1 stepped pressure equilibrium code : ganbat

Contents

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1.1 outline

1. Revised construction of vector potential.

1.1.1 coordinates

1. We shall work in toroidal coordinates, (s, θ, ζ), which will be defined inversely via a transformation to Cartesian, (x, y, z), or cylindrical coordinates, (R, ϕ, Z). Note that s is *not* a global variable: it only needs to be defined in each volume.

2. The geometry of the interfaces, $\mathbf{x}_l(\theta, \zeta)$, where θ and ζ are arbitrary poloidal and toroidal angles, is given by

   - `Lgeometry=1` : Cartesian : $\mathbf{x} \equiv \theta\, \hat{\mathbf{i}} + \zeta\, \hat{\mathbf{j}} + R\, \hat{\mathbf{k}}$;
   - `Lgeometry=2` : Cylindrical : $\mathbf{x} \equiv R\, \hat{\mathbf{r}} + \zeta\, \hat{\mathbf{k}}$;
   - `Lgeometry=3` : Toroidal : $\mathbf{x} \equiv R\, \hat{\mathbf{R}} + Z\, \hat{\mathbf{k}}$;

   where $\hat{\mathbf{r}} \equiv \cos\theta\, \hat{\mathbf{i}} + \sin\theta\, \hat{\mathbf{j}}$, and $\hat{\mathbf{R}} \equiv \cos\phi\, \hat{\mathbf{i}} + \sin\phi\, \hat{\mathbf{j}}$.

3. The l-th volume is bounded by the $\mathbf{x}_{l-1}$ and $\mathbf{x}_l$. Toroidal coordinates are constructed by linear interpolation

$$\mathbf{x}(s, \theta, \zeta) \equiv [\ (1 - s)\, \mathbf{x}_{l-1} + (1 + s)\, \mathbf{x}_l\ ]\ /\ 2. \tag{1}$$

4. Note that $s \in [-1, 1]$ in each volume.

1.1.2 gauge and boundary conditions

1. In the l-th annulus, bounded by the $(l-1)$-th and l-th interfaces, a general covariant representation of the magnetic vector-potential is written

$$\bar{\mathbf{A}} = \bar{A}_s \nabla s + \bar{A}_\theta \nabla\theta + \bar{A}_\zeta \nabla\zeta. \tag{2}$$

2. To this add $\nabla g(s,\theta,\zeta)$, where g satisfies

$$\begin{array}{rcccl} \partial_s g(s,\theta,\zeta) & = & - & \bar{A}_s(s,\theta,\zeta) & \\ \partial_\theta g(-1,\theta,\zeta) & = & - & \bar{A}_\theta(-1,\theta,\zeta) & + \quad \psi_{t,l-1} \\ \partial_\zeta g(-1,0,\zeta) & = & - & \bar{A}_\zeta(-1,0,\zeta) & + \quad \psi_{p,l-1} \end{array} \tag{3}$$

for arbitrary constants $\psi_{t,l-1}$, $\psi_{p,l-1}$, which can be identified as the toroidal and poloidal fluxes by using

$$\int_S \mathbf{B}\cdot\mathbf{ds} = \int_{\partial S} \mathbf{A}\cdot\mathbf{dl}. \tag{4}$$

3. Then $\mathbf{A} = \bar{\mathbf{A}} + \nabla g$ is given by $\mathbf{A} = A_\theta \nabla\theta + A_\zeta \nabla\zeta$ with

$$\begin{array}{rcl} A_\theta(-1,\theta,\zeta) & = & \psi_{t,l-1}, \\ A_\zeta(-1,0,\zeta) & = & \psi_{p,l-1}. \end{array} \tag{5}$$

4. This specifies the gauge: to see this, notice that no gauge term can be added without violating the conditions in Eq.(5).

5. Note that the gauge employed in each volume is distinct.

6. The magnetic field is $\sqrt{g}\mathbf{B} = (\partial_\theta A_\zeta - \partial_\zeta A_\theta)\,\mathbf{e}_s - \partial_s A_\zeta\,\mathbf{e}_\theta + \partial_s A_\theta\,\mathbf{e}_\zeta$.

7. In the annular volumes, the condition that the field is tangential to the inner interface gives $\partial_\theta A_\zeta - \partial_\zeta A_\theta = 0$. With the above gauge condition on A_θ given in Eq.(5), this entails that $\partial_\theta A_\zeta = 0$, which with Eq.(5) entails that $A_\zeta(-1,\theta,\zeta) = \psi_{p,l-1}$.

8. In the annular volumes, the condition that the field is tangential to the outer interface cannot be so simply enforced (all of the gauge freedom was exploited to simplify A_θ and A_ζ at the inner interface). At the outer interface we must constrain the vector potential to be of the form

$$A_\theta(1,\theta,\zeta) = \partial_\theta f(\theta,\zeta), \tag{6}$$

$$A_\zeta(1,\theta,\zeta) = \partial_\zeta f(\theta,\zeta), \tag{7}$$

for arbitrary f of the form

$$f = \psi_{t,l}\,\theta + \psi_{p,l}\,\zeta + \tilde{f}(\theta,\zeta), \tag{8}$$

where $\tilde{f}$ is periodic.

1.1.3 mixed Fourier, Chebyshev representation

1. Doubly periodic functions are represented as

$$f(\theta,\zeta) = \sum_{n=0}^{N} f_{0,n}\cos(-n\zeta) + \sum_{m=1}^{M}\sum_{n=-N}^{N} f_{m,n}\cos(m\theta - n\zeta) \equiv \sum_i f_i \cos\alpha_i, \tag{9}$$

   where $\alpha_i \equiv m_i\theta - n_i\zeta$.

2. The covariant components of the vector potential are written as the following sum,

$$A_\theta = \sum_i\sum_{l=0}^{L} A_{\theta,e,i,l}\, T_l(s)\cos\alpha_i + \sum_i\sum_{l=0}^{L} A_{\theta,o,i,l}\, T_l(s)\sin\alpha_i \tag{10}$$

$$A_\zeta = \sum_i\sum_{l=0}^{L} A_{\zeta,e,i,l}\, T_l(s)\cos\alpha_i + \sum_i\sum_{l=0}^{L} A_{\zeta,o,i,l}\, T_l(s)\sin\alpha_i, \tag{11}$$

   where $T_l(s)$ are the Chebyshev polynomials, and the stellarator symmetric terms are shown in red.

