0.1 stepped pressure equilibrium code : ma01af

1. Constructs matrix that represents the Beltrami linear system.

0.1.1 gauge and coordinates

1. The helicity integral, $\int \mathbf{A}\cdot\mathbf{B}\,dv$, depends explicity on the vector potential. The gauge must be constrained and the boundary conditions enforced.

2. The coordinates, (s,θ,ζ), are adapted to the interfaces, so that the interfaces coincide with coordinate surfaces $s = const.$ and θ,ζ are poloidal and toroidal angles. Here the angles remaing arbitrary.

3. The treatment of the coordinate singularity (described below) depends on the radial coordinate.

   (a) If `Lradial.eq.1`, the radial coordinate is proportional to the polar radial coordinate, $s \approx r \propto \sqrt{\psi_t}$.

   (b) If `Lradial.eq.2`, the radial coordinate is proportional to the toroidal flux, $s \approx r^2 \propto \psi_t$.

4. In each volume, a regular, radial-sub-grid, $s_{l,i} \equiv$ `ss(l)%s(i)`, for $i = 0$, `Ni`, is established, see `al00aa`.

0.1.2 annular volume - gauge

1. In the l-th annulus, bounded by the $(l-1)$-th and l-th interfaces, a general covariant representation of the magnetic vector-potential is written

$$\bar{\mathbf{A}} = \bar{A}_s\nabla s + \bar{A}_\theta\nabla\theta + \bar{A}_\zeta\nabla\zeta. \tag{1}$$

   To this add $\nabla g(s,\theta,\zeta)$, where g satisfies

$$\partial_s g(s,\theta,\zeta) = -\bar{A}_s(s,\theta,\zeta),\;\; \partial_\theta g(s_{l-1},\theta,\zeta) = -\bar{A}_\theta(s_{l-1},\theta,\zeta) + \psi_{t,l-1},\;\; \partial_\zeta g(s_{l-1},0,\zeta) = -\bar{A}_\zeta(s_{l-1},0,\zeta) + \psi_{p,l-1}, \tag{2}$$

   for arbitrary constants $\psi_{t,l-1}$, $\psi_{p,l-1}$, which are the toroidal and poloidal-fluxes on the interior of surface $l-1$. Then $\mathbf{A} = \bar{\mathbf{A}} + \nabla g$ is given by $\mathbf{A} = A_\theta\nabla\theta + A_\zeta\nabla\zeta$ with

$$A_\theta(s_{l-1},\theta,\zeta) = \psi_{t,l-1},\;\; A_\zeta(s_{l-1},0,\zeta) = \psi_{p,l-1}. \tag{3}$$

   This specifies the gauge.

2. The magnetic field is $\sqrt{g}\mathbf{B} = (\partial_\theta A_\zeta - \partial_\zeta A_\theta)\mathbf{e}_s - \partial_s A_\zeta\mathbf{e}_\theta + \partial_s A_\theta\mathbf{e}_\zeta$.

$$B^2 = B^sB^sg_{ss} + 2B^sB^\theta g_{s\theta} + 2B^sB^\zeta g_{s\zeta} + B^\theta B^\theta g_{\theta\theta} + 2B^\theta B^\zeta g_{\theta\zeta} + B^\zeta B^\zeta g_{\zeta\zeta} \tag{4}$$

$$\sqrt{g}\mathbf{A}\cdot\mathbf{B} = -A_\theta\partial_s A_\zeta + A_\zeta\partial_s A_\theta. \tag{5}$$

3. For stellarator symmetric equilibria, A_θ and A_ζ may be represented by cosine series

$$A_\theta(s,\theta,\zeta) = \sum_j A_{\theta,j}(s)\cos(m_j\theta - n_j\zeta),\;\; A_\zeta(s,\theta,\zeta) = \sum_j A_{\zeta,j}(s)\cos(m_j\theta - n_j\zeta), \tag{6}$$

   where $A_{\theta,j}(s)$ and $A_{\zeta,j}(s)$ are represented using finite-elements, as described below.

0.1.3 interface boundary condition

1. The condition that the field is tangential to the inner interface is

$$-m_jA_{\zeta,j}(s_{l-1}) - n_jA_{\theta,j}(s_{l-1}) = 0. \tag{7}$$

   Combing the gauge constraints and the flux surface condition we have

$$A_{\theta,j}(s_{l-1}) = \begin{cases} \psi_{t,l-1} & , \quad j=1, \\ 0 & , \quad j>1, \end{cases} \quad \text{and} \quad A_{\zeta,j}(s_{l-1}) = \begin{cases} \psi_{p,l-1} & , \quad j=1, \\ 0 & , \quad j>1, \end{cases} \tag{8}$$

