Stability and nonlinear saturation of reconnecting current sheets in a helicity-conserving variational model

Joaquim Loizu

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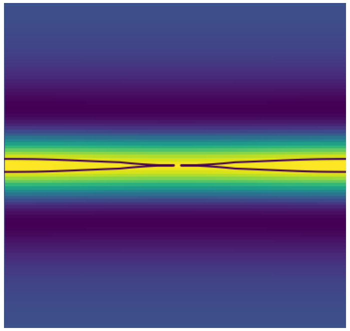
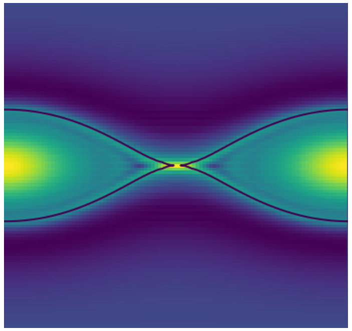


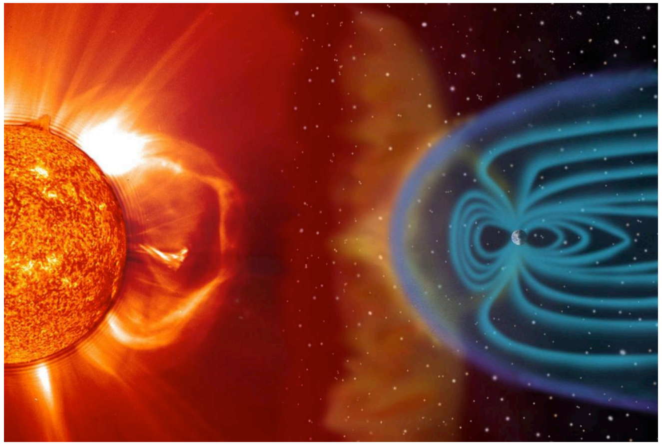


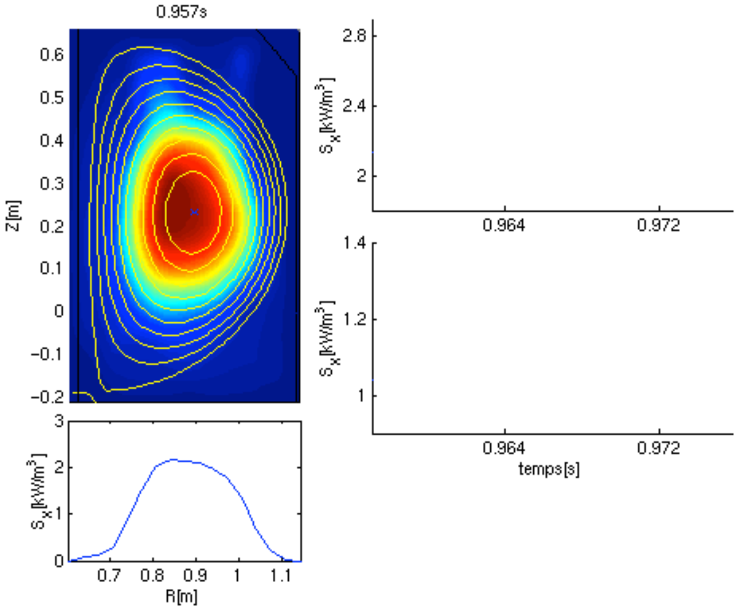
European Fusion Theory Conference, Ghent, Belgium, October 10th 2019

Magnetic reconnection is a ubiquitous phenomenon

Fast conversion of tremendous amounts of magnetic energy into kinetic energy is observed…

…in **solar flares**, as well as during the interaction between the **solar wind** and the **Earth magnetic field**

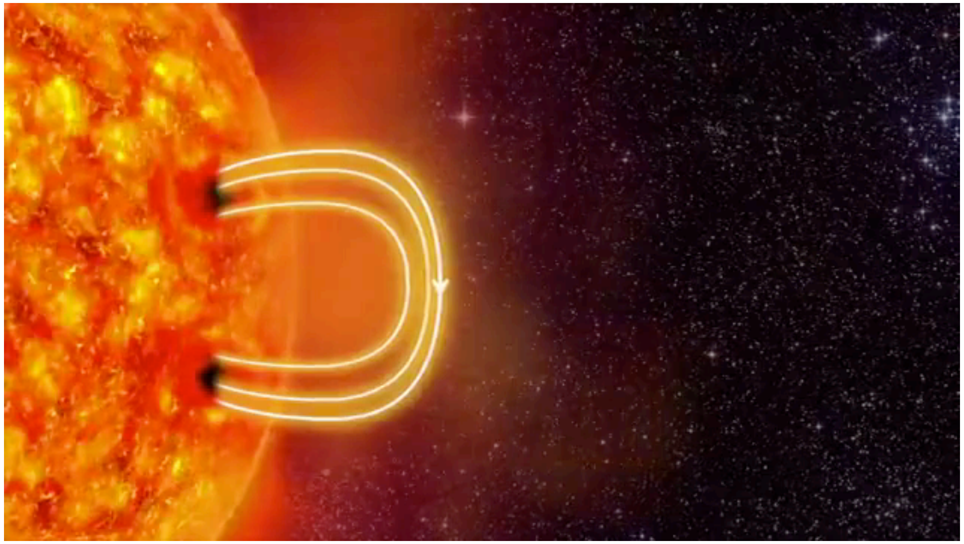




…in **magnetic fusion devices**, for example during **sawtooth crashes** in tokamaks

Magnetic reconnection breaks ideal MHD constraints

Opposite magnetic field lines, pushed together by plasma flows, break and rejoin.



Only possible if the frozen-in-flux condition (Alfven's theorem) is broken.

Resistivity and localized currents can produce reconnection.

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}$$

Reconnection is usually many orders of magnitude faster than resistive diffusion.

Tearing instabilities trigger fast reconnection

[Lütjens, JCP, 2008]

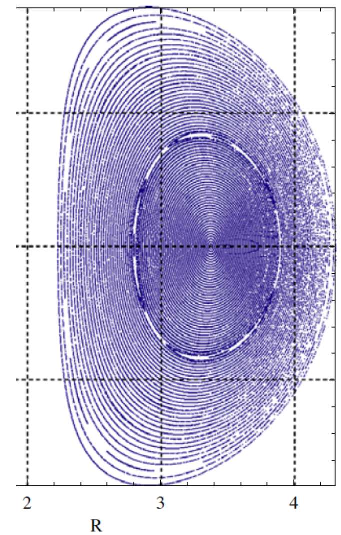
In the presence of a strong guide field ($B_\varphi >> B_\theta$ in tokamaks), reconnection typically occurs via the **tearing mode instability**.

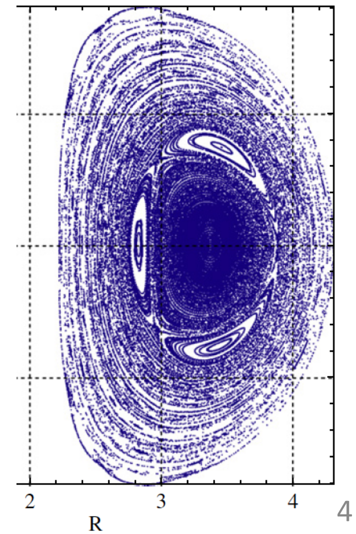
Tearing modes often develop as **resistive MHD** instabilities, appearing when $\Delta' > 0$ (depends on the equilibrium profiles).

A magnetic island typically grows with time scales between the ideal (Alfven) time τ_A and resistive (diffusion) time τ_η.

In magnetic fusion, nonlinear **saturation** is observed, and the island width $w_{\mathbf{sat}}$ is a critical parameter for machine operation.

Numerical **simulations** of this multiscale problem are expensive, primarily because of large Lundquist number $S = \tau_\eta / \tau_A >> 1$

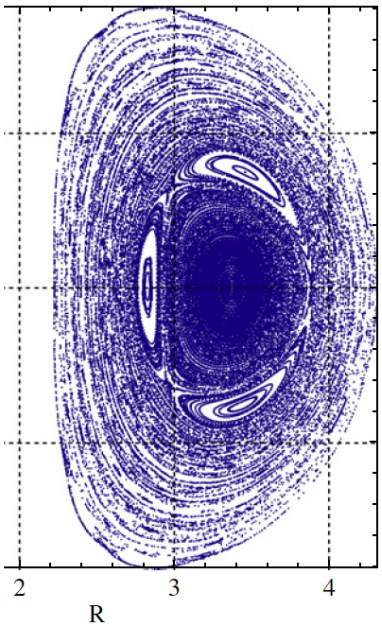




Stability and saturation are independent of resistivity

OBS 1: linear theory of the tearing mode shows that

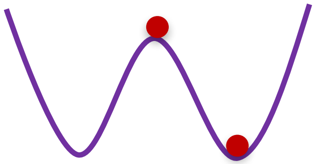
- Δ' is **independent of resistivity**
- linear growth rate depends on resistivity,



OBS 2: nonlinear theory and simulations show that, for $S >> 1$,

- nonlinear growth rate weakly depends on resistivity
- saturation size of the island is **independent of resistivity**

OBS 3: saturation suggests the existence of another MHD equilibrium



linearly unstable,
nonlinearly stable

Is there a variational principle allowing direct calculation of saturated tearing modes?