3. The Chebyshev polynomials may be defined recursively:

$$T_0(s) = 1, \tag{12}$$

$$T_1(s) = s, \tag{13}$$

$$T_{l+1}(s) = 2s\, T_l(s) - T_{l-1}(s). \tag{14}$$

4. Note that $T_l(-1) = +1$ if l is even, and $T_l(-1) = -1$ if l is odd, and that $T_l(+1) = +1$.

5. The magnetic field is, with summation over i and l hereafter implied,

$$\begin{aligned}
\sqrt{g}B^s &= + (m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\; T_l \cos\alpha_i - (m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\; T_l \sin\alpha_i \\
\sqrt{g}B^\theta &= - A_{\zeta,e,i,l}\; T_l' \cos\alpha_i - A_{\zeta,o,i,l}\; T_l' \sin\alpha_i \\
\sqrt{g}B^\zeta &= + A_{\theta,e,i,l}\; T_l' \cos\alpha_i + A_{\theta,o,i,l}\; T_l' \sin\alpha_i
\end{aligned} \tag{15}$$

1.1.4 interface boundary conditions: annular volumes

1. In terms of the Fourier harmonics we may write

$$-m_j A_{\zeta,j}(-1) - n_j A_{\theta,j}(-1) = 0. \tag{16}$$

2. Combing the gauge constraints and the flux surface condition we have

$$A_{\theta,j}(-1) = \begin{cases} \psi_{t,l-1} & , \ j = 1, \\ 0 & , \ j > 1, \end{cases} \quad \text{and} \quad A_{\zeta,j}(-1) = \begin{cases} \psi_{p,l-1} & , \ j = 1, \\ 0 & , \ j > 1, \end{cases} \tag{17}$$

3. In terms of the Fourier representation, we have

$$A_{\theta,j}(1) = \begin{cases} \psi_{t,l} & , \ j = 1, \\ m_j f_{l,j} & , \ j > 1, \end{cases} \quad \text{and} \quad A_{\zeta,j}(1) = \begin{cases} \psi_{p,l} & , \ j = 1, \\ -n_j f_{l,j} & , \ j > 1, \end{cases} \tag{18}$$

4. The boundary conditions on the inner, $s = -1$, and outer, $s = +1$, interfaces may be enforced using the Chebyshev representation, $A_{\theta,e,i}(s) \equiv f(s) = \sum f_l T_l(s)$, as follows:

$$\begin{aligned} f(-1) &= \sum_{l=0}^{L-2} f_l \ (-1)^l + f_{L-1} \ (-1)^{L-1} + f_L \ (-1)^L \\ f(+1) &= \sum_{l=0}^{L-2} f_l + f_{L-1} + f_L \end{aligned} \tag{19}$$

5. We consider the f_{L-1} and f_L to depend on the f_0, f_1, ..., f_{L-2}, as follows:

$$f_{L-1} = \begin{cases} [f(+1) - f(-1)]/2 - f_1 - f_3 \ldots - f_{L-3}, & L \text{ even} \\ [f(+1) + f(-1)]/2 - f_0 - f_2 \ldots - f_{L-3}, & L \text{ odd} \end{cases} \tag{20}$$

$$f_L = \begin{cases} [f(+1) + f(-1)]/2 - f_0 - f_2 \ldots - f_{L-2}, & L \text{ even} \\ [f(+1) - f(-1)]/2 - f_1 - f_3 \ldots - f_{L-2}, & L \text{ odd} \end{cases} \tag{21}$$

6. For L even, we have

$$\frac{\partial}{\partial f_l} \equiv \frac{\partial}{\partial f_l} - \frac{\partial}{\partial f_L}, \quad l \text{ even} \tag{22}$$

$$\frac{\partial}{\partial f_l} \equiv \frac{\partial}{\partial f_l} - \frac{\partial}{\partial f_{L-1}}, \quad l \text{ odd} \tag{23}$$

7. For L odd, we have

$$\frac{\partial}{\partial f_l} \equiv \frac{\partial}{\partial f_l} - \frac{\partial}{\partial f_{L-1}}, \quad l \text{ even} \tag{24}$$

$$\frac{\partial}{\partial f_l} \equiv \frac{\partial}{\partial f_l} - \frac{\partial}{\partial f_L}, \quad l \text{ odd} \tag{25}$$

1.1.5 integrands

1. The integrands are

$$
\begin{aligned}
\sqrt{g}\,\mathbf{B}\cdot\mathbf{B} \;=\;
&+ \quad (m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;(m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p})\;T_l\,T_p\,g_{ss}\,\cos\alpha_i\,\cos\alpha_j \\
&- 2\;(m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;(m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p})\;T_l\,T_p\,g_{ss}\,\cos\alpha_i\,\sin\alpha_j \\
&+ \quad (m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\;(m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p})\;T_l\,T_p\,g_{ss}\,\sin\alpha_i\,\sin\alpha_j \\
&- 2\;(m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;A_{\zeta,e,j,p}\;T_l\,T'_p\,g_{s\theta}\,\cos\alpha_i\,\cos\alpha_j \\
&- 2\;(m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;A_{\zeta,o,j,p}\;T_l\,T'_p\,g_{s\theta}\,\cos\alpha_i\,\sin\alpha_j \\
&+ 2\;(m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\;A_{\zeta,e,j,p}\;T_l\,T'_p\,g_{s\theta}\,\sin\alpha_i\,\cos\alpha_j \\
&+ 2\;(m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\;A_{\zeta,o,j,p}\;T_l\,T'_p\,g_{s\theta}\,\sin\alpha_i\,\sin\alpha_j \\
&+ 2\;(m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;A_{\theta,e,j,p}\;T_l\,T'_p\,g_{s\zeta}\,\cos\alpha_i\,\cos\alpha_j \\
&+ 2\;(m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;A_{\theta,o,j,p}\;T_l\,T'_p\,g_{s\zeta}\,\cos\alpha_i\,\sin\alpha_j \\
&- 2\;(m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\;A_{\theta,e,j,p}\;T_l\,T'_p\,g_{s\zeta}\,\sin\alpha_i\,\cos\alpha_j \\
&- 2\;(m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\;A_{\theta,o,j,p}\;T_l\,T'_p\,g_{s\zeta}\,\sin\alpha_i\,\sin\alpha_j \\
&+ \quad A_{\zeta,e,i,l}\;A_{\zeta,e,j,p}\;T'_l\,T'_p\,g_{\theta\theta}\,\cos\alpha_i\,\cos\alpha_j \\
&+ 2\;A_{\zeta,e,i,l}\;A_{\zeta,o,j,p}\;T'_l\,T'_p\,g_{\theta\theta}\,\cos\alpha_i\,\sin\alpha_j \\
&+ \quad A_{\zeta,o,i,l}\;A_{\zeta,o,j,p}\;T'_l\,T'_p\,g_{\theta\theta}\,\sin\alpha_i\,\sin\alpha_j \\
&- 2\;A_{\zeta,e,i,l}\;A_{\theta,e,j,p}\;T'_l\,T'_p\,g_{\theta\zeta}\,\cos\alpha_i\,\cos\alpha_j \\
&- 2\;A_{\zeta,e,i,l}\;A_{\theta,o,j,p}\;T'_l\,T'_p\,g_{\theta\zeta}\,\cos\alpha_i\,\sin\alpha_j \\
&- 2\;A_{\zeta,o,i,l}\;A_{\theta,e,j,p}\;T'_l\,T'_p\,g_{\theta\zeta}\,\sin\alpha_i\,\cos\alpha_j \\
&- 2\;A_{\zeta,o,i,l}\;A_{\theta,o,j,p}\;T'_l\,T'_p\,g_{\theta\zeta}\,\sin\alpha_i\,\sin\alpha_j \\
&+ \quad A_{\theta,e,i,l}\;A_{\theta,e,j,p}\;T'_l\,T'_p\,g_{\zeta\zeta}\,\cos\alpha_i\,\cos\alpha_j \\
&+ 2\;A_{\theta,e,i,l}\;A_{\theta,o,j,p}\;T'_l\,T'_p\,g_{\zeta\zeta}\,\cos\alpha_i\,\sin\alpha_j \\
&+ \quad A_{\theta,o,i,l}\;A_{\theta,o,j,p}\;T'_l\,T'_p\,g_{\zeta\zeta}\,\sin\alpha_i\,\sin\alpha_j
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\sqrt{g}\,\mathbf{A}\cdot\mathbf{B} \;=\;
&- \quad A_{\zeta,e,i,l}\;A_{\theta,e,j,p}\;T'_l\,T_p\,\cos\alpha_i\,\cos\alpha_j \\
&- \quad A_{\zeta,e,i,l}\;A_{\theta,o,j,p}\;T'_l\,T_p\,\cos\alpha_i\,\sin\alpha_j \\
&- \quad A_{\zeta,o,i,l}\;A_{\theta,e,j,p}\;T'_l\,T_p\,\sin\alpha_i\,\cos\alpha_j \\
&- \quad A_{\zeta,o,i,l}\;A_{\theta,o,j,p}\;T'_l\,T_p\,\sin\alpha_i\,\sin\alpha_j \\
&+ \quad A_{\theta,e,i,l}\;A_{\zeta,e,j,p}\;T'_l\,T_p\,\cos\alpha_i\,\cos\alpha_j \\
&+ \quad A_{\theta,e,i,l}\;A_{\zeta,o,j,p}\;T'_l\,T_p\,\cos\alpha_i\,\sin\alpha_j \\
&+ \quad A_{\theta,o,i,l}\;A_{\zeta,e,j,p}\;T'_l\,T_p\,\sin\alpha_i\,\cos\alpha_j \\
&+ \quad A_{\theta,o,i,l}\;A_{\zeta,o,j,p}\;T'_l\,T_p\,\sin\alpha_i\,\sin\alpha_j
\end{aligned}
\tag{27}
$$

