1 stepped pressure equilibrium code : fu00aa

Contents

1.1 outline

1. Evaluates magnetic pressure and helicity volume integrals, and their derivatives with respect to interface geometry.

1.1.1 The magnetic pressure, $\int_{\mathcal{V}} B^2 dv$, and the helicity, $\int_{\mathcal{V}} \mathbf{A}\cdot\mathbf{B} dv$.

2. In the annular regions, the volume integrals are given by the following:

$$
\begin{aligned}
& \int_{s_{l-1}}^{s_l} ds \ \oint\oint d\theta d\zeta \ \sqrt{g}\ B^s B^s g_{ss} \\
= \ & \sum_{i=1}^{N_i} \sum_{j,k} \int_{s_{i-1}}^{s_i} ds \ (-m_j A_{\zeta,l,j} - n_j A_{\theta,l,j})(-m_k A_{\zeta,l,k} - n_k A_{\theta,l,k}) \left\langle s_j \left| \sqrt{g}^{-1} g_{ss} \right| s_k \right\rangle \\
= \ & m_j \ A_{\zeta,l,j,p,i-1+l} \ m_k \ A_{\zeta,l,k,q,i-1+r} \int_{s_{i-1}}^{s_i} ds \ \varphi_{l,p}\varphi_{r,q} \left\langle s_j \left| \sqrt{g}^{-1} g_{ss} \right| s_k \right\rangle \\
+ \ & m_j \ A_{\zeta,l,j,p,i-1+l} \ n_k \ A_{\theta,l,k,q,i-1+r} \int_{s_{i-1}}^{s_i} ds \ \varphi_{l,p}\varphi_{r,q} \left\langle s_j \left| \sqrt{g}^{-1} g_{ss} \right| s_k \right\rangle \\
+ \ & n_j \ A_{\theta,l,j,p,i-1+l} \ m_k \ A_{\zeta,l,k,q,i-1+r} \int_{s_{i-1}}^{s_i} ds \ \varphi_{l,p}\varphi_{r,q} \left\langle s_j \left| \sqrt{g}^{-1} g_{ss} \right| s_k \right\rangle \\
+ \ & n_j \ A_{\theta,l,j,p,i-1+l} \ n_k \ A_{\theta,l,k,q,i-1+r} \int_{s_{i-1}}^{s_i} ds \ \varphi_{l,p}\varphi_{r,q} \left\langle s_j \left| \sqrt{g}^{-1} g_{ss} \right| s_k \right\rangle
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
& \int_{s_{l-1}}^{s_l} ds \ \oint\oint d\theta d\zeta \ \sqrt{g}\ B^s B^\theta g_{s\theta} \\
= \ & \sum_{i=1}^{N_i} \sum_{j,k} \int_{s_{i-1}}^{s_i} ds \ \ (-m_j A_{\zeta,l,j} - n_j A_{\theta,l,j})(-A'_{\zeta,l,k}) \left\langle s_j \left| \sqrt{g}^{-1} g_{s\theta} \right| c_k \right\rangle \\
= \ & m_j A_{\zeta,l,j,p,i-1+l} A_{\zeta,l,k,q,i-1+r} \int_{s_{i-1}}^{s_i} ds \ \varphi_{l,p}\varphi'_{r,q} \left\langle s_j \left| \sqrt{g}^{-1} g_{s\theta} \right| c_k \right\rangle \\
+ \ & n_j A_{\theta,l,j,p,i-1+l} A_{\zeta,l,k,q,i-1+r} \int_{s_{i-1}}^{s_i} ds \ \varphi_{l,p}\varphi'_{r,q} \left\langle s_j \left| \sqrt{g}^{-1} g_{s\theta} \right| c_k \right\rangle
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
& \int_{s_{l-1}}^{s_l} ds \ \oint\oint d\theta d\zeta \ \sqrt{g}\ B^s B^\zeta g_{s\zeta} \\
= \ & \sum_{i=1}^{N_i} \sum_{j,k} \int_{s_{i-1}}^{s_i} ds \ \ (-m_j A_{\zeta,l,j} - n_j A_{\theta,l,j})(A'_{\theta,l,k}) \left\langle s_j \left| \sqrt{g}^{-1} g_{s\zeta} \right| c_k \right\rangle \\
= \ & -m_j A_{\zeta,l,j,p,i-1+l} A_{\theta,l,k,q,i-1+r} \int_{s_{i-1}}^{s_i} ds \ \varphi_{l,p}\varphi'_{r,q} \left\langle s_j \left| \sqrt{g}^{-1} g_{s\zeta} \right| c_k \right\rangle \\