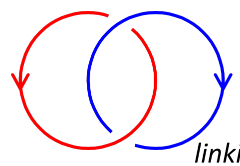
On the menu today

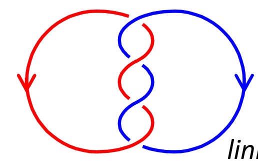
- Magnetic helicity as a good **invariant** in reconnecting plasmas
- **Variational principle** for partially reconnected plasmas
- Numerical implementation in the **SPEC code**
- Application to **linear** tearing stability of “textbook” tearing mode
- Demonstration of **nonlinear** island saturation predictions in slab
- Summary

Magnetic helicity is often a good invariant in MHD

- Magnetic helicity is defined as: [Berger, PPCF, 1999]

$$K \equiv \int_V \mathbf{A} \cdot \mathbf{B} \; dV$$


linking number = 1


linking number = 2

and is an integrated measure of the Gauss linking number for all pairs of field lines.

- For an infinitely conductive fluid (ideal MHD), K is conserved in time:

$$\frac{\partial K}{\partial t} = \int_V \frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{B} \; dV + \int_V \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial t} \; dV + \int_{\partial V} (\mathbf{A} \cdot \mathbf{B}) \mathbf{v} \cdot \mathbf{ds} = 0$$

using Faraday-Ohm's law, **B . n** = 0 on the boundary, and a single-valued gauge for **A**.

- For finite conductivity (resistive MHD), K is dissipated, but often much slower than energy

$$\frac{\partial K}{\partial t} = -2\eta \int_V \mathbf{J} \cdot \mathbf{B} \; dV \quad \sim \quad -2\eta \sum_k k B_k^2$$

$$\frac{\partial W}{\partial t} = -\eta \int_V \mathbf{J}^2 \; dV \quad \sim \quad -\eta \sum_k k^2 B_k^2$$

$$\frac{\Delta K}{K} \leq \sqrt{\frac{\tau_{\mathrm{rec}}}{\tau_\eta}}$$

e.g. for small scale turbulence or for fast reconnection time scales. [Taylor 1986 , Berger 1999]

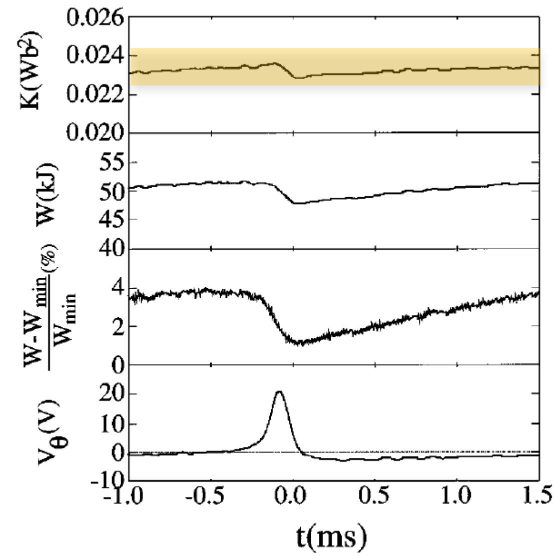
There is experimental evidence for the invariance of K

Conservation of Magnetic Helicity during Plasma Relaxation

H. Ji,* S. C. Prager, and J. S. Sarff
Department of Physics, University of Wisconsin, Madison, Wisconsin 53706
(Received 9 July 1994)

The change in magnetic energy and magnetic helicity has been measured during the sawtooth relaxation in the Madison Symmetric Torus reversed-field pinch. The larger decay of the energy (4.0%–10.5%), relative to helicity decay (1.3%–5.1%), modestly supports the helicity conservation hypothesis in Taylor's relaxation theory. However, the observed helicity change is larger than the simple magnetohydrodynamics prediction. Enhanced fluctuation-induced helicity transport during the relaxation is observed.

[Ji et al, PRL, 1995]

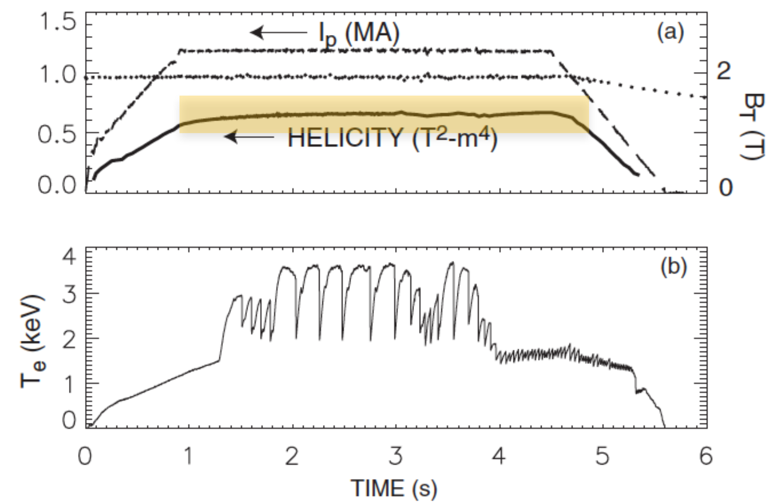


Magnetic helicity is conserved at a tokamak sawtooth crash

W W Heidbrink and T H Dang
University of California, Irvine, CA 92697, USA

Received 25 July 2000

Abstract. Sawtooth instabilities cause sudden changes in the magnetic topology during combined neutral beam and fast wave heating in the DIII-D tokamak. Measurements with a motional Stark effect diagnostic provide accurate determination of the equilibria before and after the sawtooth reconnection events. The global magnetic helicity $\int \boldsymbol{A} \cdot \boldsymbol{B}\, dV$ changes $0.2 \pm 0.9\%$ at a sawtooth crash. The local change in the helical flux, χ, is roughly consistent with the Kadomtsev model within large errors. The volume in which the helical flux changes is $85 \pm 15\%$ of the volume predicted by Kadomtsev, while the central value of χ is within 1% of the predicted value.



[Heidbrink and Dang, PPCF, 2000]

Equilibria can be obtained from a variational principle

minimize energy, $W = \int_V \left(\frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0}\right) dV$, with constraints on helicity, $K \equiv \int_V \left(\mathbf{A} \cdot \mathbf{B}\right) dV$

more constraints

Ideal MHD
[Kruskal, Kulsrud, 1958]

K conserved **continuously** $\Longrightarrow$ $\mathbf{j} \times \mathbf{B} = \nabla p$

N = ∞

MRxMHD
[Dewar et al, 2006]

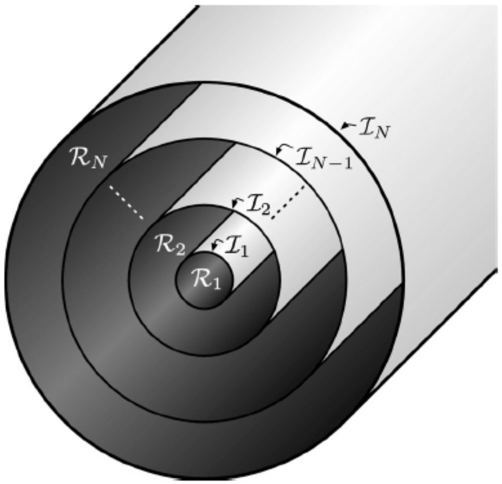
K conserved **discretely** $\Longrightarrow$ $\nabla \times \mathbf{B} = \mu_l \mathbf{B}$, $[[p + \frac{B^2}{2\mu_0}]] = 0$

N = 1

Taylor's theory
[Taylor, 1974]

K conserved **globally** $\Longrightarrow$ $\nabla \times \mathbf{B} = \mu \mathbf{B}$

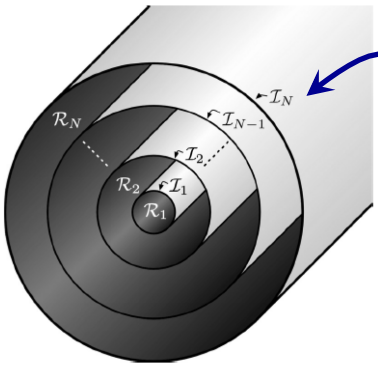
fewer constraints



N relaxation volumes
N-1 ideal interfaces

MRxMHD allows exploring energetically favorable **reconnection events** while **constraining** profiles.