1.1.6 first derivatives with respect to $A_{\theta,e,i,l}$ and $A_{\theta,o,i,l}$

1. The first derivatives with respect to $A_{\theta,e,i,l}$ and $A_{\theta,o,i,l}$ are

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& - \; 2 \;\; n_i \;\; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \;\; \int ds \;\; T_p\, T_l \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{ss} \;\; \cos\alpha_j \;\; \sin\alpha_i \\
& + \; 2 \;\; n_i \;\; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \;\; \int ds \;\; T_l\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{ss} \;\; \sin\alpha_i \;\; \sin\alpha_j \\
& + \; 2 \;\; n_i \;\; A_{\zeta,e,j,p} \;\; \int ds \;\; T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\theta} \;\; \sin\alpha_i \;\; \cos\alpha_j \\
& + \; 2 \;\; n_i \;\; A_{\zeta,o,j,p} \;\; \int ds \;\; T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\theta} \;\; \sin\alpha_i \;\; \sin\alpha_j \\
& + \; 2 \;\; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \;\; \int ds \;\; T_p\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \cos\alpha_j \;\; \cos\alpha_i \\
& - \; 2 \;\; n_i \;\; A_{\theta,e,j,p} \;\; \int ds \;\; T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \sin\alpha_i \;\; \cos\alpha_j \\
& - \; 2 \;\; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \;\; \int ds \;\; T_p\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \sin\alpha_j \;\; \cos\alpha_i \\
& - \; 2 \;\; n_i \;\; A_{\theta,o,j,p} \;\; \int ds \;\; T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \sin\alpha_i \;\; \sin\alpha_j \\
& - \; 2 \;\; A_{\zeta,e,j,p} \;\; \int ds \;\; T_p'\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\theta\zeta} \;\; \cos\alpha_j \;\; \cos\alpha_i \\
& - \; 2 \;\; A_{\zeta,o,j,p} \;\; \int ds \;\; T_p'\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\theta\zeta} \;\; \sin\alpha_j \;\; \cos\alpha_i \\
& + \; 2 \;\; A_{\theta,e,j,p} \;\; \int ds \;\; T_l'\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\zeta\zeta} \;\; \cos\alpha_i \;\; \cos\alpha_j \\
& + \; 2 \;\; A_{\theta,o,j,p} \;\; \int ds \;\; T_l'\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\zeta\zeta} \;\; \cos\alpha_i \;\; \cos\alpha_j
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& - \; A_{\zeta,e,j,p} \;\; \int ds \;\; T_p'\, T_l \;\; \oint\!\!\oint d\theta d\zeta \;\; \cos\alpha_j \;\; \cos\alpha_i \\
& - \; A_{\zeta,o,j,p} \;\; \int ds \;\; T_p'\, T_l \;\; \oint\!\!\oint d\theta d\zeta \;\; \sin\alpha_j \;\; \cos\alpha_i \\
& + \; A_{\zeta,e,j,p} \;\; \int ds \;\; T_l'\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; \cos\alpha_i \;\; \cos\alpha_j \\
& + \; A_{\zeta,o,j,p} \;\; \int ds \;\; T_l'\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; \cos\alpha_i \;\; \sin\alpha_j
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& + \; 2 \;\; n_i \;\; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \;\; \int ds \;\; T_l\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{ss} \;\; \cos\alpha_i \;\; \cos\alpha_j \\
& - \; 2 \;\; n_i \;\; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \;\; \int ds \;\; T_l\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{ss} \;\; \cos\alpha_i \;\; \sin\alpha_j \\
& - \; 2 \;\; n_i \;\; A_{\zeta,e,j,p} \;\; \int ds \;\; T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\theta} \;\; \cos\alpha_i \;\; \cos\alpha_j \\
& - \; 2 \;\; n_i \;\; A_{\zeta,o,j,p} \;\; \int ds \;\; T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\theta} \;\; \cos\alpha_i \;\; \sin\alpha_j \\
& + \; 2 \;\; n_i \;\; A_{\theta,e,j,p} \;\; \int ds \;\; T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \cos\alpha_i \;\; \cos\alpha_j \\
& + \; 2 \;\; n_i \;\; A_{\theta,o,j,p} \;\; \int ds \;\; T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \cos\alpha_i \;\; \sin\alpha_j \\
& + \; 2 \;\; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \;\; \int ds \;\; T_p\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \cos\alpha_j \;\; \sin\alpha_i \\
& - \; 2 \;\; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \;\; \int ds \;\; T_p\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \sin\alpha_j \;\; \sin\alpha_i \\
& - \; 2 \;\; A_{\zeta,e,j,p} \;\; \int ds \;\; T_p'\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\theta\zeta} \;\; \cos\alpha_j \;\; \sin\alpha_i \\
& - \; 2 \;\; A_{\zeta,o,j,p} \;\; \int ds \;\; T_p'\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\theta\zeta} \;\; \sin\alpha_j \;\; \sin\alpha_i \\
& + \; 2 \;\; A_{\theta,e,j,p} \;\; \int ds \;\; T_p'\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\zeta\zeta} \;\; \cos\alpha_j \;\; \sin\alpha_i \\
& + \; 2 \;\; A_{\theta,o,j,p} \;\; \int ds \;\; T_l'\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\zeta\zeta} \;\; \sin\alpha_i \;\; \sin\alpha_j
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
? \quad & - \; A_{\zeta,e,j,p} \;\; \int ds \;\; T_p'\, T_l \;\; \oint\!\!\oint d\theta d\zeta \;\; \cos\alpha_i \;\; \sin\alpha_i \\
? \quad & - \; A_{\zeta,o,j,p} \;\; \int ds \;\; T_p'\, T_l \;\; \oint\!\!\oint d\theta d\zeta \;\; \sin\alpha_j \;\; \sin\alpha_i \\
? \quad & + \; A_{\zeta,e,j,p} \;\; \int ds \;\; T_l'\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; \sin\alpha_i \;\; \cos\alpha_j \\
? \quad & + \; A_{\zeta,o,j,p} \;\; \int ds \;\; T_l'\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; \sin\alpha_i \;\; \sin\alpha_j
\end{aligned}
\tag{31}
$$

