Differentiating the optimal coil geometry with respect to the target surface

S. R. Hudson
Princeton Plasma Physics Laboratory, PO Box 451, Princeton NJ 08543, USA

C. Zhu
University of Science and Technology of China, 96 JinZhai Road, Hefei, Anhui 230026, P. R. China

D. Pffeferlé
University of the Other Side of the World, Way Waaay Out West, Oz

L. Gunderson
Princeton Plasma Physics Laboratory, P.O. Box 451, Princeton NJ 08543, USA

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The task of designing the geometry of a set of current-carrying coils that produce the magnetic field required to confine a given plasma equilibrium is expressed as a minimization principle, namely that the coils minimize a suitably defined error expressed as a surface integral, which is recognized as the quadratic-flux. A penalty on the coil length is included to avoid pathological solutions.

A simple expression for how the quadratic-flux and length vary as the coil geometry varies is derived, and an expression describing how this varies with variations in the surface geometry is derived. These expressions allow efficient coil-design algorithms to be implemented, and also enable efficient algorithms for varying the surface in order to simplify the coil geometry.

I. INTRODUCTION

The conventional approach [1] to designing stellarators [2] is to first determine the desired plasma state via an equilibrium optimization, and then to determine the geometry of a number of closed, current-carrying "coils" that produce the required vacuum field. Together with the field produced by plasma currents that accompany finite-pressure plasmas, the vacuum field must create the "magnetic bottle" that confines the plasma.

Stellarators have the particular advantage that the magnetic bottle is primarily produced by the externally applied field, and there is not much that plasmas can do to "break" the bottle. To use more formal terminology, most macroscopic instabilities are benign. Stellarators traditionally, however, have had the disadvantage that the necessarily complicated, so-called three-dimensional geometry means that there are additional "hills" in the magnetic field strength along the magnetic fieldlines, between which charged particles bounce back and forth leading to enhanced losses. There are, however, encouraging recent developments that suggest that these losses can be minimized (see the recent overview by Gates *et al.* [3] and references therein). This paper addresses how to design coils that confine a given equilibrium, and how changing the shape of the equilibrium will change the shape of the coils.

We first, in Sec. II, consider the problem of designing the coils for a given "target" equilibrium. The coils must produce the required magnetic field inside some toroidal volume, $\mathcal{V}$, that encompasses the plasma domain.

In Sec. III we consider how the shape of the optimal coils change as the target surface changes. This is accomplished by the calculus of variations of surface integrals. The magnetic field produced by the plasma in equilibrium with a given boundary will change as the boundary changes, but herein we restrict attention to vacuum fields.

Finally, recognizing the growing, practical realization that designing plasmas and designing coils are not really separate problems but must be considered together, in Sec. IV, we describe an constrained optimization principle that simplifies a measure of coil complexity under the constraint of conserved plasma properties.

II. THE PROBLEM OF COIL DESIGN

Laplace showed that vacuum fields in a given $\mathcal{V}$, with boundary $\mathcal{S} \equiv \partial\mathcal{V}$, are unique if appropriate boundary conditions are provided. We choose a Neumann boundary condition: we require a set of coils that produces a given normal magnetic field, $B_n^T \equiv \mathbf{B}^T \cdot \mathbf{n}$, on $\bar{\mathbf{x}}(\theta,\zeta) \equiv \mathcal{S}$, where θ and ζ parameterize position on the surface. This must satisfy $\oint_{\mathcal{S}} \mathbf{B}^T \cdot d\mathbf{s} = 0$ for the net flux of fieldlines to be consistent with a divergence-free field.

For brevity, herein we primarily restrict attention to the case that $B_n^T = 0$. It is straightforward to generalize the following to accommodate arbitrary B_n^T. The unique solution for the vacuum field must also be constrained by a loop integral, e.g. the enclosed toroidal flux, $\Psi \equiv \oint_{\mathcal{L}} \mathbf{A} \cdot d\mathbf{l}$, where $\mathcal{L}$ is a "poloidal loop".