2. The condition that the field is tangential to the outer interface is similar to Eq.(7), but it cannot be simplified further and so at the outer interface we must constrain the vector potential to be of the form

$$A_\theta(s_l) = \partial_\theta f(\theta,\zeta)\,, \quad A_\zeta(s_l) = \partial_\zeta f(\theta,\zeta), \tag{9}$$

for arbitrary f of the form

$$f = \psi_{t,l}\theta + \psi_{p,l}\zeta + \sum_j f_{l,j}\sin(m_j\theta - n_j\zeta) \tag{10}$$

and $\psi_{t,l} \equiv A_{\theta,1}(s_l)$ and $\psi_{p,l} \equiv A_{\zeta,1}(s_l)$. We have

$$A_{\theta,j}(s_l) = \begin{cases} \psi_{t,l} & , \quad j=1, \\ m_j f_{l,j} & , \quad j>1, \end{cases} \quad \text{and} \quad A_{\zeta,j}(s_l) = \begin{cases} \psi_{p,l} & , \quad j=1, \\ -n_j f_{l,j} & , \quad j>1, \end{cases} \tag{11}$$

0.1.4 finite element radial basis functions

1. In the region $s \in [s_{i-1}, s_i]$, the vector potential is $\mathbf{A} = A_\theta \nabla\theta + A_\zeta \nabla\zeta$ where

$$A_\theta(s,\theta,\zeta) = \sum_{j,l,p} A_{\theta,j,i-1+l,p}\ \varphi_{l,p}(s)\ \cos(m_j\theta - n_j\zeta), \tag{12}$$

$$A_\zeta(s,\theta,\zeta) = \sum_{j,l,p} A_{\zeta,j,i-1+l,p}\ \varphi_{l,p}(s)\ \cos(m_j\theta - n_j\zeta), \tag{13}$$

where j labels the Fourier harmonic, and $l = 0, 1$ and p identify the radial basis functions, $\varphi_{l,p}(s)$.

2. Near the coordinate singularity (described in more detail below) this representation needs to be modified.

0.1.5 metric information

1. The geometric information required in the following is

$$\bar{g}_{ss} \equiv g_{ss}/\sqrt{g} = \sum_i \bar{g}_{ss,i}(s)\cos(m_i\theta - n_i\zeta), \tag{14}$$

$$\bar{g}_{s\theta} \equiv g_{s\theta}/\sqrt{g} = \sum_i \bar{g}_{s\theta,i}(s)\sin(m_i\theta - n_i\zeta), \tag{15}$$

$$\bar{g}_{s\zeta} \equiv g_{s\zeta}/\sqrt{g} = \sum_i \bar{g}_{s\zeta,i}(s)\sin(m_i\theta - n_i\zeta), \tag{16}$$

$$\bar{g}_{\theta\theta} \equiv g_{\theta\theta}/\sqrt{g} = \sum_i \bar{g}_{\theta\theta,i}(s)\cos(m_i\theta - n_i\zeta), \tag{17}$$

$$\bar{g}_{\theta\zeta} \equiv g_{\theta\zeta}/\sqrt{g} = \sum_i \bar{g}_{\theta\zeta,i}(s)\cos(m_i\theta - n_i\zeta), \tag{18}$$

$$\bar{g}_{\zeta\zeta} \equiv g_{\zeta\zeta}/\sqrt{g} = \sum_i \bar{g}_{\zeta\zeta,i}(s)\cos(m_i\theta - n_i\zeta). \tag{19}$$

2. For any function $f(s,\theta,\zeta)$, in particular the metrics, we write

$$\begin{aligned}
\langle s_j\,|f|\,s_k\rangle &= \oint\!\!\oint d\theta d\zeta\ \ \sin(m_j\theta - n_j\zeta)\, f(s,\theta,\zeta)\ \sin(m_k\theta - n_k\zeta) \\
\langle s_j\,|f|\,c_k\rangle &= \oint\!\!\oint d\theta d\zeta\ \ \sin(m_j\theta - n_j\zeta)\, f(s,\theta,\zeta)\ \cos(m_k\theta - n_k\zeta) \\
\langle c_j\,|f|\,c_k\rangle &= \oint\!\!\oint d\theta d\zeta\ \ \cos(m_j\theta - n_j\zeta)\, f(s,\theta,\zeta)\ \cos(m_k\theta - n_k\zeta)
\end{aligned}$$

Note that the second of these is ***not*** symmetric with respect to j and k.