1.1.7 first derivatives with respect to $A_{\zeta,e,i,l}$ and $A_{\zeta,o,i,l}$

1. The first derivatives with respect to $A_{\zeta,e,i,l}$ and $A_{\zeta,o,i,l}$ are

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& - \ 2 \ m_i \ (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \ \int ds \ T_p\, T_l \ \oint\!\!\oint d\theta d\zeta \ g_{ss} \ \cos\alpha_j \ \sin\alpha_i \\
& + \ 2 \ m_i \ (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \ \int ds \ T_l\, T_p \ \oint\!\!\oint d\theta d\zeta \ g_{ss} \ \sin\alpha_i \ \sin\alpha_j \\
& - \ 2 \ (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \ \int ds \ T_p\, T_l' \ \oint\!\!\oint d\theta d\zeta \ g_{s\theta} \ \cos\alpha_j \ \cos\alpha_i \\
& + \ 2 \ m_i \ A_{\zeta,e,j,p} \ \int ds \ T_l\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{s\theta} \ \sin\alpha_i \ \cos\alpha_j \\
& + \ 2 \ (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \ \int ds \ T_p\, T_l' \ \oint\!\!\oint d\theta d\zeta \ g_{s\theta} \ \sin\alpha_j \ \cos\alpha_i \\
& + \ 2 \ m_i \ A_{\zeta,o,j,p} \ \int ds \ T_l\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{s\theta} \ \sin\alpha_i \ \sin\alpha_j \\
& - \ 2 \ m_i \ A_{\theta,e,j,p} \ \int ds \ T_l\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{s\zeta} \ \sin\alpha_i \ \cos\alpha_j \\
& - \ 2 \ m_i \ A_{\theta,o,j,p} \ \int ds \ T_l\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{s\zeta} \ \sin\alpha_i \ \sin\alpha_j \\
& + \ 2 \ A_{\zeta,e,j,p} \ \int ds \ T_l'\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{\theta\theta} \ \cos\alpha_j \ \cos\alpha_j \\
& + \ 2 \ A_{\zeta,o,j,p} \ \int ds \ T_l'\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{\theta\theta} \ \cos\alpha_i \ \sin\alpha_j \\
& - \ 2 \ A_{\theta,e,j,p} \ \int ds \ T_l'\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{\theta\zeta} \ \cos\alpha_i \ \cos\alpha_j \\
& - \ 2 \ A_{\theta,o,j,p} \ \int ds \ T_l'\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{\theta\zeta} \ \cos\alpha_i \ \sin\alpha_j
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& - \ A_{\theta,e,j,p} \ \int ds \ T_l'\, T_p \ \oint\!\!\oint d\theta d\zeta \ \cos\alpha_i \ \cos\alpha_j \\
& - \ A_{\theta,o,j,p} \ \int ds \ T_l'\, T_p \ \oint\!\!\oint d\theta d\zeta \ \cos\alpha_i \ \sin\alpha_j \\
& + \ A_{\theta,e,j,p} \ \int ds \ T_p'\, T_l \ \oint\!\!\oint d\theta d\zeta \ \cos\alpha_j \ \cos\alpha_i \\
& + \ A_{\theta,o,j,p} \ \int ds \ T_p'\, T_l \ \oint\!\!\oint d\theta d\zeta \ \sin\alpha_j \ \cos\alpha_i
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& + \ 2 \ m_i \ (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \ \int ds \ T_l\, T_p \ \oint\!\!\oint d\theta d\zeta \ g_{ss} \ \cos\alpha_i \ \cos\alpha_j \\
& - \ 2 \ m_i \ (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \ \int ds \ T_l\, T_p \ \oint\!\!\oint d\theta d\zeta \ g_{ss} \ \cos\alpha_i \ \sin\alpha_j \\
& - \ 2 \ m_i \ A_{\zeta,e,j,p} \ \int ds \ T_l\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{s\theta} \ \cos\alpha_i \ \cos\alpha_j \\
& - \ 2 \ m_i \ A_{\zeta,o,j,p} \ \int ds \ T_l\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{s\theta} \ \cos\alpha_i \ \sin\alpha_j \\
& - \ 2 \ (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \ \int ds \ T_p\, T_l' \ \oint\!\!\oint d\theta d\zeta \ g_{s\theta} \ \cos\alpha_j \ \sin\alpha_i \\
& + \ 2 \ (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p} \ \int ds \ T_p\, T_l' \ \oint\!\!\oint d\theta d\zeta \ g_{s\theta} \ \sin\alpha_j \ \sin\alpha_i \\
& + \ 2 \ m_i \ A_{\theta,e,j,p} \ \int ds \ T_l\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{s\zeta} \ \cos\alpha_i \ \cos\alpha_j \\
& + \ 2 \ m_i \ A_{\theta,o,j,p} \ \int ds \ T_l\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{s\zeta} \ \cos\alpha_i \ \sin\alpha_j \\
& + \ 2 \ A_{\zeta,e,j,p} \ \int ds \ T_p'\, T_l' \ \oint\!\!\oint d\theta d\zeta \ g_{\theta\theta} \ \cos\alpha_j \ \sin\alpha_i \\
& + \ 2 \ A_{\zeta,o,j,p} \ \int ds \ T_l'\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{\theta\theta} \ \sin\alpha_i \ \sin\alpha_j \\
& - \ 2 \ A_{\theta,e,j,p} \ \int ds \ T_l'\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{\theta\zeta} \ \sin\alpha_i \ \cos\alpha_j \\
& - \ 2 \ A_{\theta,o,j,p} \ \int ds \ T_l'\, T_p' \ \oint\!\!\oint d\theta d\zeta \ g_{\theta\zeta} \ \sin\alpha_i \ \sin\alpha_j
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& - \ A_{\theta,e,j,p} \ \int ds \ T_l'\, T_p \ \oint\!\!\oint d\theta d\zeta \ \sin\alpha_i \ \cos\alpha_j \\
& - \ A_{\theta,o,j,p} \ \int ds \ T_l'\, T_p \ \oint\!\!\oint d\theta d\zeta \ \sin\alpha_i \ \sin\alpha_j \\
& + \ A_{\theta,e,j,p} \ \int ds \ T_p'\, T_l \ \oint\!\!\oint d\theta d\zeta \ \cos\alpha_j \ \sin\alpha_i \\
& + \ A_{\theta,o,j,p} \ \int ds \ T_p'\, T_l \ \oint\!\!\oint d\theta d\zeta \ \sin\alpha_j \ \sin\alpha_i
\end{aligned}
\tag{35}
$$