Let $\mathbf{x}_i(l)$ represent the geometry of a set of $i = 1, \dots, N_C$ closed one-dimensional curves, hereafter called "coils", which are parameterized by l, each carrying current I_i. Herein we shall treat the number of coils, N_C, as being fixed, but generally N_C is a degree-of-freedom. No constraints are imposed on the coil geometry, other than requiring that each coil be closed, $\mathbf{x}(l+2\pi) = \mathbf{x}(l)$. The magnetic field is given by the Biot-Savart law,

$$\mathbf{B}_i(\bar{\mathbf{x}}) \ = \ I_i \oint_i \mathbf{x}_i' \times \mathbf{r}/r^3 \, dl, \tag{1}$$

where $\mathbf{r}(\theta,\zeta,l) \equiv \bar{\mathbf{x}}(\theta,\zeta) - \mathbf{x}_i(l)$, and the prime denotes differentiation with respect to l.

With a finite number of finite-length coils, we cannot generally expect to obtain a coil set that *exactly* produces the required field. So, we must seek instead a coil set that minimizes a suitably defined error.

In 1987, Merkel [4] presented a method that determined the continuous current potential on a prescribed "winding" surface lying outside the plasma that minimized the squared normal field; thirty years later Landreman [5] regularized this method. Drawing upon these ideas, the *minimally* constrained solution [6] for the coil geometry minimizes the functional

$$F(\mathbf{x}_i, \bar{\mathbf{x}}) \equiv \frac{1}{2} \oint_{\mathcal{S}} B_n^2 \, ds + \omega L. \tag{2}$$

The first term is called [7] the quadratic flux, φ_2. A penalty on the total length of the coils, $L \equiv \sum_i \oint |\mathbf{x}_i'| \, dl$, is included, and ω is a user-supplied "weight".

More elaborate functionals can be introduced that recognize that some distributions, B^n_{mn}, of the normal field on the boundary are more important than others, namely those that resonate with internal rational rotational-transform surfaces and thereby create magnetic islands, or that resonate with plasma oscillations. The particularly important distributions should be suitably weighted in the minimization, achieved by replacing φ_2 with $\oint \omega_{mn} |B^n_{mn}| \, ds$, for example.

The penalty on the length is a regularization term. In the limit that $\omega \to 0$ the minimization problem is ill-posed and the coils can become infinitely long. As ω is increased, the extremizing coils become shorter, and φ_2 will typically increase. Including additional constraints or penalties in F, e.g. penalizing the inter-coil electromagnetic forces that increase the cost of the support structures, will also typically serve to compromise the minimization of φ_2.

Upon varying the i-th current, the first variation in F is

$$\delta F = \delta I_i \, \frac{\partial F}{\partial I_i} \tag{3}$$

where

$$\frac{\partial F}{\partial I_i} = \oint_i \oint_{\mathcal{S}} B_n \, \mathbf{x}_i' \times \mathbf{r} \cdot \mathbf{n} / r^3 \, ds \, dl. \tag{4}$$

Generally [6], a constraint on the toroidal flux must be included in F to avoid the trivial solution that all the coil currents are zero, $I_i = 0$. In this paper this solution is avoided by setting each $I_i = 1$. This has some practical advantage, as it means that the coils can be energized in series.

Upon varying the geometry of the i-th coil, the first variation in the magnetic field is

$$\delta \mathbf{B}(\bar{\mathbf{x}}) \; = \; \oint_i (\delta \mathbf{x}_i \times \mathbf{x}_i') \cdot \mathbf{R}_i \, dl, \tag{5}$$

where $\mathbf{R} = 3\,\mathbf{r}\,\mathbf{r}/r^5 - \mathbf{I}/r^3$, where $\mathbf{I}$ is the "idemfactor", e.g., $\mathbf{I} \; = \; \mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k}$, which has the property that $\mathbf{v} \cdot \mathbf{I} = \mathbf{v}$ and $\mathbf{I} \cdot \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$. We have used $\mathbf{r} \times [\mathbf{r} \times (\delta\mathbf{x} \times \mathbf{x}')] = (\mathbf{r} \times \delta\mathbf{x})\,(\mathbf{r} \cdot \mathbf{x}') - (\mathbf{r} \times \mathbf{x}')\,(\mathbf{r} \cdot \delta\mathbf{x})$ to obtain an expression that explicity shows that variations tangential to the curve, which do not alter the geometry of the coils, do not alter the magnetic field. The first variation in F is

$$\delta F = \oint_i \delta \mathbf{x}_i \cdot \frac{\delta F}{\delta \mathbf{x}_i} \, dl, \tag{6}$$

where

$$\frac{\delta F}{\delta \mathbf{x}_i} \equiv \mathbf{x}_i' \times \left(\oint_{\mathcal{S}} \mathbf{R}_{i,n} \, B_n \, ds + \omega \, \boldsymbol{\kappa}_i \right), \tag{7}$$

and $\boldsymbol{\kappa}$ is the coil curvature, $\boldsymbol{\kappa} \equiv \mathbf{x}' \times \mathbf{x}''/|\mathbf{x}'|^3$.