+ \ & -n_j A_{\theta,l,j,p,i-1+l} A_{\theta,l,k,q,i-1+r} \int_{s_{i-1}}^{s_i} ds \ \varphi_{l,p}\varphi'_{r,q} \left\langle s_j \left| \sqrt{g}^{-1} g_{s\zeta} \right| c_k \right\rangle
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&\int_{s_{l-1}}^{s_l} ds \ \oint\oint d\theta d\zeta \ \sqrt{g}\ B^\theta B^\theta g_{\theta\theta} \\
&= \sum_{i=1}^{N_i}\sum_{j,k}\int_{s_{i-1}}^{s_i} ds\ (-A'_{\zeta,l,j})(-A'_{\zeta,l,k})\left\langle c_j \left|\sqrt{g}^{-1} g_{\theta\theta}\right| c_k\right\rangle \\
&= A_{\zeta,l,j,p,i-1+l}A_{\zeta,l,k,q,i-1+r}\int_{s_{i-1}}^{s_i} ds\ \varphi'_{l,p}\varphi'_{r,q}\left\langle c_j \left|\sqrt{g}^{-1} g_{\theta\theta}\right| c_k\right\rangle
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
&\int_{s_{l-1}}^{s_l} ds \ \oint\oint d\theta d\zeta \ \sqrt{g}\ B^\theta B^\zeta g_{\theta\zeta} \\
&= \sum_{i=1}^{N_i}\sum_{j,k}\int_{s_{i-1}}^{s_i} ds\ \ (-A'_{\zeta,l,j})(A'_{\theta,l,k})\left\langle c_j \left|\sqrt{g}^{-1} g_{\theta\zeta}\right| c_k\right\rangle \\
&= -A_{\zeta,l,j,p,i-1+l}A_{\theta,l,k,q,i-1+r}\int_{s_{i-1}}^{s_i} ds\ \varphi'_{l,p}\varphi'_{r,q}\left\langle c_j \left|\sqrt{g}^{-1} g_{\theta\zeta}\right| c_k\right\rangle
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
&\int_{s_{l-1}}^{s_l} ds \ \oint\oint d\theta d\zeta \ \sqrt{g}\ B^\zeta B^\zeta g_{\zeta\zeta} \\
&= \sum_{i=1}^{N_i}\sum_{j,k}\int_{s_{i-1}}^{s_i} ds\ (A'_{\theta,l,j})(A'_{\theta,l,k})\left\langle c_j \left|\sqrt{g}^{-1} g_{\zeta\zeta}\right| c_k\right\rangle \\
&= A_{\theta,l,j,p,i-1+l}A_{\theta,l,k,q,i-1+r}\int_{s_{i-1}}^{s_i} ds\ \varphi'_{l,p}\varphi'_{r,q}\left\langle c_j \left|\sqrt{g}^{-1} g_{\zeta\zeta}\right| c_k\right\rangle
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&\int_{s_{l-1}}^{s_l} ds \ \oint\oint d\theta d\zeta \ \sqrt{g}\ \mathbf{A}\cdot\mathbf{B} \\
&= \sum_{i=1}^{N_i}\int_{s_{i-1}}^{s_i} ds\ \oint\oint d\theta d\zeta\ \ [-A_\theta \partial_s A_\zeta + A_\zeta \partial_s A_\theta] \\
&= \sum_{i=1}^{N_i}\sum_{j,k}\int_{s_{i-1}}^{s_i} ds\ \left[-A_{\theta,l,j}A'_{\zeta,l,k} + A_{\zeta,l,j}A'_{\theta,l,k}\right]\langle c_j\,|1|\,c_k\rangle \\
&= -A_{\theta,l,j,p,i-1+l}A_{\zeta,l,k,q,i-1+r}\int_{s_{i-1}}^{s_i} ds\ \varphi_{l,p}\varphi'_{r,q}\,\langle c_j\,|1|\,c_k\rangle \\
&\quad +A_{\zeta,l,j,p,i-1+l}A_{\theta,l,k,q,i-1+r}\int_{s_{i-1}}^{s_i} ds\ \varphi_{l,p}\varphi'_{r,q}\,\langle c_j\,|1|\,c_k\rangle
\end{aligned} \tag{7}
$$

3. In the innermost volume, there are additional radial factor $s^{m/2}$, but these are incorporated into the `igss`, `igst`, `igsz`, `igtt`, `igtz`, `igzz`, and `igoo`.