1.1.8 second derivatives

1. The second derivatives are

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad &= \\
&+ \; 2 \quad n_i \quad n_j \quad \int ds \quad T_l\, T_p \; \oint\!\!\oint d\theta d\zeta \; g_{ss} \; \sin\alpha_i \; \sin\alpha_j \\
&- \; 2 \quad n_i \quad\quad \int ds \quad T_l\, T_p' \; \oint\!\!\oint d\theta d\zeta \; g_{s\zeta} \; \sin\alpha_i \; \cos\alpha_j \\
&- \; 2 \quad\quad n_j \quad \int ds \quad T_p\, T_l' \; \oint\!\!\oint d\theta d\zeta \; g_{s\zeta} \; \sin\alpha_j \; \cos\alpha_i \\
&+ \; 2 \quad\quad\quad \int ds \quad T_l'\, T_p' \; \oint\!\!\oint d\theta d\zeta \; g_{\zeta\zeta} \; \cos\alpha_i \; \cos\alpha_j
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad &= \\
&- \; 2 \quad n_i \quad n_j \quad \int ds \quad T_p\, T_l \; \oint\!\!\oint d\theta d\zeta \; g_{ss} \; \cos\alpha_j \; \sin\alpha_i \\
&+ \; 2 \quad\quad n_j \quad \int ds \quad T_p\, T_l' \; \oint\!\!\oint d\theta d\zeta \; g_{s\zeta} \; \cos\alpha_j \; \cos\alpha_i \\
&- \; 2 \quad n_i \quad\quad \int ds \quad T_l\, T_p' \; \oint\!\!\oint d\theta d\zeta \; g_{s\zeta} \; \sin\alpha_i \; \sin\alpha_j \\
&+ \; 2 \quad\quad\quad \int ds \quad T_l'\, T_p' \; \oint\!\!\oint d\theta d\zeta \; g_{\zeta\zeta} \; \cos\alpha_i \; \cos\alpha_j
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad &= \\
&+ \; 2 \quad n_i \quad m_j \quad \int ds \quad T_l\, T_p \; \oint\!\!\oint d\theta d\zeta \; g_{ss} \; \sin\alpha_i \; \sin\alpha_j \\
&+ \; 2 \quad n_i \quad\quad \int ds \quad T_l\, T_p' \; \oint\!\!\oint d\theta d\zeta \; g_{s\theta} \; \sin\alpha_i \; \cos\alpha_j \\
&- \; 2 \quad\quad m_j \quad \int ds \quad T_p\, T_l' \; \oint\!\!\oint d\theta d\zeta \; g_{s\zeta} \; \sin\alpha_j \; \cos\alpha_i \\
&- \; 2 \quad\quad\quad \int ds \quad T_p'\, T_l' \; \oint\!\!\oint d\theta d\zeta \; g_{\theta\zeta} \; \cos\alpha_j \; \cos\alpha_i
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad &= \\
&- \; 2 \quad n_i \quad m_j \quad \int ds \quad T_p\, T_l \; \oint\!\!\oint d\theta d\zeta \; g_{ss} \; \cos\alpha_i \; \sin\alpha_i \\
&+ \; 2 \quad n_i \quad\quad \int ds \quad T_l\, T_p' \; \oint\!\!\oint d\theta d\zeta \; g_{s\theta} \; \sin\alpha_i \; \sin\alpha_j \\
&+ \; 2 \quad\quad m_j \quad \int ds \quad T_p\, T_l' \; \oint\!\!\oint d\theta d\zeta \; g_{s\zeta} \; \cos\alpha_j \; \cos\alpha_i \\
&- \; 2 \quad\quad\quad \int ds \quad T_p'\, T_l' \; \oint\!\!\oint d\theta d\zeta \; g_{\theta\zeta} \; \sin\alpha_j \; \cos\alpha_i
\end{aligned}
\tag{39}
$$

$$
\frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad 0
\tag{40}
$$

$$
\frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad 0
\tag{41}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad &= \\
&- \; \int ds \quad T_p'\, T_l \; \oint\!\!\oint d\theta d\zeta \; \cos\alpha_j \; \cos\alpha_i \\
&+ \; \int ds \quad T_l'\, T_p \; \oint\!\!\oint d\theta d\zeta \; \cos\alpha_i \; \cos\alpha_j
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad &= \\
&- \; \int ds \quad T_p'\, T_l \; \oint\!\!\oint d\theta d\zeta \; \sin\alpha_j \; \cos\alpha_i \\
&+ \; \int ds \quad T_l'\, T_p \; \oint\!\!\oint d\theta d\zeta \; \cos\alpha_i \; \sin\alpha_j
\end{aligned}
\tag{43}
$$