The SPEC code finds MRxMHD equilibria



Input

Boundary geometry
+
Constraints

Output

Shape of ideal interfaces
+
B in each volume

Possible constraints

pressure, helicity, and fluxes

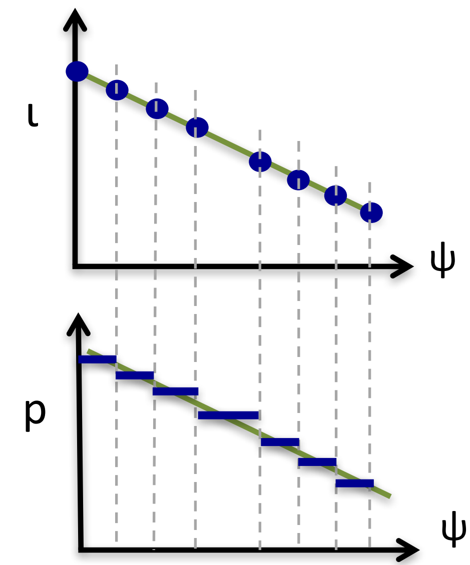
$\{ \mathbf{p} , \mathbf{K} , \boldsymbol{\Psi}_{pol} , \boldsymbol{\Psi}_{tor} \}_{l=1...N}$

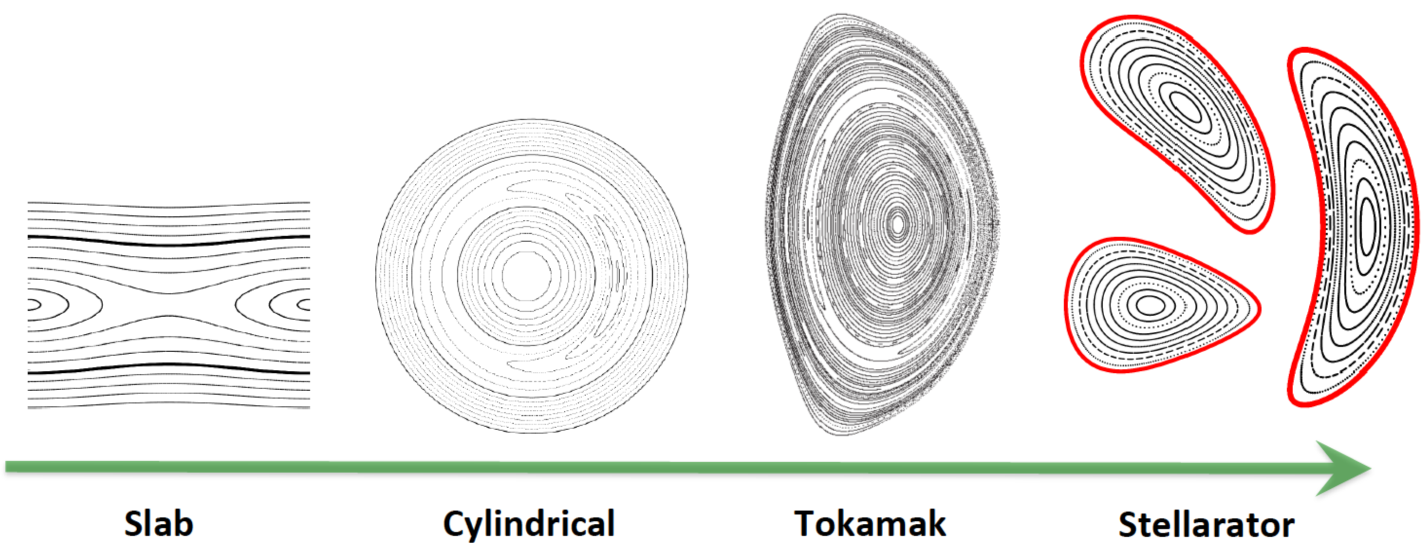
pressure, current, and fluxes

$\{ \mathbf{p} , \boldsymbol{\mu} , \boldsymbol{\Psi}_{pol} , \boldsymbol{\Psi}_{tor} \}_{l=1...N}$

pressure, iota, and flux

$\{ \mathbf{p} , \boldsymbol{\iota}^{+} , \boldsymbol{\iota}^{-} , \boldsymbol{\Psi}_{tor} \}_{l=1...N}$