A local minimum may be found from an initial guess by integrating $\partial \mathbf{x}_i / \partial \tau = -\delta F / \delta \mathbf{x}_i$, where τ is an arbitrary integration parameter. This "descent" algorithm is certainly not the fastest; but, because the coils are continuously deformed, the coils cannot pass through the surface without producing infinities in F, and therefore the Gauss linking number of the coils with respect to the plasma,

$$\frac{1}{4\pi} \oint_i \oint_a \frac{\mathbf{x}_i - \mathbf{x}_a}{|\mathbf{x}_i - \mathbf{x}_a|^3} \cdot d\mathbf{x}_i \times d\mathbf{x}_a, \tag{8}$$

is conserved, where $\mathbf{x}_a$ is the magnetic axis, for example. The trivial solution that the coils become arbitrarily far removed is avoided if the initial geometry of the coils is suitably linked. (This article shall not address the problems associated with finding global minima.)

Coils that link N times, where N is an integer greater than 1, are commonly called "helical", and if $N = 1$ the coils are called "modular". Coils that do not link the plasma may provide a vertical field; or if they are used for fine-tuning plasma performance they are called "trim coils", or "saddle coils", or "resonant magnetic perturbation coils". If the plasma itself has a non-trivial knottedness [8], different linking arrangements are possible, but this is as-yet largely unexplored. The theoretical and numerical methods described in this paper are applicable to any type of coil.

A Newton method may be used to find extremizing coils. The second variation in F resulting from variations in the coil geometry are, for $j \neq i$, given by

$$\delta^2 F = \oint_i \oint_j \delta \mathbf{x}_i \cdot \frac{\delta^2 F}{\delta \mathbf{x}_i \delta \mathbf{x}_j} \cdot \delta \mathbf{x}_j \, dl_i \, dl_j, \tag{9}$$

where

$$\frac{\delta^2 F}{\delta \mathbf{x}_i \delta \mathbf{x}_j} = \oint_{\mathcal{S}} (\mathbf{x}_i' \times \mathbf{R}_{i,n})\,(\mathbf{x}_j' \times \mathbf{R}_{j,n}) \, ds. \tag{10}$$

For $j = i$, the expression is more complicated and not particularly insightful. The algebra becomes concise if we write $\mathbf{x}_i(l) \equiv \sum_k x_{i,k} \, \boldsymbol{\varphi}_k(l)$, where the $\boldsymbol{\varphi}_k$ comprise set of basis functions; e.g., $\varphi_1 = \mathbf{i}$, $\varphi_2 = \mathbf{j}$, $\varphi_3 = \mathbf{k}$, $\varphi_4 = \cos(l)\mathbf{i}$, $\varphi_4 = \cos(l)\mathbf{j}$, and so on; and the $x_{i,k}$ are the independent degrees of freedom that describe the geometry of the coils. Then, $F = F(\mathbf{c}, \mathbf{s})$, where $\mathbf{c} \equiv \{x_{i,k}\}$, and $\mathbf{s} \equiv \{\bar{x}_k\}$ represents a similar parameterization of the surface. The variation in F resulting from variations in the coil geometry is

$$F(\mathbf{c} + \delta\mathbf{c}, \mathbf{s}) \approx F(\mathbf{c}, \mathbf{s}) + \nabla_{\mathbf{c}} F \cdot \delta\mathbf{c} + \frac{1}{2} \delta\mathbf{c}^T \cdot \nabla^2_{\mathbf{cc}} F \cdot \delta\mathbf{c}.$$

Newton iterations proceed by inverting the Hessian, $\delta \mathbf{c} = -\nabla^2_{\mathbf{cc}} F(\mathbf{c},\mathbf{s})^{-1} \cdot \nabla_{\mathbf{c}} F(\mathbf{c},\mathbf{s})$. This approach has been implemented [9] in the recently developed FOCUS code [6].