1.1.9 second derivatives

1. The second derivatives are

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& - \; 2 \quad n_i \quad n_j \quad \int ds \quad T_l\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{ss} \;\; \cos\alpha_i \;\; \sin\alpha_j \\
& + \; 2 \quad n_i \quad\quad \int ds \quad T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \cos\alpha_i \;\; \cos\alpha_j \\
& + \; 2 \quad\quad n_j \quad \int ds \quad T_p\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \sin\alpha_j \;\; \sin\alpha_i \\
& - \; 2 \quad\quad\quad \int ds \quad T_p'\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\zeta\zeta} \;\; \cos\alpha_j \;\; \sin\alpha_i
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& + \; 2 \quad n_i \quad n_j \quad \int ds \quad T_l\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{ss} \;\; \cos\alpha_i \;\; \cos\alpha_j \\
& + \; 2 \quad n_i \quad\quad \int ds \quad T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \cos\alpha_i \;\; \sin\alpha_j \\
& + \; 2 \quad\quad n_j \quad \int ds \quad T_p\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \cos\alpha_j \;\; \sin\alpha_i \\
& + \; 2 \quad\quad\quad \int ds \quad T_l'\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\zeta\zeta} \;\; \sin\alpha_i \;\; \sin\alpha_j
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& - \; 2 \quad n_i \quad m_j \quad \int ds \quad T_l\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{ss} \;\; \cos\alpha_i \;\; \sin\alpha_j \\
& - \; 2 \quad n_i \quad\quad \int ds \quad T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\theta} \;\; \cos\alpha_i \;\; \cos\alpha_j \\
& - \; 2 \quad\quad m_j \quad \int ds \quad T_p\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \sin\alpha_j \;\; \sin\alpha_i \\
& - \; 2 \quad\quad\quad \int ds \quad T_p'\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\theta\zeta} \;\; \cos\alpha_j \;\; \sin\alpha_i
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& + \; 2 \quad n_i \quad m_j \quad \int ds \quad T_l\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{ss} \;\; \cos\alpha_i \;\; \cos\alpha_j \\
& - \; 2 \quad n_i \quad\quad \int ds \quad T_l\, T_p' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\theta} \;\; \cos\alpha_i \;\; \sin\alpha_j \\
& + \; 2 \quad\quad m_j \quad \int ds \quad T_p\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{s\zeta} \;\; \cos\alpha_j \;\; \sin\alpha_i \\
& - \; 2 \quad\quad\quad \int ds \quad T_p'\, T_l' \;\; \oint\!\!\oint d\theta d\zeta \;\; g_{\theta\zeta} \;\; \sin\alpha_j \;\; \sin\alpha_i
\end{aligned}
\tag{47}
$$

$$
\frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad 0
\tag{48}
$$

$$
\frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad 0
\tag{49}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& ? \quad \int ds \quad T_p'\, T_l \;\; \oint\!\!\oint d\theta d\zeta \;\; \cos\alpha_i \;\; \sin\alpha_i \\
& ? \quad \int ds \quad T_l'\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; \sin\alpha_i \;\; \cos\alpha_j
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& ? \quad \int ds \quad T_p'\, T_l \;\; \oint\!\!\oint d\theta d\zeta \;\; \sin\alpha_j \;\; \cos\alpha_i \\
& ? \quad \int ds \quad T_l'\, T_p \;\; \oint\!\!\oint d\theta d\zeta \;\; \cos\alpha_i \;\; \sin\alpha_j
\end{aligned}
\tag{51}
$$

1.1.10 second derivatives

1. The second derivatives are

$$\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} = \begin{array}{lllllllll} ? & 2 & n_i & n_j & \int ds & T_l\,T_p & \oint\oint d\theta d\zeta & g_{ss} & \cos\alpha_i\ \sin\alpha_j \\ ? & 2 & n_i & & \int ds & T_l\,T_p' & \oint\oint d\theta d\zeta & g_{s\zeta} & \cos\alpha_i\ \cos\alpha_j \\ ? & 2 & & n_j & \int ds & T_p\,T_l' & \oint\oint d\theta d\zeta & g_{s\zeta} & \sin\alpha_j\ \sin\alpha_i \\ ? & 2 & & & \int ds & T_p'\,T_l' & \oint\oint d\theta d\zeta & g_{\zeta\zeta} & \cos\alpha_j\ \sin\alpha_i \end{array} \tag{52}$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} = \begin{array}{lllllllll} ? & 2 & n_i & n_j & \int ds & T_l\,T_p & \oint\oint d\theta d\zeta & g_{ss} & \cos\alpha_i\ \sin\alpha_j \\ ? & 2 & n_i & & \int ds & T_l\,T_p' & \oint\oint d\theta d\zeta & g_{s\zeta} & \cos\alpha_i\ \cos\alpha_j \\ ? & 2 & & n_j & \int ds & T_p\,T_l' & \oint\oint d\theta d\zeta & g_{s\zeta} & \sin\alpha_j\ \sin\alpha_i \\ ? & 2 & & & \int ds & T_p'\,T_l' & \oint\oint d\theta d\zeta & g_{\zeta\zeta} & \cos\alpha_j\ \sin\alpha_i \end{array} \tag{53}$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} = \begin{array}{lllllllll} ? & 2 & n_i & n_j & \int ds & T_l\,T_p & \oint\oint d\theta d\zeta & g_{ss} & \cos\alpha_i\ \sin\alpha_j \\ ? & 2 & n_i & & \int ds & T_l\,T_p' & \oint\oint d\theta d\zeta & g_{s\zeta} & \cos\alpha_i\ \cos\alpha_j \\ ? & 2 & & n_j & \int ds & T_p\,T_l' & \oint\oint d\theta d\zeta & g_{s\zeta} & \sin\alpha_j\ \sin\alpha_i \\ ? & 2 & & & \int ds & T_p'\,T_l' & \oint\oint d\theta d\zeta & g_{\zeta\zeta} & \cos\alpha_j\ \sin\alpha_i \end{array} \tag{54}$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} = \begin{array}{lllllllll} ? & 2 & n_i & n_j & \int ds & T_l\,T_p & \oint\oint d\theta d\zeta & g_{ss} & \cos\alpha_i\ \sin\alpha_j \\ ? & 2 & n_i & & \int ds & T_l\,T_p' & \oint\oint d\theta d\zeta & g_{s\zeta} & \cos\alpha_i\ \cos\alpha_j \\ ? & 2 & & n_j & \int ds & T_p\,T_l' & \oint\oint d\theta d\zeta & g_{s\zeta} & \sin\alpha_j\ \sin\alpha_i \\ ? & 2 & & & \int ds & T_p'\,T_l' & \oint\oint d\theta d\zeta & g_{\zeta\zeta} & \cos\alpha_j\ \sin\alpha_i \end{array} \tag{55}$$

$$\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} = \begin{array}{lllllll} ? & \int ds & T_p'\,T_l & \oint\oint d\theta d\zeta & \sin\alpha_j\ \cos\alpha_i \\ ? & \int ds & T_l'\,T_p & \oint\oint d\theta d\zeta & \cos\alpha_i\ \sin\alpha_j \end{array} \tag{56}$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} = \begin{array}{lllllll} ? & \int ds & T_p'\,T_l & \oint\oint d\theta d\zeta & \sin\alpha_j\ \cos\alpha_i \\ ? & \int ds & T_l'\,T_p & \oint\oint d\theta d\zeta & \cos\alpha_i\ \sin\alpha_j \end{array} \tag{57}$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} = 0 \tag{58}$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} = 0 \tag{59}$$

