Quadratic Flux-Minimizing Surfaces for Toroidal Magnetic Fields

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1 Magnetic Islands, Flux coordinates

It is well known that fully three-dimensional magnetic confinement geometries in general will have island chains and stochastic regions [1]. Such islands may impose a severe limit on β, the plasma pressure [2, 3], though in practice it is possible to control the size of a particular island, by varying the magnetic configuration in order to minimize a measure of the 'non-integrability' [4] . Recently, it has been observed that magnetic islands may enable 'self-healing' of the plasma surfaces [5, 6], if the phase of islands in the vacuum is opposite to the phase of islands induced by finite plasma pressure. Thus in order to improve plasma confinement, it may be required to manipulate islands in the vacuum field, either reduce the island width or to set islands to a particular phase and size.

In regions where the magnetic fields lines remain on nested surfaces magnetic, straight field line coordinates may be defined which enable the following representation

$$\mathbf{B} = \nabla\psi \times \nabla\theta + \iota(\psi)\nabla\phi \times \nabla\psi. \tag{1}$$

The simplicity of straight field line coordinates derives, in part, from utilizing what invariant motion exists, in this case the flux surfaces themselves, as a coordinate. This is similiar to the use of action-angle coordinates [7] in Hamiltonian mechanics. In fact, the problems are identical as toroidal magnetic field line flow is equivalent to a Hamiltonian system [8].

Unfortunately, magnetic coordinates are destroyed by island chains and are not defined in chaotic regions. To overcome this problem, whilst maintaining the usefulness of straight field line coordinates, we attempt to introduce generalized magnetic coordinates. We would like coordinates that are defined in all regions of phase space. In 'nice' regions with nested flux surfaces, we require the coordinates to reduce to straight field line coordinates.

2 Quadratic-flux minimizing surfaces

We follow [9, 10] and construct *quadratic-flux minimizing surfaces* to replace flux surfaces as a coordinate for the magnetic coordinate system. Quadratic-flux minimizing surfaces are defined by

$$\varphi_2 = \frac{1}{2}\int_\Gamma \frac{B_n^2}{C_n} d\sigma. \tag{2}$$

The Euler-Lagrange of this functional is $\mathbf{B}_* \cdot \nabla\nu = 0$; where we have introduced the pseudo field $\mathbf{B}_* \equiv \mathbf{B} - \nu\mathbf{C}$, $\mathbf{C} = \nabla\theta \times \nabla\zeta$ and the non-integrability parameter $\nu \equiv B_n/C_n$. As seen by [9], quadratic-flux minimizing surfaces are the natural generalization of flux surfaces. We may note that quadratic-flux minimizing surfaces reduce to flux surfaces ($B_n = 0$) if they exist, and are well defined in regions of chaos. As such, they may be used as a robust framework for a generalized coordinate system.

We construct the surfaces numerically by locating a periodic magnetic field line. Typically, there will exist two such closed field lines for each rational rotational transform, corresponding the the X and O points of the Poincaré cross section. The X point is unstable and difficult to follow using field line tracing, so we perform a two dimensional search to locate the stable periodic curve. To the real magnetic field $\mathbf{B}$, we add a small radial field $\nu\mathbf{C}$. The new (pseudo) periodic orbit is found nearby the real periodic orbit, but shifted across in the poloidal angle. We may continue until a periodic surface is constructed.

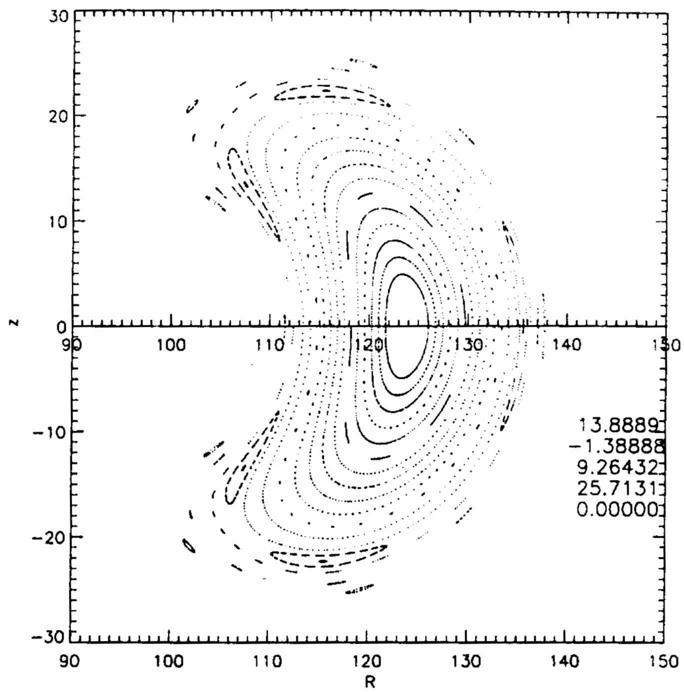

Figure 1: Poincaré plot of H-1 vacuum field, showing island chain with O point on outward symmetry line.