A suitably constrained equal-arc parameterization, for example, will eliminate the purely "numerical" nullspace of the Hessian associated with tangential variations. An eigenvalue analysis of the Hessian determines sensitivity to coil misplacement errors [10], and this determines the construction tolerances. Bifurcations in the coil geometry are particularly interesting and are associated with zero eigenvalues. Hereafter, we consider that the coil geometry is a function of the boundary, i.e., $\mathbf{x}_i = \mathbf{x}_i(\bar{\mathbf{x}})$.

III. VARIATIONS IN THE SURFACE

Imagine that the coil geometry that minimizes F for a given surface has been found and consider a variation, $\delta\bar{\mathbf{x}}$, in the geometry of the surface. A variation in the coil geometry is generally required if the condition $\delta F/\delta \mathbf{x}_i = 0$ is to be preserved. The variation in F resulting from variations, $\delta \mathbf{x}_i$ and $\delta\bar{\mathbf{x}}$, in the geometry of the i-th coil and the surface is

$$\delta^2 F = \oint_i \delta \mathbf{x}_i \cdot \oint_{\mathcal{S}} \frac{\delta^2 F}{\delta \mathbf{x}_i \delta \bar{\mathbf{x}}} \cdot \delta \bar{\mathbf{x}} \;\, ds\, dl, \tag{11}$$

where

$$\frac{\delta^2 F}{\delta \mathbf{x}_i \delta \bar{\mathbf{x}}} \equiv \mathbf{x}_i' \times (\mathbf{R}_S \cdot \nabla B_n + \mathbf{B}_S \cdot \nabla \mathbf{R}_n + B_n \mathbf{R} \cdot \mathbf{H})\, \mathbf{n}, \tag{12}$$

where $\mathbf{B}_S \equiv \mathbf{B} - B_n \mathbf{n}$ is the projection of $\mathbf{B}$ in the tangent plane to $\bar{\mathbf{x}}$, and similarly for $\mathbf{R}_S \equiv \mathbf{R} - \mathbf{R}_n \mathbf{n}$. The mean curvature can be written $\mathbf{H} \equiv -\mathbf{n}\,(\nabla \cdot \mathbf{n})$. In deriving Eqn. 12, we have followed the mathematical formalism for variations in surface integrals such as the quadratic flux with respect to surface variations described by Dewar *et al.* [7], and we have used $\nabla_{\bar{\mathbf{x}}} \cdot \mathbf{R} = 0$.

Only variations in the boundary that are *normal* to the boundary are relevant, and only derivatives that are tangential to the surface appear. The latter are most conveniently computed using the tangential dual space to $\mathbf{e}_\theta$ and $\mathbf{e}_\zeta$ given by $\nabla\theta \equiv \mathbf{e}_\zeta \times \mathbf{n}/(\mathbf{n} \cdot \mathbf{e}_\theta \times \mathbf{e}_\zeta)$ and $\nabla\zeta \equiv \mathbf{n} \times \mathbf{e}_\theta/(\mathbf{n} \cdot \mathbf{e}_\theta \times \mathbf{e}_\zeta)$, and the tangential directional derivative is $\mathbf{B}_S \cdot \nabla = \mathbf{B} \cdot \nabla\theta\; \partial_\theta + \mathbf{B} \cdot \nabla\zeta\; \partial_\zeta$, and similarly for $\mathbf{R}_S \cdot \nabla$.

The initial *direction* in which the coils will change under the descent algorithm is given by

$$\frac{\partial \mathbf{x}_i}{\partial \tau} = -\oint_{\mathcal{S}} \frac{\delta^2 F}{\delta \mathbf{x}_i \delta \bar{\mathbf{x}}} \cdot \delta \bar{\mathbf{x}}\, ds. \tag{13}$$

To determine the true change in the coil geometry, however, it is required to invert the Hessian matrix. The condition that $\delta F/\delta \mathbf{x}_i$ remains zero as the surface changes is

$$\delta \mathbf{c} = -\nabla^2_{\mathbf{cc}} F^{-1} \cdot \nabla^2_{\mathbf{cs}} F \cdot \delta \mathbf{s} = 0. \tag{14}$$

Similar expressions that describe how the coils change with changes in the length penalty, ω, can be derived.

These mixed second variations with respect to the coil and surface geometry, $\nabla^2_{\mathbf{cs}} F \sim \delta^2 F/\delta \mathbf{x}_i \delta \bar{\mathbf{x}}$ given in Eqn. 12, have been implemented in FOCUS.