1.1.11 second derivatives

1. The second derivatives are

$$\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} = \begin{array}{lllllllll} ? & 2 & n_i & n_j & \int ds & T_l\,T_p & \oint\oint d\theta d\zeta & g_{ss} & \cos\alpha_i\ \sin\alpha_j \\ ? & 2 & n_i & & \int ds & T_l\,T_p' & \oint\oint d\theta d\zeta & g_{s\zeta} & \cos\alpha_i\ \cos\alpha_j \\ ? & 2 & & n_j & \int ds & T_p\,T_l' & \oint\oint d\theta d\zeta & g_{s\zeta} & \sin\alpha_j\ \sin\alpha_i \\ ? & 2 & & & \int ds & T_p'\,T_l' & \oint\oint d\theta d\zeta & g_{\zeta\zeta} & \cos\alpha_j\ \sin\alpha_i \end{array} \tag{60}$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} = \begin{array}{lllllllll} ? & 2 & n_i & n_j & \int ds & T_l\,T_p & \oint\oint d\theta d\zeta & g_{ss} & \cos\alpha_i\ \sin\alpha_j \\ ? & 2 & n_i & & \int ds & T_l\,T_p' & \oint\oint d\theta d\zeta & g_{s\zeta} & \cos\alpha_i\ \cos\alpha_j \\ ? & 2 & & n_j & \int ds & T_p\,T_l' & \oint\oint d\theta d\zeta & g_{s\zeta} & \sin\alpha_j\ \sin\alpha_i \\ ? & 2 & & & \int ds & T_p'\,T_l' & \oint\oint d\theta d\zeta & g_{\zeta\zeta} & \cos\alpha_j\ \sin\alpha_i \end{array} \tag{61}$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} = \begin{array}{lllllllll} ? & 2 & n_i & n_j & \int ds & T_l\,T_p & \oint\oint d\theta d\zeta & g_{ss} & \cos\alpha_i\ \sin\alpha_j \\ ? & 2 & n_i & & \int ds & T_l\,T_p' & \oint\oint d\theta d\zeta & g_{s\zeta} & \cos\alpha_i\ \cos\alpha_j \\ ? & 2 & & n_j & \int ds & T_p\,T_l' & \oint\oint d\theta d\zeta & g_{s\zeta} & \sin\alpha_j\ \sin\alpha_i \\ ? & 2 & & & \int ds & T_p'\,T_l' & \oint\oint d\theta d\zeta & g_{\zeta\zeta} & \cos\alpha_j\ \sin\alpha_i \end{array} \tag{62}$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} = \begin{array}{lllllllll} ? & 2 & n_i & n_j & \int ds & T_l\,T_p & \oint\oint d\theta d\zeta & g_{ss} & \cos\alpha_i\ \sin\alpha_j \\ ? & 2 & n_i & & \int ds & T_l\,T_p' & \oint\oint d\theta d\zeta & g_{s\zeta} & \cos\alpha_i\ \cos\alpha_j \\ ? & 2 & & n_j & \int ds & T_p\,T_l' & \oint\oint d\theta d\zeta & g_{s\zeta} & \sin\alpha_j\ \sin\alpha_i \\ ? & 2 & & & \int ds & T_p'\,T_l' & \oint\oint d\theta d\zeta & g_{\zeta\zeta} & \cos\alpha_j\ \sin\alpha_i \end{array} \tag{63}$$

$$\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} = \begin{array}{lllllll} ? & \int ds & T_p'\,T_l & \oint\oint d\theta d\zeta & \sin\alpha_j\ \cos\alpha_i \\ ? & \int ds & T_l'\,T_p & \oint\oint d\theta d\zeta & \cos\alpha_i\ \sin\alpha_j \end{array} \tag{64}$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} = \begin{array}{lllllll} ? & \int ds & T_p'\,T_l & \oint\oint d\theta d\zeta & \sin\alpha_j\ \cos\alpha_i \\ ? & \int ds & T_l'\,T_p & \oint\oint d\theta d\zeta & \cos\alpha_i\ \sin\alpha_j \end{array} \tag{65}$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} = 0 \tag{66}$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} = 0 \tag{67}$$

1.1.12 reduction using FFTs and double angle formulae

1. The metric elements may be represented as a Fourier series using fast Fourier transforms

$$\bar{g}_{\mu\nu}(s,\theta,\zeta) = \sum_k \bar{g}_{\mu\nu,e,k}(s)\ \cos(m_k\theta - n_k\zeta) + \sum_k \bar{g}_{\mu\nu,o,k}(s)\ \sin(m_k\theta - n_k\zeta) \tag{68}$$

2. The integrals over the angles can be computed using double angle formulae:

$$\oint\oint d\theta d\zeta\ \ g_{\mu\nu}\ \cos\alpha_i\ \cos\alpha_j = \frac{1}{2}\oint\oint d\theta d\zeta\ \ g_{\mu\nu}(\cos\alpha_{k^+_{ij}} + \cos\alpha_{k^-_{ij}}) = g_{\mu\nu,e,k^+_{ij}} + g_{\mu\nu,e,k^-_{ij}} \tag{69}$$

$$\oint\oint d\theta d\zeta\ \ g_{\mu\nu}\ \cos\alpha_i\ \sin\alpha_j = \frac{1}{2}\oint\oint d\theta d\zeta\ \ g_{\mu\nu}(\sin\alpha_{k^+_{ij}} + \sin\alpha_{k^-_{ij}}) = g_{\mu\nu,o,k^+_{ij}} + g_{\mu\nu,o,k^-_{ij}} \tag{70}$$

$$\oint\oint d\theta d\zeta\ \ g_{\mu\nu}\ \sin\alpha_i\ \sin\alpha_j = \frac{1}{2}\oint\oint d\theta d\zeta\ \ g_{\mu\nu}(\cos\alpha_{k^+_{ij}} + \cos\alpha_{k^-_{ij}}) = g_{\mu\nu,e,k^+_{ij}} + g_{\mu\nu,e,k^-_{ij}} \tag{71}$$

$$\oint\oint d\theta d\zeta\ \ \cos\alpha_i\ \cos\alpha_j = \frac{1}{2}\oint\oint d\theta d\zeta\ \ (\cos\alpha_{k^+_{ij}} + \cos\alpha_{k^-_{ij}}) = \tag{72}$$

$$\oint\oint d\theta d\zeta\ \ \cos\alpha_i\ \sin\alpha_j = \frac{1}{2}\oint\oint d\theta d\zeta\ \ (\sin\alpha_{k^+_{ij}} + \sin\alpha_{k^-_{ij}}) = \tag{73}$$

$$\oint\oint d\theta d\zeta\ \ \sin\alpha_i\ \sin\alpha_j = \frac{1}{2}\oint\oint d\theta d\zeta\ \ (\cos\alpha_{k^+_{ij}} + \cos\alpha_{k^-_{ij}}) = \tag{74}$$

where $(m_{k^+}, n_{k^+}) = (m_i + m_j, n_i + n_j)$ and $(m_{k^-}, n_{k^-}) = (m_i - m_j, n_i - n_j)$.