4. The above integrals are returned in

   `lBBintegral(lvol)=`$\int_{\mathcal{V}} dv\ \mathbf{B}\cdot\mathbf{B}$

   `lABintegral(lvol)=`$\int_{\mathcal{V}} dv\ \mathbf{A}\cdot\mathbf{B}$

1.1.2 gauge invariant helicity

5. A gauge invariant form for the helicity is derived as follows. The change in helicity, δH, where the helicity is defined $H \equiv \int \mathbf{A}\cdot\mathbf{B}\ dv$, resulting from a change in the gauge, e.g. $\mathbf{A}' = \mathbf{A} + \nabla g$, is given by

$$
\delta H = \int_{\mathcal{V}} \nabla g\cdot\mathbf{B}\ dv = \int_{\mathcal{V}} \nabla\cdot(g\mathbf{B})\ dv = \int_{\partial\mathcal{V}} g\ \mathbf{B}\cdot d\mathbf{s}. \tag{8}
$$

6. The gauge function, g, is not required to be a single valued function of postion, i.e. $g(s,\theta+2\pi,\zeta) \neq g(s,\theta,\zeta)$ and $g(s,\theta,\zeta+2\pi) \neq g(s,\theta,\zeta)$. In contrast, the magnetic field, and Jacobian, are periodic.

7. Using $\mathbf{B}\cdot\mathbf{n}=0$ on the ideal interfaces that bound the volume, $\mathcal{V}$, the expression for the variation in helicity can be reduced to

$$\delta H = \iint [g(s,\theta,2\pi)-g(s,\theta,0)]\,\mathbf{B}\cdot d\mathbf{s} + \iint [g(s,2\pi,\zeta)-g(s,0,\zeta)]\,\mathbf{B}\cdot d\mathbf{s}, \tag{9}$$

$$\tag{10}$$

where the first integral is over the region of the plane $\zeta=0$ bounded by the adjacent interfaces, and the second integral over the region of the $\theta=0$ plane.

8. HOW DO THE TERMS INVOLVING g GET PULLED OUT OF THE INTEGRALS?
Is $g(s,\theta,2\pi)-g(s,\theta,0)=const.$ and $g(s,2\pi,\zeta)-g(s,0,\zeta)=const.$?

9. If the vector potential is required to be single valued, the representation of the gauge function is restricted to be of the form

$$g = G_\theta\,\theta + G_\zeta\,\zeta + \tilde{g}(s,\theta,\zeta) \tag{11}$$

where G_θ and G_ζ are constants and $\tilde{g}$ is periodic. In this case, $g(s,\theta,2\pi)-g(s,\theta,0)=2\pi G_\zeta$ and $g(s,2\pi,\zeta)-g(s,0,\zeta)=2\pi G_\theta$.

10. By identifying the toroidal and poloidal fluxes enclosed between the interfaces, $\delta\psi \equiv \iint \mathbf{B}\cdot d\mathbf{s}$, we obtain

$$\delta H = 2\pi G_\zeta\,\delta\psi_t + 2\pi G_\theta\,\delta\psi_p. \tag{12}$$

11. Some logic missing . . .

12. A gauge invariant form for the helicity is

$$H = \int \mathbf{A}\cdot\mathbf{B}\,dv - \delta\psi_p \int_0^{2\pi} A_\theta(s_{l-1},\theta,\zeta)d\theta - \delta\psi_t \int_0^{2\pi} A_\zeta(s_l,\theta,\zeta)d\zeta, \tag{13}$$

where s_{l-1} coincides with the inner interface, and s_l coincides with the outer interface.

13. Note that the gauge and boundary conditions employed give $A_\theta(s_{l-1},\theta,\zeta)=\psi_{t,l-1}$ and $A_\zeta(s_l,\theta,\zeta)=\psi_{p,l}$. After using $\delta\psi_t \equiv \psi_{t,l}-\psi_{t,l-1}$ and $\delta\psi_p \equiv \psi_{p,l}-\psi_{p,l-1}$, the final expression for the helicity is

$$H = \int \mathbf{A}\cdot\mathbf{B}\,dv + \psi_{t,l}\psi_{p,l} - \psi_{t,l-1}\psi_{p,l-1}. \tag{14}$$

fu00aa.h last modified on 2012-12-18 ;