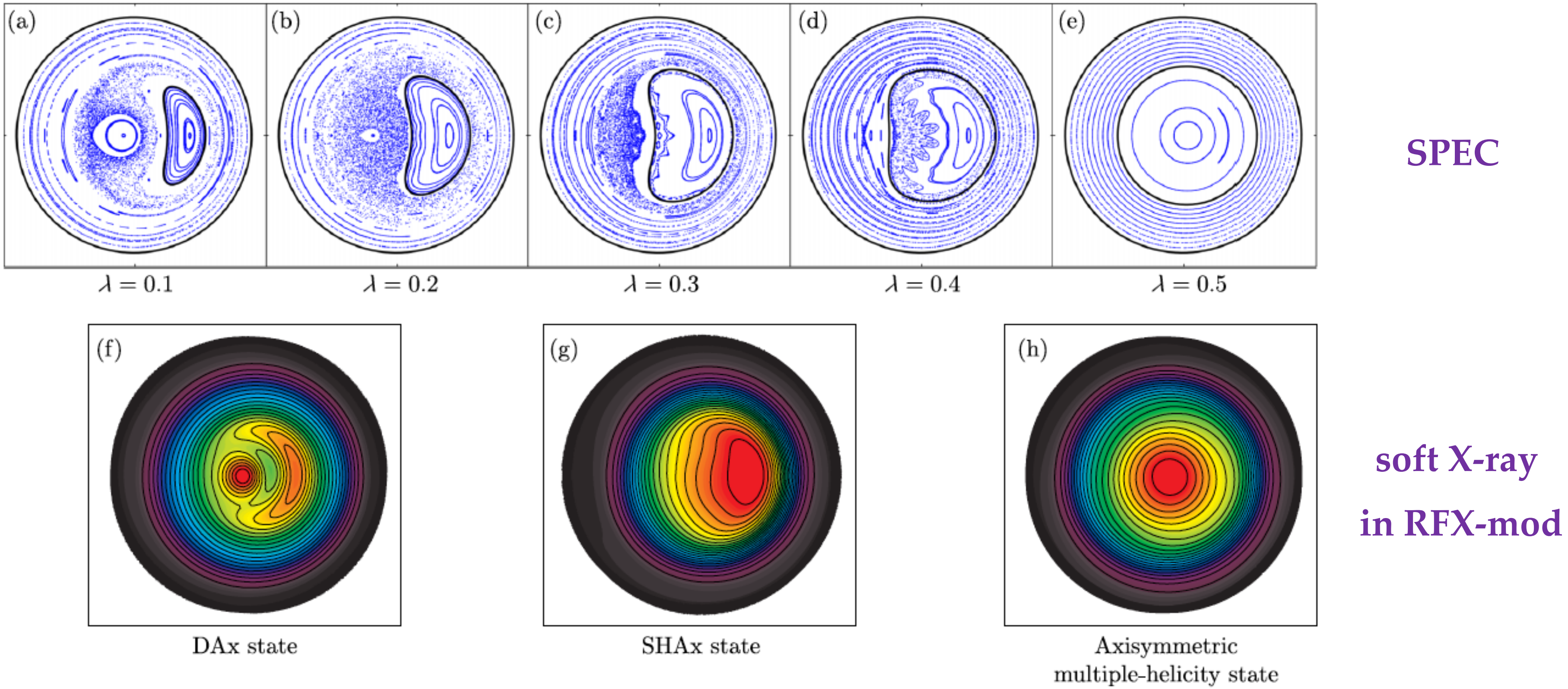




MRxMHD can describe equilibria with islands & chaos

EXAMPLE: Reversed Field Pinch configuration

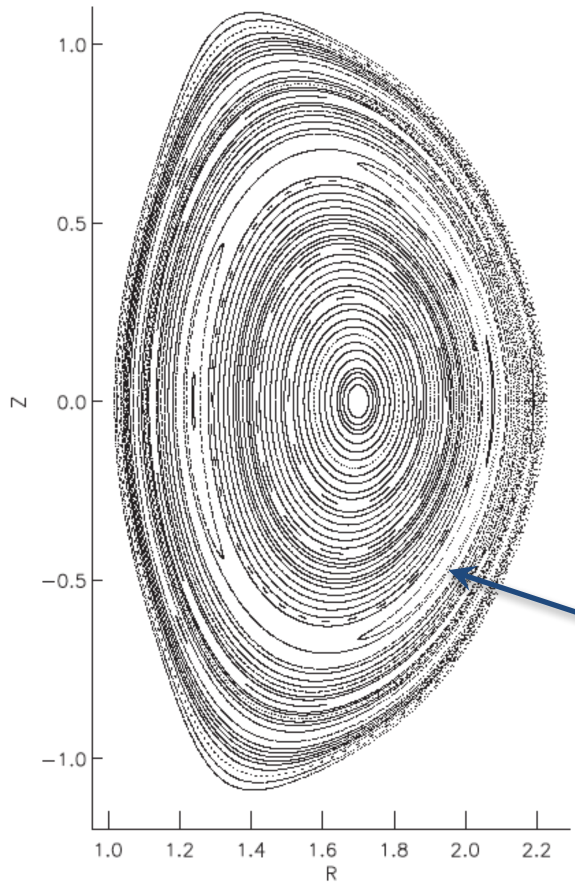
Reproduced experimentally observed 3D states with SPEC and only $\boldsymbol{N = 2}$. [Dennis et al, PRL, 2013]

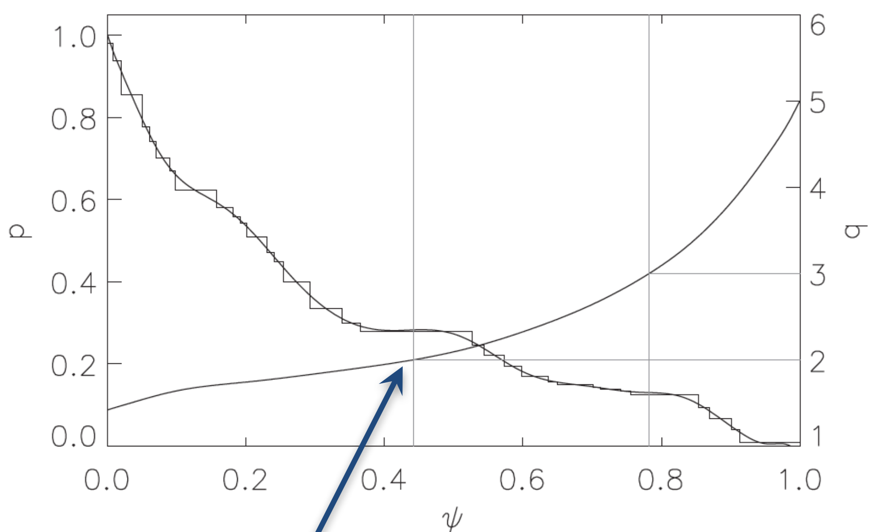


MRxMHD can describe equilibria with islands & chaos

EXAMPLE: Tokamak with resonant magnetic perturbations

Equilibrium reconstructed from DIII-D experimental profiles with ***N* = *32*.** [Hudson et al, PoP, 2012]



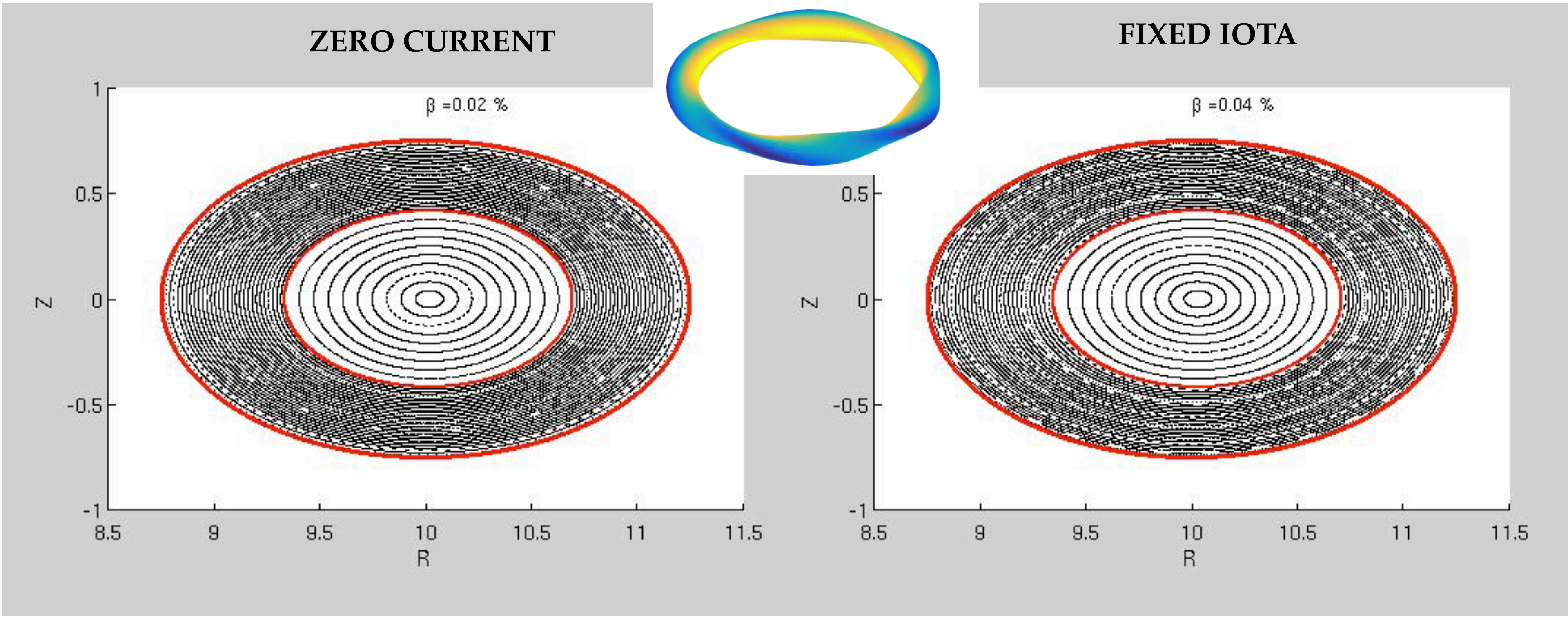


Magnetic island at q = 2 rational surface

MRxMHD can describe equilibria with islands & chaos

EXAMPLE: Classical stellarator at fixed current or fixed iota

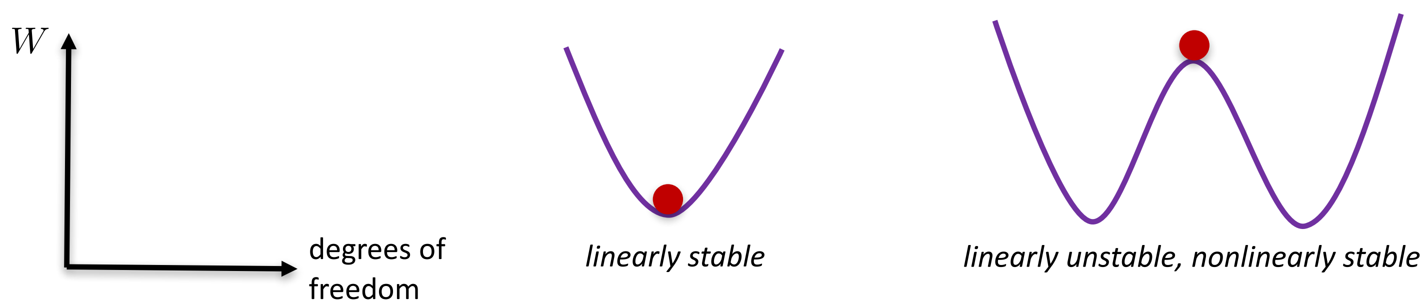
Equilibrium β-limits studied with SPEC and $N = 2$. [Loizu et al, JPP, 2017]



Stability obtained by exploiting a variational principle

- Equilibrium states are those for which $\delta W = 0$.

- Their **stability** can be evaluated by studying the sign of δW for a finite perturbation.



