minimizing total energy with constant mass, entropy and fluid helicity (or, equivalently, minimizing fluid helicity at constant mass, entropy and energy). This leads to a compressible Beltrami equation, $\nabla \times \mathbf{v} = \alpha_0 \exp\left(-v^2/2\tau\right)\mathbf{v}$, where $\alpha_0$ and $\tau$ are constant in each region, $\tau$ being the square of the isothermal sound speed in that region. The simplest case is $\alpha_0 = 0$, i.e. the flow has *zero vorticity*, but, because our relaxation regions are not simply connected, non-trivial rotation profiles can still be treated.

**References:**

1. R.L. Dewar, *et al.*, J. Plasma Phys., **81**, 515810604-1–22, (2015).
2. R.L. Dewar, S.R. Hudson *et al.*, Phys. Plasmas, **24**, 042507-1–18, (2017).
3. S.R. Hudson, R.L. Dewar *et al.*, Phys. Plasmas **19**, 112502-1–18, (2012).
4. G.R. Dennis, S.R. Hudson, R.L. Dewar and M.J. Hole, Phys. Plasmas **19**, 042501-1–9, (2014).
5. N. Sato and R.L. Dewar, *Relaxation of Compressible Euler Flow in a Toroidal Domain* https://arxiv.org/pdf/1708.06193.pdf.

# Derived from MRxMHD Lagrangian

## *kinetic energy – MHD potential energy* + Lagrange multiplier constraint terms:

- MHD Lagrangian density in region *i*

$$\mathcal{L}^{\mathrm{MHD}} = \rho \frac{v^2}{2} - \frac{p}{\gamma - 1} - \frac{B^2}{2\mu_0}$$

- Constrained Lagrangian in region *i*

$$L_i = \int_{\Omega_i} \mathcal{L}^{\mathrm{MHD}} dV + \tau_i (S_i - S_{i0}) + \mu_i (K_i - K_{i0})$$

- Helicity and entropy *macroscopic* invariants

$$K_i \equiv \int_{\Omega_i} \frac{\mathbf{A}\cdot\mathbf{B}}{2\mu_0} dV \qquad S_i \equiv \int_{\Omega_i} \frac{\rho}{\gamma - 1} \ln\left(\kappa \frac{p}{\rho^\gamma}\right) dV$$