Alternatively (and more reliably) we may instead specify at which poloidal angle θ we require a periodic-pseudo orbit. We are then left with a two dimensional search, but now in the variables (ν, r), where r is some radial variable. If nested flux surfaces exist, then we are guaranteed of locating a real periodic orbit at every θ, and we may set $\nu = 0$ and search in one dimension to find r. In chaotic regions, we observe that ν displays sinuisoidal behaviour with θ.

A Poincaré section of H1 is shown in fig.(1), in which islands are clearly observed. A set of quadratic-flux minimizing surfaces for this field is shown in fig.(2). Each surface has been Fourier decomposed, with each of the coefficients being interpolated with cubic splines in the radial direction. This defines a continuous coordinate system, which we call generalized magnetic coordinates. Note that as yet a canonical angle has not been implemented, and as such the magnetic field lines, even in good regions, will not necessarily be straight. This is the topic of present research. A Poincaré section of H1 in these coordinates is shown in fig.(3). Where good flux surfaces exist, the new coordinates ensure the motion maintains a constant coordinate, indicated by constant curves on the Poincar'e section. Near the islands, the X point is expanded into an X line which separates individual islands within the island chain. To capture this feature adequately, several nearby quadratic-flux minimizing surfaces are required to be accurately resolved, thus many Fourier modes are needed, and care must be taken in the interpolation of the modes. The deviation of the curves shown in fig.(3) near the islands from being constant is a result of the truncation of the Fourier decomposition of the surfaces.

3 Manipulation of islands

The parameter ν is an indicator of the size of the island and provides a technique by which the islands may be manipulated. The typical behaviour of ν parameter on each surface is sinusoidal. Noting that the phase of the islands is locked due to the discrete stellarator symmetry (note that the Poincaré sections displayed are symmetric about $\theta = 0$), and that $\nu = 0$ on both the X and O points, we may define θ_{max} as that angle at which ν attains a maximum on a specified surface. With careful choice of the auxillary field $\mathbf{C}$ it is possible to identify the magnitude of the resonant perturbation H_{nm} to an integrable field,

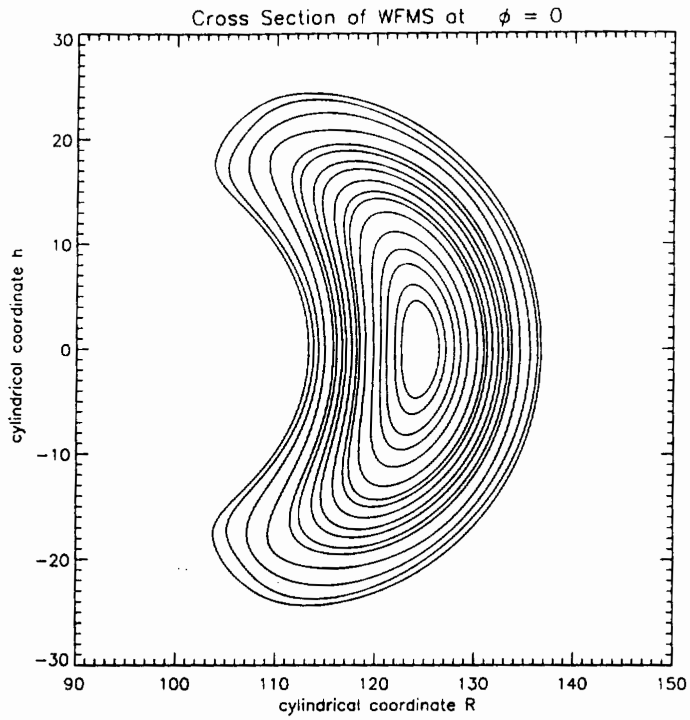

Figure 2: Cross section of quadratic-flux minimizing surfaces.

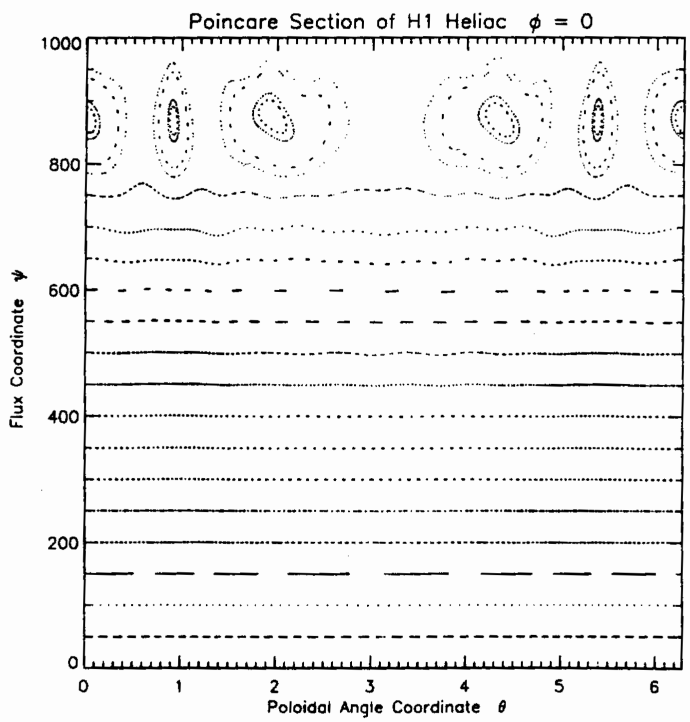

Figure 3: Poincaré section of H1 in quadratic-flux minimizing coordinates.