The above equations, Eqn. 6 and Eqn. 12, have revealed the role played by the curvature of the coils and the mean curvature of the surface, $\boldsymbol{\kappa}$ and $\mathbf{H}$.

IV. VARYING BOUNDARY TO SIMPLIFY COILS

Turning now to the topic of combined plasma-coil design, we note that most properties of the plasma depend on the magnetohydrodynamic (MHD) equilibrium. The equilibrium depends on the geometry of the boundary, $\mathcal{S}$, the normal field on the boundary, and two "profile" functions usually taken as the pressure and rotational-transform (or parallel current-density) as functions of the enclosed toroidal flux. For brevity, this paper will ignore the dependence on the profiles and assume that all properties of the plasma depend on the boundary.

The "optimized stellarator" design algorithm splits the problem of stellarator design into two steps: first, identify via iterations the boundary that yields the desired equilibrium; and second, determine the geometry of the coils that provide the required field. This two-step approach was used to design W7-X, which was successfully built [11] and is now operating [12] at the Max-Planck-Institut für Plasmaphysik in Griefswald, Germany. This approach was also used to design NCSX [13], which was built (but not assembled) at Princeton Plasma Physics Laboratory, USA.

Because of the ill-posed nature of coil design, a small change in the plasma boundary may require an unfortunately large change in the coil geometry; a possibly trivial improvement in the plasma performance may result in an incommensurate increase in the construction cost. Simple, easy-to-build and inexpensive coil sets are preferable whenever possible, and engineering properties should be more intimately brought into the fold of the design process. We can imagine algorithms that simultaneously optimize the plasma performance and simplify the coil geometry. We present one example of such an optimization algorithm.

We introduce a measure of the coil "complexity", $\mathcal{C}(\mathbf{x}_i)$. Preferably, this should reflect the financial cost of building a given set of coils to the required tolerances. For example, we may consider the total integrated torsion of the coils,

$$\mathcal{C}(\mathbf{x}_i) = \sum_i \oint_i \frac{\mathbf{x}_i' \cdot \mathbf{x}_i'' \times \mathbf{x}_i'''}{|\mathbf{x}_i' \times \mathbf{x}_i''|^2} dl, \tag{15}$$

which measures how "non-planar" the coils are. W7-X has proved-by-construction that it is possible to construct stellarators with non-planar coils, but it is fair to say that planar coils are simpler to build than non-planar coils (furthermore, convex coils, which can be wound under tension, are easier to build than non-convex coils). The following mathematical description is valid for any differentiable coil complexity function, e.g. the strength of the inter-coil electromagnetic forces.

Let $\mathcal{P}(\bar{\mathbf{x}})$ represent the "properties" of the plasma that are important. The following mathematical description is valid for any property that is a differentiable function of the plasma boundary. We seek to minimize $\mathcal{C}$ subject to the constraint that $\mathcal{P} = \mathcal{P}_0$, and so we seek extrema of

$$G(\bar{\mathbf{x}}) \equiv C(\mathbf{x}_i(\bar{\mathbf{x}})) + \lambda\,[\mathcal{P}(\bar{\mathbf{x}}) - \mathcal{P}_0]\,, \tag{16}$$

where λ is a Lagrange multiplier. Solutions satisfy

$$\frac{\partial \mathbf{x}_i}{\partial \bar{\mathbf{x}}} \cdot \frac{\partial C}{\partial \mathbf{x}_i} + \lambda \frac{\partial \mathcal{P}}{\partial \bar{\mathbf{x}}} = 0. \tag{17}$$

For an illustration, we examine the rotational-transform on the magnetic axis. It has long been known [14, 15] that rotational-transform in vacuum fields can be produced either by the “rotating ellipticitiness” of the boundary, or by the integrated torsion of the magnetic axis, or both. We construct a two-parameter family of vacuum fields parameterized by torsion and ellipticity as follows. The Fundamental Theory of Curves shows that one-dimensional curves, $\mathbf{x}(\zeta)$, embedded in three-dimensional space are characterized by their torsion, $\tau(\zeta) \equiv \mathbf{x}' \cdot \mathbf{x}'' \times \mathbf{x}'''/|\mathbf{x}' \times \mathbf{x}''|^2$, and curvature, $\kappa(\zeta) \equiv |\mathbf{x}' \times \mathbf{x}''|/|\mathbf{x}'|^3$. We construct a closed curve, $\mathbf{x}_a(\zeta) \equiv \sum_m \mathbf{x}_m \exp(im\zeta)$, that will serve as a proxy magnetic axis with prescribed integrated torsion and minimum integrated curvature squared by seeking extrema of