1.1.13 geometric quantities to be determined numerically

1. The following quantities must be computed numerically:

$$\texttt{iTTgsse(l,p,i,j,0:1)} = \int ds \;\; T_l \, T_p \;\; g_{ss,e,k_{ij}^{\pm}} \tag{75}$$

$$\texttt{iTTgsso(l,p,i,j,0:1)} = \int ds \;\; T_l \, T_p \;\; g_{ss,o,k_{ij}^{\pm}} \tag{76}$$

$$\texttt{iTTgste(l,p,i,j,0:1)} = \int ds \;\; T_l \, T_p' \;\; g_{s\theta,e,k_{ij}^{\pm}} \tag{77}$$

$$\texttt{iTTgsto(l,p,i,j,0:1)} = \int ds \;\; T_l \, T_p' \;\; g_{s\theta,o,k_{ij}^{\pm}} \tag{78}$$

$$\texttt{iTTgsze(l,p,i,j,0:1)} = \int ds \;\; T_l \, T_p' \;\; g_{s\zeta,e,k_{ij}^{\pm}} \tag{79}$$

$$\texttt{iTTgszo(l,p,i,j,0:1)} = \int ds \;\; T_l \, T_p' \;\; g_{s\zeta,o,k_{ij}^{\pm}} \tag{80}$$

$$\texttt{iTTgtte(l,p,i,j,0:1)} = \int ds \;\; T_l' \, T_p' \;\; g_{\theta\theta,e,k_{ij}^{\pm}} \tag{81}$$

$$\texttt{iTTgtto(l,p,i,j,0:1)} = \int ds \;\; T_l' \, T_p' \;\; g_{\theta\theta,o,k_{ij}^{\pm}} \tag{82}$$

$$\texttt{iTTgtze(l,p,i,j,0:1)} = \int ds \;\; T_l' \, T_p' \;\; g_{\theta\zeta,e,k_{ij}^{\pm}} \tag{83}$$

$$\texttt{iTTgtzo(l,p,i,j,0:1)} = \int ds \;\; T_l' \, T_p' \;\; g_{\theta\zeta,o,k_{ij}^{\pm}} \tag{84}$$

$$\texttt{iTTgzze(l,p,i,j,0:1)} = \int ds \;\; T_l' \, T_p' \;\; g_{\zeta\zeta,e,k_{ij}^{\pm}} \tag{85}$$

$$\texttt{iTTgzzo(l,p,i,j,0:1)} = \int ds \;\; T_l' \, T_p' \;\; g_{\zeta\zeta,o,k_{ij}^{\pm}} \tag{86}$$

2. Note that there are various symmetries that could/should be exploited.

1.1.14 constructing Beltrami fields

1. The energy, $W \equiv \int dv\ \mathbf{B}\cdot\mathbf{B}$, and helicity, $K \equiv \int dv\ \mathbf{A}\cdot\mathbf{B}$, functionals may be written

$$W = \frac{1}{2}\mathbf{a}^T \cdot A \cdot \mathbf{a} + B \cdot \mathbf{a} + C \tag{87}$$

$$K = \frac{1}{2}\mathbf{a}^T \cdot D \cdot \mathbf{a} + E \cdot \mathbf{a} + F, \tag{88}$$

   where the independent degrees of freedom in the vector potential are written as a vector, $\mathbf{a}$.

2. Note the following dependencies:

   - The matrix A depends only on the geometry.
   - The matrix B and the vector C depend on the geometry and the enclosed toroidal and poloidal fluxes.
   - The matrix D is constant.
   - The matrix E and the vector F depend on the enclosed toroidal and poloidal fluxes.

3. The MRXMHD approach is to seek minima of W subject to $K = K_0$. The following approaches may be used:

   (a) Sequential Quadratic Programming : see the NAG routine `E04UFF`.

   (b) Newton iterations : construct the constrained energy functional

$$F \equiv W - \mu(K - K_0)/2, \tag{89}$$

   and seek a solution that sets the first derivatives of F to zero,

$$\frac{\partial}{\partial \mathbf{a}} F = 0, \qquad \frac{\partial}{\partial \mu} F = 0 \tag{90}$$

   where the Lagrange multiplier, μ, is to be considered an independent degree of freedom. The efficiency of the Newton method is greatly enhanced by exploiting the second derivatives of F with respect to $\mathbf{a}$ and μ.

   (c) Reduction to linear system : in the case that μ is given, it is only required to find solutions that satisfy $\partial F/\partial \mathbf{a} = 0$, and in this case Eq.(90) reduces to a linear system,

$$(A - \mu D/2) \cdot \mathbf{a} = (B - \mu E/2). \tag{91}$$

4. It would seem, given the matrices A, B, ..., that each of these methods would be similarly efficient. The 'Linear' method would appear to be the fastest method, but this may be deceptive: generally, μ is *not* given but must be adjusted to enforce the helicity constraint. Furthermore, it is not always the case the μ parameterizes different solutions (it does not when there are bifurcations).

5. Frequently, it may be preferable to adjust μ and the enclosed poloidal flux, $\Delta\psi_p$, to enforce the rotational-transform constraint. In this case, the linear method is probably fastest (as it is not required to iterate on μ to satisfy the helicity constraint $K = K_0$, and then iterate on K_0 to satisfy the transform constraint, but instead it is possible to iterate on μ to satisfy the transform constraint directly.)

1.1.15 constructing perturbed Beltrami fields

1. To enable an efficient global optimization, the derivatives of the Beltrami fields with respect to interface geometry are required.

2. The construction of the perturbed fields using the 'Linear' method is achieved by solving

$$(A + D) \cdot \delta\mathbf{a} = (\delta B + \delta E) - (\delta A + \delta D) \cdot \mathbf{a} \tag{92}$$

ganbat.h last modified on 2012-12-14 ;