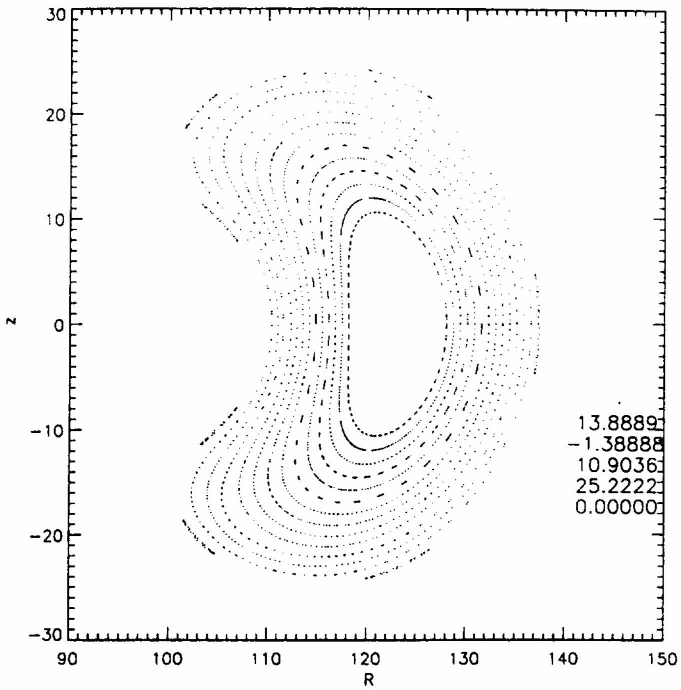

Figure 4: Poincaré section of H-1 vacuum field, showing island chain dissappearing.

to ν_{max}. This information along with the shear, ι, enables the island half width to be estimated using $\Delta = 2\sqrt{H_{nm}/\iota}$. Thus, by locating a single periodic pseudo orbit at θ_{max}, the value of ν, ν_{max} indicates both the size (through $|\nu_{max}|$) and phase (through $\mathrm{sgn}(\nu_{max})$ of the associated island chain. We may vary the coil currents (but not the coil geometry, as this would upset the symmetry) so as to achieve the required value of ν. By varying the upper and lower vertical field currents, using a standard minimization routine, a configuration was found that eliminates the island chain appearing in fig.(1) and this is shown in fig.(4). Also, a configuration was found in which the island chain reappears, but with opposite phase, and this is shown in fig.(5).

4 Conclusion

Quadratic-flux minimizing surfaces have successfully been implemented for the vacuum magnetic field of H1. A new concept, that of pseudo periodic orbits simplifies the construction of the rational surfaces and produces a measure of the size of the island chain. This parameter enables an efficient technique, by which islands in the vacuum field may be controlled, to be implemented.

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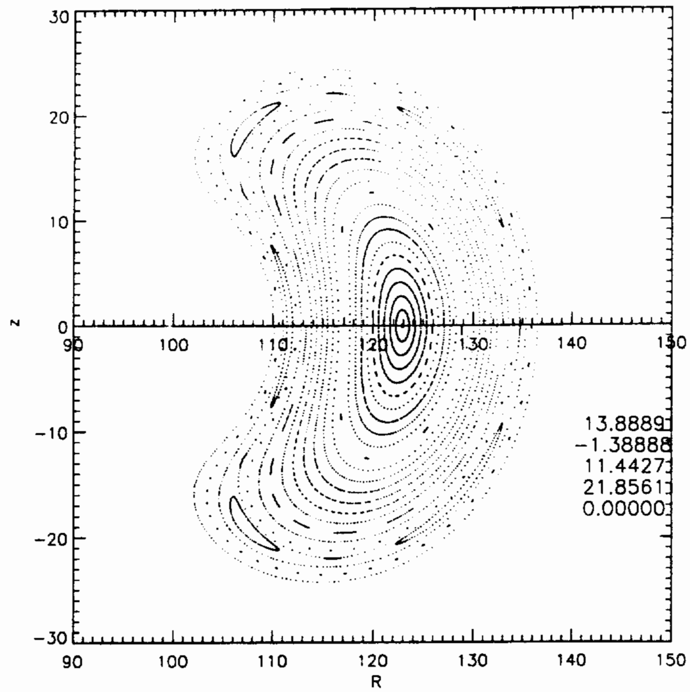

Figure 5: Poincaré section of H-1 vacuum field, showing X point on outward symmetry line.

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