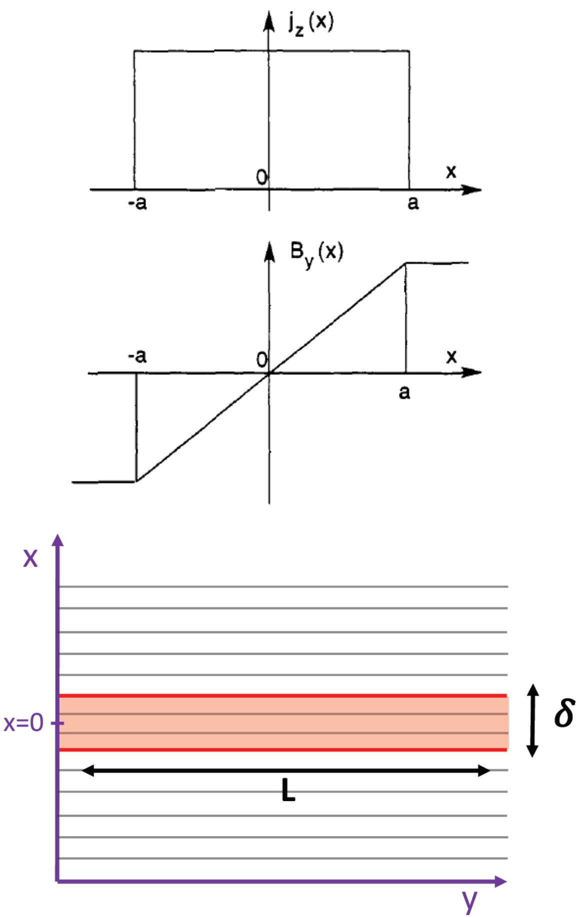
- In the ideal limit, $N \rightarrow \infty$, MRxMHD stability analysis retrieves ideal stability.

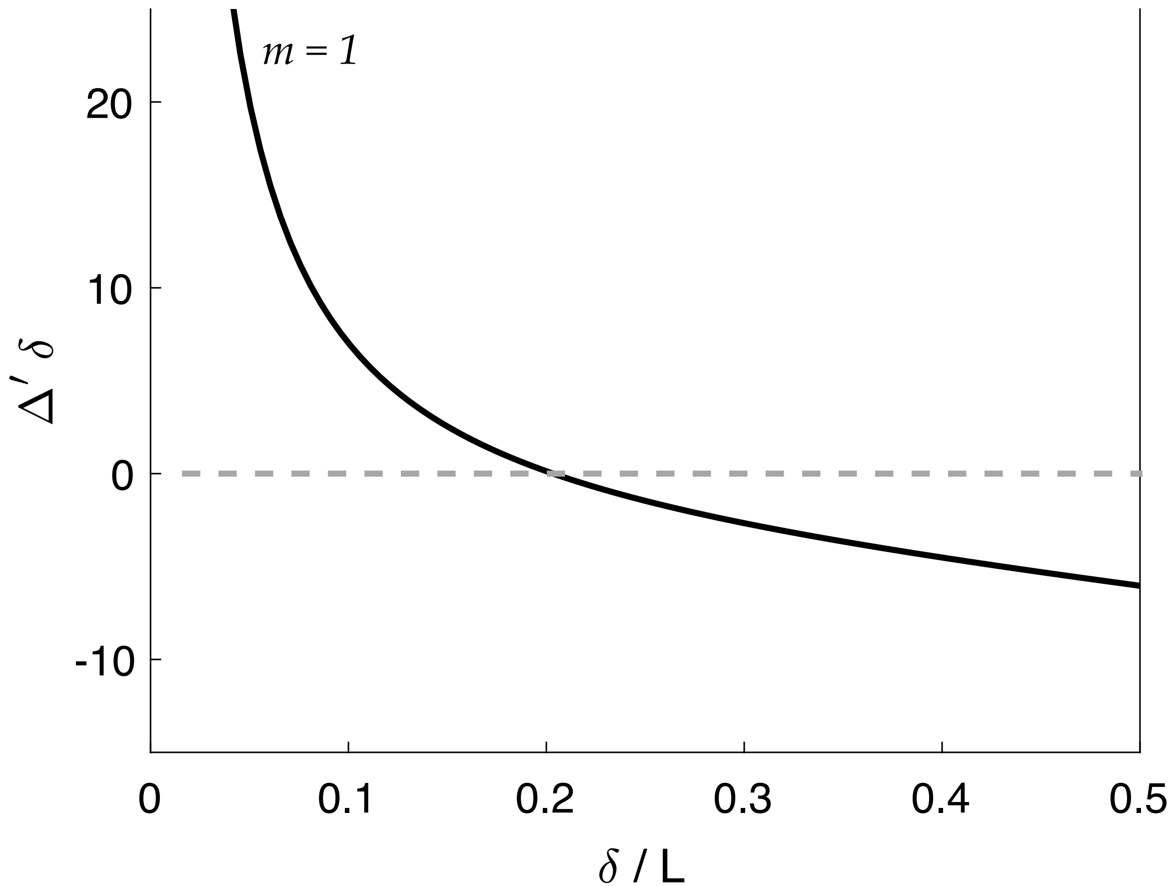
- For finite N : retrieve instabilities developing via **spontaneous magnetic reconnection**.

- **Is tearing mode linear stability and nonlinear saturation described by MRxMHD?**

A slab current sheet is ideally stable & tearing unstable

Consider a "textbook" force-free current sheet equilibrium:

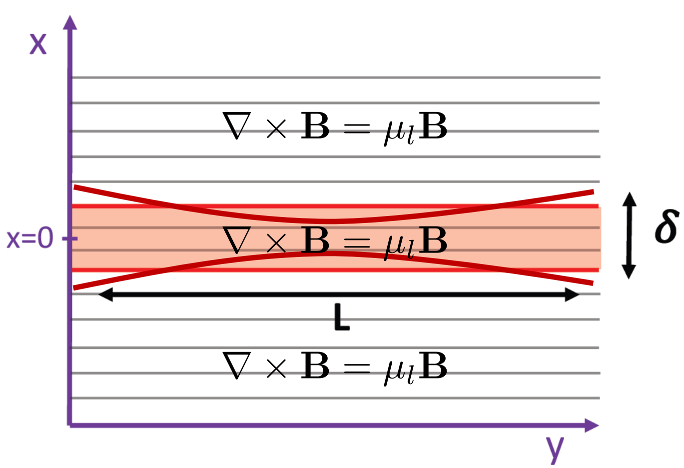


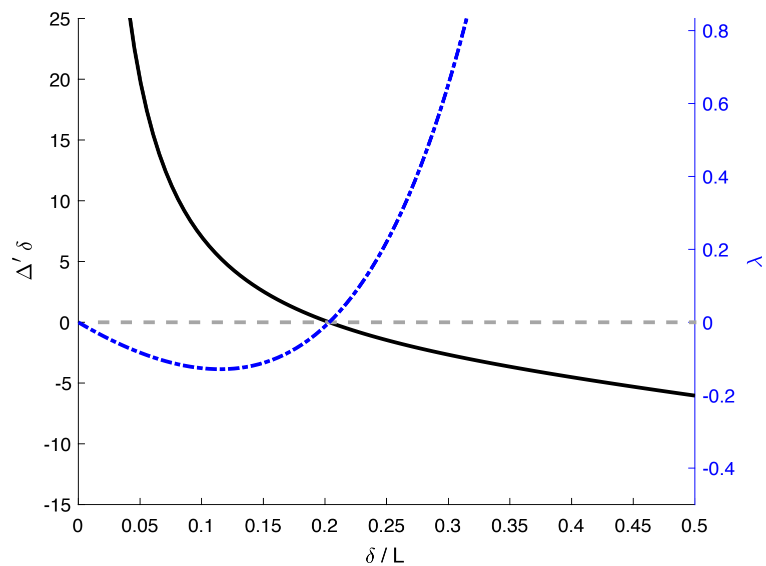


MRxMHD retrieves linear tearing stability threshold

- **Equilibrium** described by MRxMHD with **N = 3**.
- **Perturbation** of interfaces allows constructing Hessian $H_{ij} = \dfrac{\partial f_{l_i,m_i}}{\partial x_{l_j,m_j}}$ "matrix of force derivatives"
- **Eigenvalues** of H provide stability information, **$\lambda < 0$ means instability**
- Here we can calculate **analytically** the eigenvalue λ for the m = 1 mode , [Loizu and Hudson, PoP, 2019]