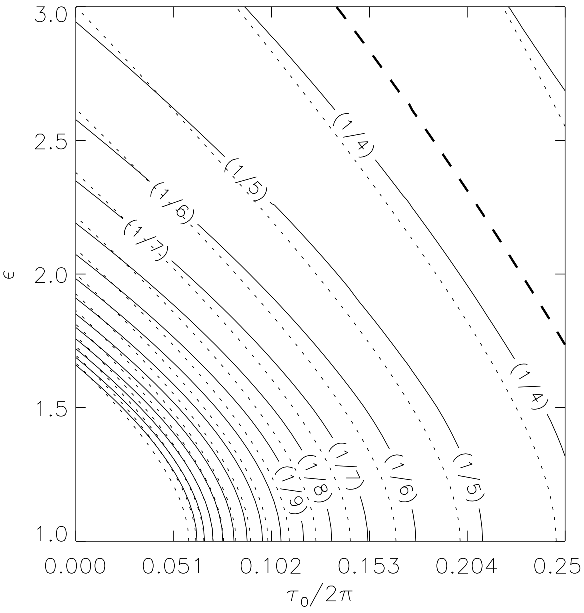

FIG. 1: Contours of low-order rational rotational transform as a function of integrated torsion and ellipticity: solid lines are from the numerical calculation; dotted lines are from Eqn. 20. The thick dashed line has $\iota = 0.276$.

$$\mathcal{F} \equiv \int \kappa^2 \, d\zeta + \mu \left(\int \tau \, d\zeta - \tau_0 \right), \tag{18}$$

where μ is a Lagrange multiplier. Additional constraints are included to constrain the parameterization so that $|\mathbf{x}'(\zeta)| = 1$, from which it follows that the length of the curve is 2π, and to constrain the curve with respect to rigid shifts and rotations. [Should we mention that this is the two-period, figure-eight branch? David: do you have a paper that we can refer to on this issue?]

A two dimensional surface, $\bar{\mathbf{x}}(\theta, \zeta)$, defined by a rotating ellipse in the plane perpendicular to $\mathbf{x}_a'$ is

$$\bar{\mathbf{x}} = \mathbf{x}_a(\zeta) + \rho\,(\,\epsilon^{1/2}\cos\theta\,\mathbf{v}_1 + \epsilon^{-1/2}\sin\theta\,\mathbf{v}_2\,), \tag{19}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the rotations of the normal and binormal, $\mathbf{v}_1 = \cos\alpha\,\mathbf{n} + \sin\alpha\,\mathbf{b}$ and $\mathbf{v}_2 = -\sin\alpha\,\mathbf{n} + \cos\alpha\,\mathbf{b}$, and $\alpha'(\zeta) = N/2 - \tilde{\tau}$ where $\tilde{\tau} \equiv \tau - \bar{\tau}$ is the oscillating part of the torsion and $\bar{\tau} \equiv \tau_0/2\pi$ is the average. Choosing $N = -1$, so that the ellipse makes a half rotation every 2π in ζ and that this increases rather than decreases the rotational-transform, and $\rho = 0.2$, the family of surfaces is parameterized by τ_0 and ϵ.

For each target surface we construct a set of coils using FOCUS. We compute the rotational-transform of the *true* magnetic axis, which is located by fieldline following methods, and this is shown in Fig. 1. In this figure are shown the contours for the low-order rationals, which are associated with the formation of magnetic islands and are best avoided. Shown as the thicker dashed line is the contour for the strongly irrational $\iota = (1 + \gamma)/(3 + 4\gamma) \approx 0.276$, where $\gamma = (1 + \sqrt{5})/2$ is the golden mean.

In the small aspect ratio limit, the rotational-transform is given by

$$\iota \;=\; \frac{(\epsilon - 1)^2}{\epsilon^2 + 1}\frac{N}{2} + \frac{2\epsilon}{\epsilon^2 + 1}\bar{\tau}, \tag{20}$$

and contours of ι are shown as the dotted lines in Fig. 1. [Lee: please check this formula once more.] There is a small discrepancy because the true magnetic axis will not exactly coincide with the proxy magnetic axis for non-zero ρ, and we have confirmed that the discrepancy descreases as ρ decreases.