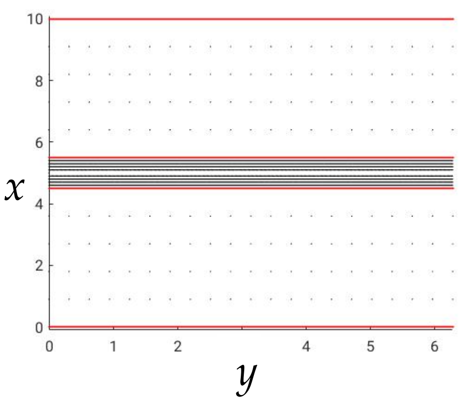
$$\lambda = \frac{\delta}{L}\left(\cosh\left(2\pi\frac{\delta}{L}\right) - 1\right) + \left(\frac{\delta}{L} - \frac{1}{\pi}\right)\sinh\left(2\pi\frac{\delta}{L}\right)$$

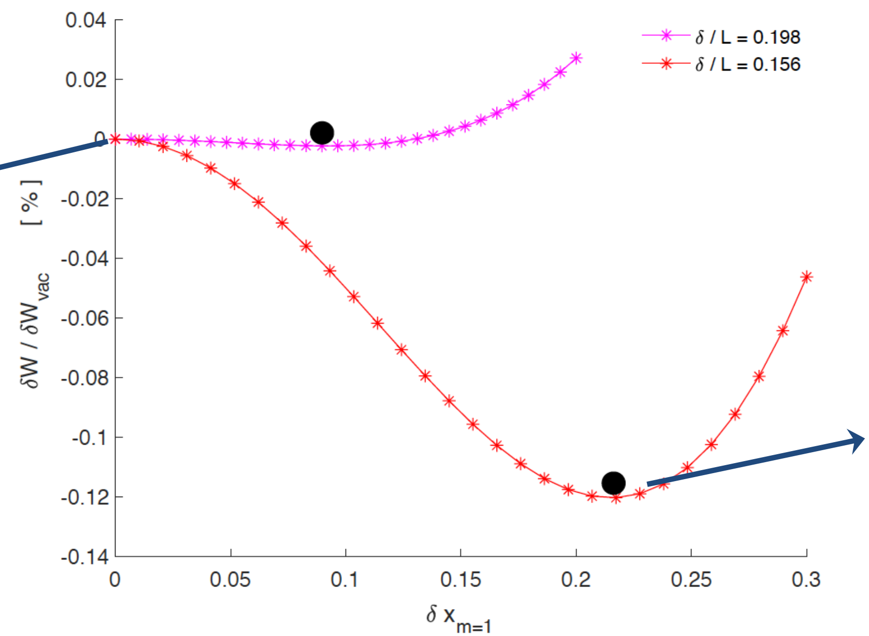


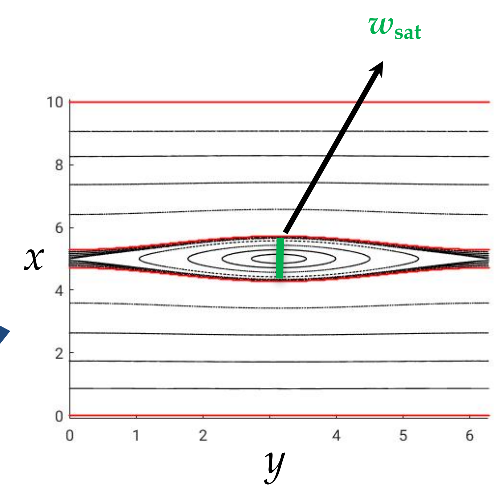


A lower energy state with an island exists

- Explored the space of W(**x**) where **x** is the amplitude of the *m = 1* perturbation.
- Energy valley is observed, suggesting the existence of a saturated island.





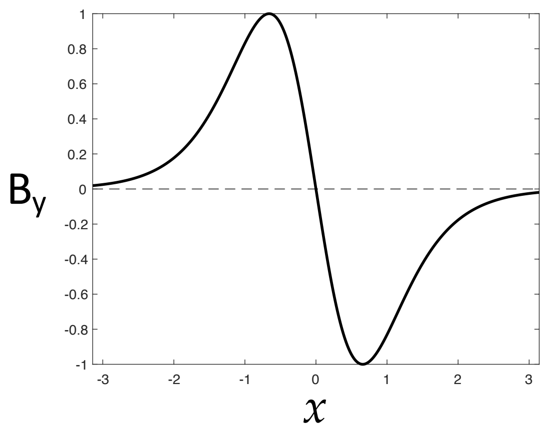


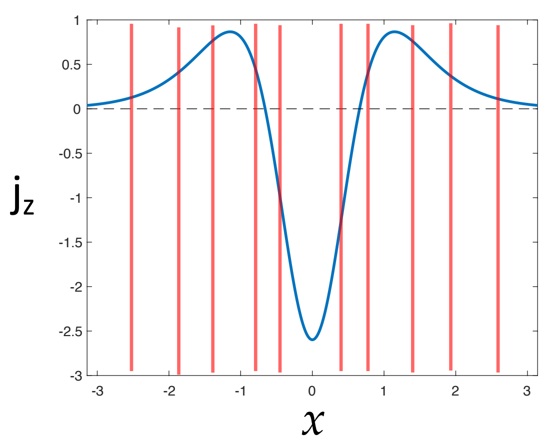
- **Can we use a variational principle to predict w_{sat} without resolving the dynamics?**
- **Can we validate predictions against resisitive MHD theory / simulations?**

SPEC can be used for more general equilibria

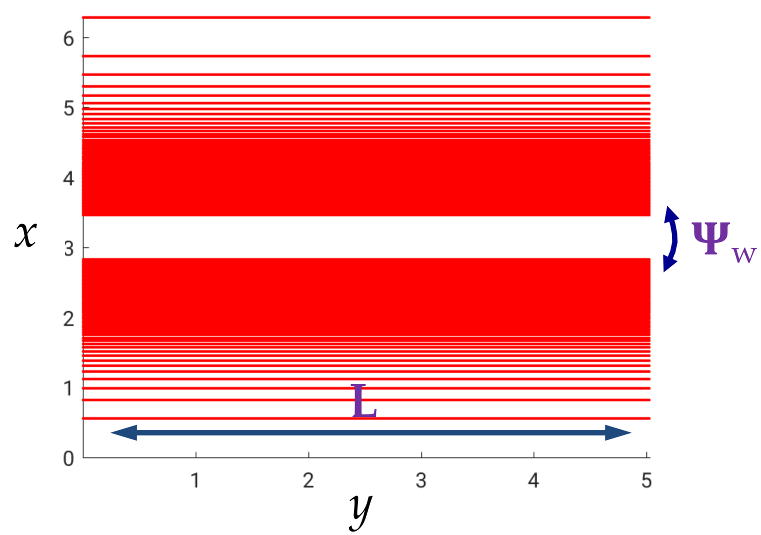
- Consider a slab equilibrium as in [Loureiro, PRL, 2005] , $\mathbf{B} = B_z\, \mathbf{e_z} + \mathbf{e_z} \times \nabla \psi(x)$, with

$$\psi(x) = \frac{\psi_0}{\cosh^2 x}$$



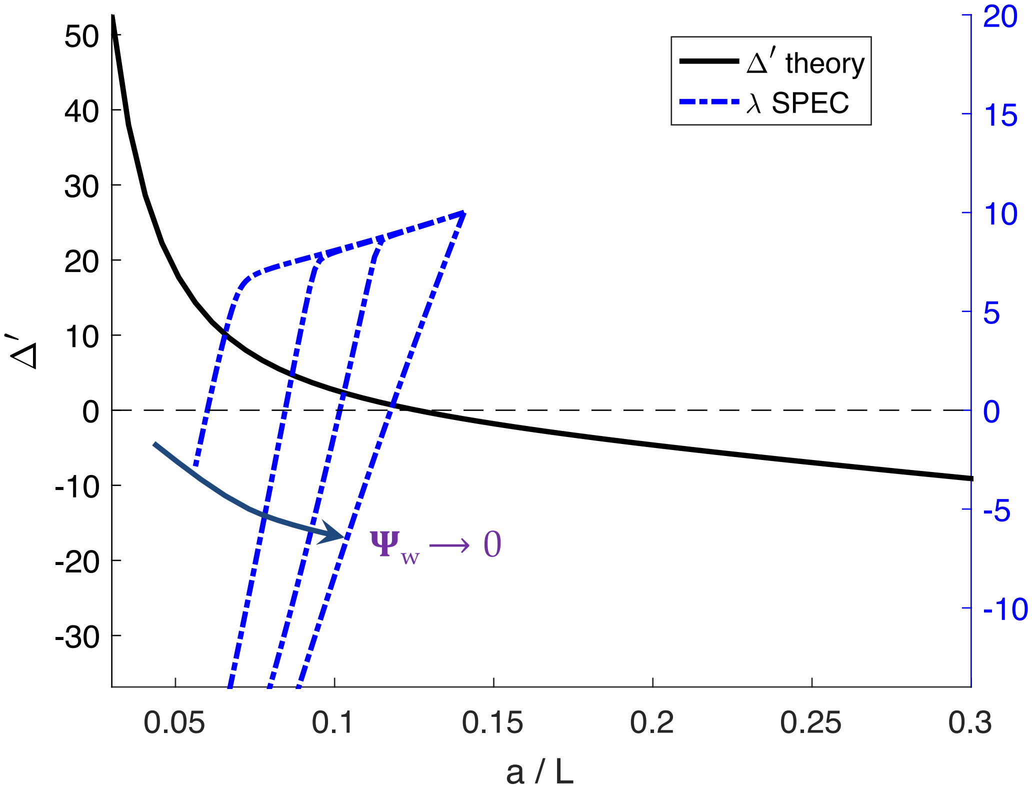


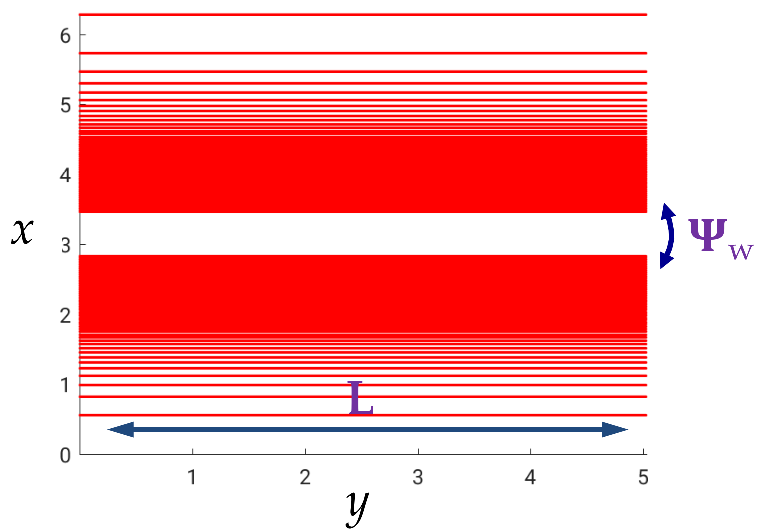
- Construct SPEC equilibrium with **N >> 1**.
- Interfaces packed around resonant surface.
- $\{\, \mathbf{p}\, ,\ \boldsymbol{\mu}\, ,\ \boldsymbol{\Psi}_{\mathrm{pol}}\, ,\ \boldsymbol{\Psi}_{\mathrm{tor}} \}_{i=1\ldots N}$ extracted analytically.
- Two control parameters: $\boldsymbol{\Psi}_w$ and **L**



SPEC retrieves linear tearing stability for $\Psi_w \longrightarrow 0$

- Linear tearing stability parameter is [Porcelli, PPCF, 2002] : $\Delta' = \frac{2(5-k^2)(3+k^2)}{k^2\sqrt{4+k^2}}$
- For the $m = 1$ tearing mode, $k = 2\pi/L$.





Analytical nonlinear theory exists in slab

- Quasi-linear theory of the tearing mode was developed in 1977. [White, PoF, 1977]

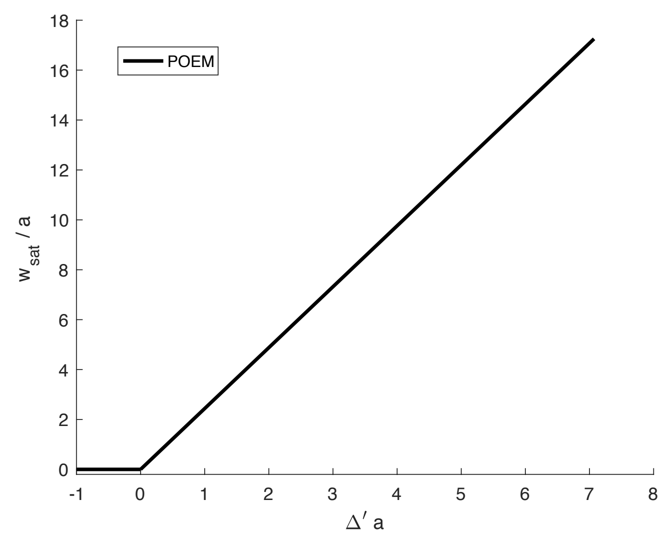
$$\frac{dw}{dt} \propto (\Delta'(w) - \alpha w) \quad , \text{ where } \alpha \text{ depends on current and resistivity profiles}$$

- Exact nonlinear theory (**POEM**) was developed in 2004. [**E**scande and **O**ttaviani, PLA, 2004] [**M**ilitello and **P**orcelli, PoP, 2005]

$$w_{sat} = 2.44\ \Delta' a_{eq}^2 \quad , \text{ where } \quad a_{eq}^2 = \frac{B_y'(0)}{B_y'''(0)}$$

- POEM has a universal character and w_{sat} is independent of resistivity.

- Both theories are only supposed to be valid for sufficiently **small Δ′** .



Resistive MHD simulations provide w_{sat} at large Δ’

Solved the compressible, visco-resistive, MHD equations in 2D [Huang and Bhattacharjee, AJ, 2010]

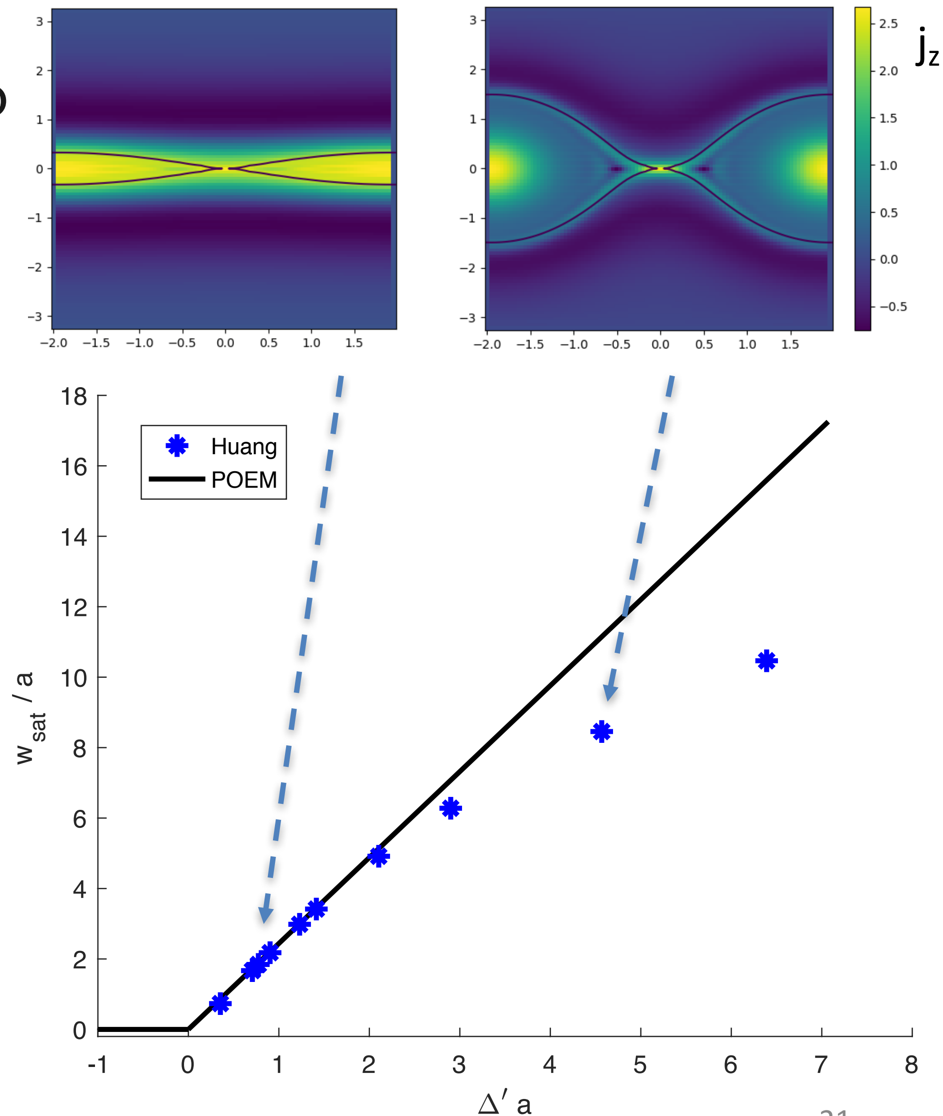
$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\partial_t (\rho \boldsymbol{v}) + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v}) = -\nabla\left(p + \frac{B^2}{2}\right) + \nabla \cdot (\boldsymbol{B}\boldsymbol{B}) + \nu \nabla^2 (\rho \boldsymbol{v})$$