There is ample opportunity to vary the shape of the boundary at fixed transform on axis, and so we may investigate whether this freedom can be exploited to simplify the coil complexity, and how shaping the boundary to produce transform shapes the coils. Shown in Fig. 2 is a configuration with $\tau_0/2\pi = 0.137$ and $\epsilon = 3.00$, and in Fig. 3 one with $\tau_0/2\pi = 0.255$ and $\epsilon = 1.73$. (These values coincide with where the dashed line in Fig. 1 intersects the plot domain.) Both have 18 coils, and for each the weight penalty is $\omega = 20$, the enclosed volume is equal to $0.799m^3$ and the rotational-transform on axis is 0.276. The average length and complexity of the coils is $\langle L \rangle = 3.07m$ and $\langle \mathcal{C} \rangle = 0.66m^{-1}$ for the first case, and $\langle L \rangle = 2.88m$ and $\langle \mathcal{C} \rangle = 0.12m^{-1}$ for the second. Poincaré plots (not shown) confirm that the coils produce the required magnetic field.

We finish this article with a comment regarding the $\partial\mathcal{P}/\partial\bar{\mathbf{x}}$ term in Eqn. 17. This should really be expressed

$$\frac{\partial \mathcal{P}}{\partial \bar{\mathbf{x}}} = \frac{\partial \mathbf{B}}{\partial \bar{\mathbf{x}}} \cdot \frac{\partial \mathcal{P}}{\partial \mathbf{B}}, \tag{21}$$

where the plasma property is assumed to be a differentiable function of the equilibrium magnetic field, $\mathbf{B}$, which is expressed as a function of the boundary, $\bar{\mathbf{x}}$. One may argue that *only* properties that are differentiable functions of the equilibrium should be be included, as extrema are *defined* by setting the dervative to zero.

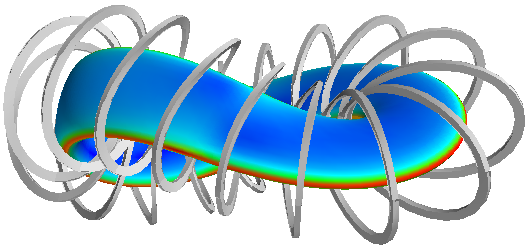

FIG. 2: Coils for case with $\tau_0 = 0.859$ and $\epsilon = 3.00$ in Eqn. 19. The color shows the mean curvature, from $|H| = 0.0$ (blue) to $|H| \approx 24.5$ (red).

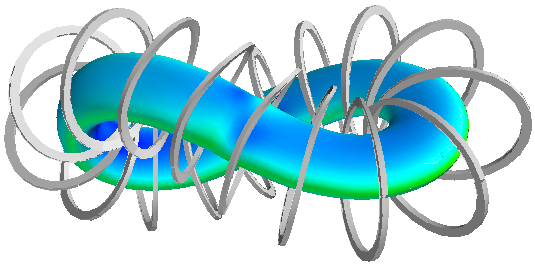

FIG. 3: Coils for case with $\tau_0 = 1.600$ and $\epsilon = 1.73$ in Eqn. 19. The color scale is the same as in Fig. 2, and for this case $|H|$ varies between 0.0 and 12.4.

We also require an MHD equilibrium model that yields solutions that are differentiable functions of the boundary; and ideal-MHD equilibria with rational rotational-transform surfaces are not. Rosenbluth *et al.* [16] described how discontinuities in the first-order ideal-MHD displacements near rational surfaces destroy analyticity. Stepped-pressure equilibria [17] are analytic functions of the boundary, as are stepped-transform [18] equilibria, as are mixed ideal-relaxed equilibria [19]. The required derivatives, namely $\partial\mathbf{B}/\partial\bar{\mathbf{x}}$, have already been implemented in the Stepped Pressure Equilibrium Code (SPEC) [20].

A final comment: given that we have free-boundary MHD equilibrium codes, we can perform free-boundary optimizations, for which the independent degrees of freedom in the optimization describe the geometry of the coils; and thereby penalties on the coil complexity can simultaneously be computed and optimized alongside measures of plasma performance. The analytical expressions presented herein can be used to enhance the numerical efficiency and accuracy of these algorithms.

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