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B} - \eta \boldsymbol{J})$$

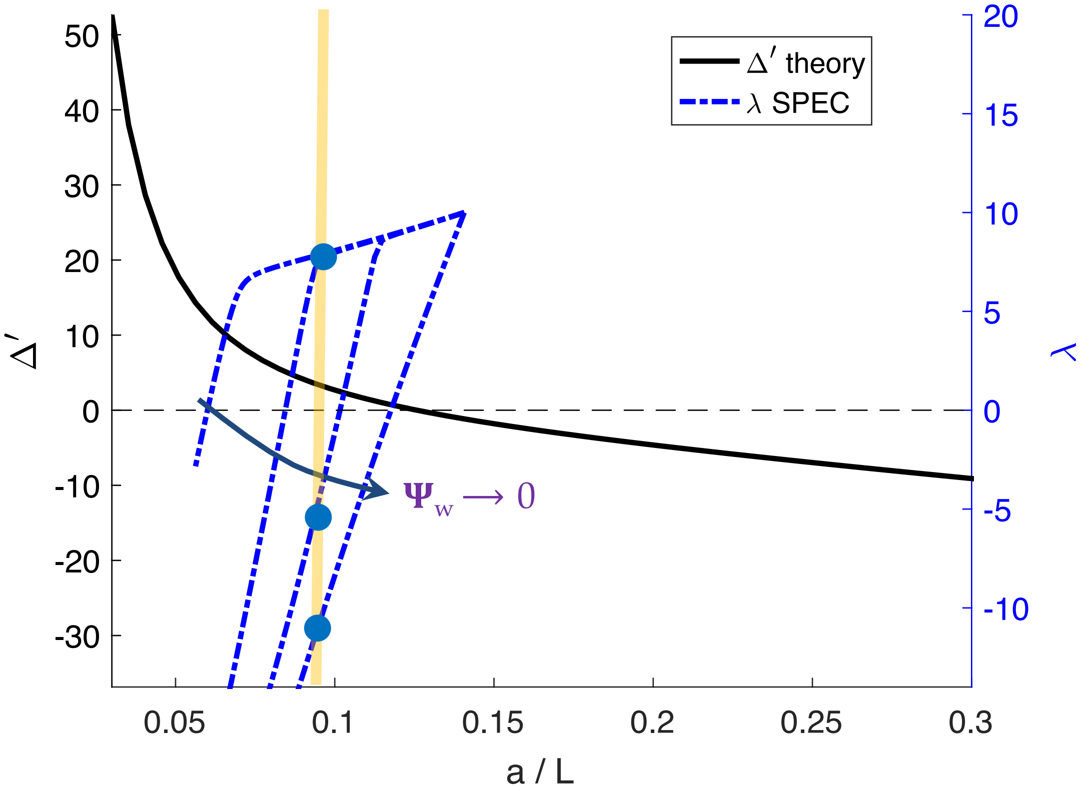
with same initial equilibrium,

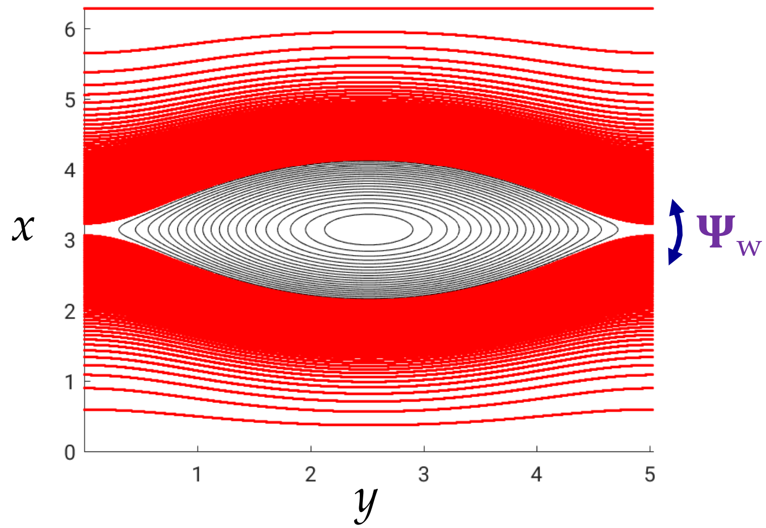
$$\mathbf{B} = B_z\, \mathbf{e_z} + \mathbf{e_z} \times \nabla \psi(x), \text{ with } \psi(x) = \frac{\psi_0}{\cosh^2 x}$$



Quasi-linear argument used to find critical Ψ_w

In SPEC, we scan Δ' by scanning **L**. But which value shall we choose for Ψ_w?





Recipe: for each **L**,

- find $\lambda(\Psi_w) = 0$
- run nonlinear SPEC

SPEC nonlinear calculations retrieve w_{sat} scaling

➢ Nonlinear calculations at fixed helicity produce reasonable yet different scaling.

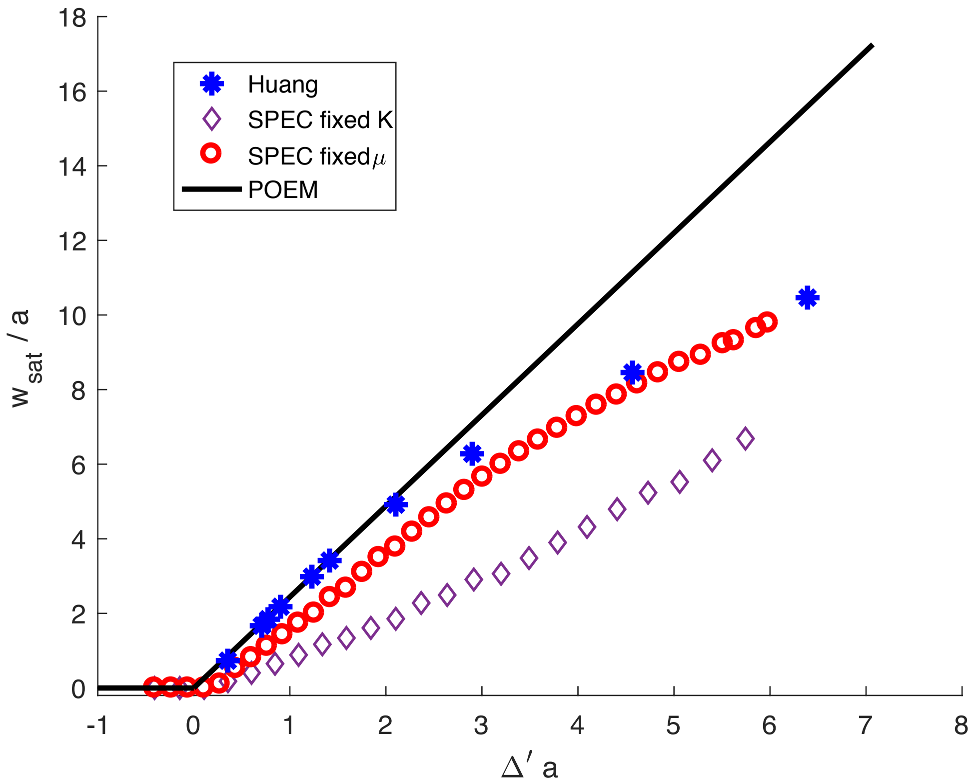
➢ In POEM, the magnetic helicity is not exactly (although quite well) conserved.

➢ What is constrained to be constant is the flux-surface averaged current $<\mathbf{j_z}>_\psi$.

➢ Nonlinear calculations at fixed current reproduce well expected scaling!

➢ Results do not depend on **N**.

➢ A **blue star** takes ~ 1h and a **red circle** takes ~ 5 sec.



Summary and perpectives

The **fundamental results** presented here are:

- MRxMHD can be used to predict linear resistive stability of force-free equilibria.

- MRxMHD is able to obtain nonlinearly saturated tearing modes.

First demonstration that tearing mode saturation in strong guide fields can be directly obtained from a variational principle!

A **direct implication** is that numerical codes like SPEC, which calculate MRxMHD equilibria in toroidal geometry, can be used to (quickly) predict the resistive stability and tearing mode saturation in **tokamaks and stellarators**.

An **open question** is whether MRxMHD can also correctly calculate stability and saturation of tearing modes in the presence of pressure gradients.