0.1.6 volume integral

1. Consider the integral $P = \sum_{i=1}^{N_i} P_i$, where

$$
\begin{aligned}
P_i \quad = \quad & \\
& + \frac{1}{2} \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\theta A_\zeta \quad \partial_\theta A_\zeta \quad \bar{g}_{ss} \\
& - \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\theta A_\zeta \quad \partial_\zeta A_\theta \quad \bar{g}_{ss} \\
& + \frac{1}{2} \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\zeta A_\theta \quad \partial_\zeta A_\theta \quad \bar{g}_{ss} \\
& - \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\theta A_\zeta \quad \partial_s A_\zeta \quad \bar{g}_{s\theta} \\
& + \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\zeta A_\theta \quad \partial_s A_\zeta \quad \bar{g}_{s\theta} \\
& + \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\theta A_\zeta \quad \partial_s A_\theta \quad \bar{g}_{s\zeta} \\
& - \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\zeta A_\theta \quad \partial_s A_\theta \quad \bar{g}_{s\zeta} \\
& + \frac{1}{2} \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\zeta \quad \partial_s A_\zeta \quad \bar{g}_{\theta\theta} \\
& - \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\zeta \quad \partial_s A_\theta \quad \bar{g}_{\theta\zeta} \\
& + \frac{1}{2} \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\theta \quad \partial_s A_\theta \quad \bar{g}_{\zeta\zeta} \\
& + \frac{\mu}{2} \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad A_\theta \quad \partial_s A_\zeta \\
& - \frac{\mu}{2} \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad A_\zeta \quad \partial_s A_\theta \quad .
\end{aligned}
\tag{20}
$$

0.1.7 derivatives of volume integral – annular regions

1. The first derivatives are as follows:

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,j,i-1+l,p}} P_i \quad = \quad & \\
& - \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\theta A_\zeta \quad n_j \quad \varphi_{l,p} \quad s_j \quad \bar{g}_{ss} \\
& + \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\zeta A_\theta \quad n_j \quad \varphi_{l,p} \quad s_j \quad \bar{g}_{ss} \\
& + \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\zeta \quad n_j \quad \varphi_{l,p} \quad s_j \quad \bar{g}_{s\theta} \\
& + \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\theta A_\zeta \quad \varphi'_{l,p} \quad c_j \quad \bar{g}_{s\zeta} \\
& - \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\theta \quad n_j \quad \varphi_{l,p} \quad s_j \quad \bar{g}_{s\zeta} \\
& - \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\zeta A_\theta \quad \varphi'_{l,p} \quad c_j \quad \bar{g}_{s\zeta} \\
& - \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\zeta \quad \varphi'_{l,p} \quad c_j \quad \bar{g}_{\theta\zeta} \\
& + \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\theta \quad \varphi'_{l,p} \quad c_j \quad \bar{g}_{\zeta\zeta} \\
& + \frac{\mu}{2} \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\zeta \quad \varphi_{l,p} \quad c_j \\
& - \frac{\mu}{2} \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad A_\zeta \quad \varphi'_{l,p} \quad c_j
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,j,i-1+l,p}} P_i \quad = \quad & \\
- \quad & \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\theta A_\zeta \quad m_j \quad \varphi_{l,p} \quad s_j \quad \bar{g}_{ss} \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\zeta A_\theta \quad m_j \quad \varphi_{l,p} \quad s_j \quad \bar{g}_{ss} \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\zeta \quad m_j \quad \varphi_{l,p} \quad s_j \quad \bar{g}_{s\theta} \\
- \quad & \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\theta A_\zeta \quad \varphi'_{l,p} \quad c_j \quad \bar{g}_{s\theta} \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_\zeta A_\theta \quad \varphi'_{l,p} \quad c_j \quad \bar{g}_{s\theta} \\
- \quad & \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\theta \quad m_j \quad \varphi_{l,p} \quad s_j \quad \bar{g}_{s\zeta} \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\zeta \quad \varphi'_{l,p} \quad c_j \quad \bar{g}_{\theta\theta} \\
- \quad & \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\theta \quad \varphi'_{l,p} \quad c_j \quad \bar{g}_{\theta\zeta} \\
+ \quad \tfrac{\mu}{2} \quad & \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad A_\theta \quad \varphi'_{l,p} \quad c_j \\
- \quad \tfrac{\mu}{2} \quad & \int_{s_{i-1}}^{s_i} ds \oint\!\!\oint d\theta d\zeta \quad \partial_s A_\theta \quad \varphi_{l,p} \quad c_j
\end{aligned}
\tag{22}
$$

2. The second derivatives are required for construction of the Beltrami matrix, which are given

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,k,i-1+r,q}} \frac{\partial}{\partial A_{\theta,j,i-1+l,p}} P_i \quad = \quad & \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad n_k \quad n_j \quad \varphi_{r,q} \quad \varphi_{l,p} \quad \langle s_j | \bar{g}_{ss} | s_k \rangle \\
- \quad & \int_{s_{i-1}}^{s_i} ds \quad n_j \quad \varphi'_{r,q} \quad \varphi_{l,p} \quad \langle s_j | \bar{g}_{s\zeta} | c_k \rangle \\
- \quad & \int_{s_{i-1}}^{s_i} ds \quad n_k \quad \varphi_{r,q} \quad \varphi'_{l,p} \quad \langle c_j | \bar{g}_{s\zeta} | s_k \rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad \varphi'_{r,q} \quad \varphi'_{l,p} \quad \langle c_j | \bar{g}_{\zeta\zeta} | c_k \rangle
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,k,i-1+r,q}} \frac{\partial}{\partial A_{\theta,j,i-1+l,p}} P_i \quad = \quad & \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad m_k \quad n_j \quad \varphi_{r,q} \quad \varphi_{l,p} \quad \langle s_j | \bar{g}_{ss} | s_k \rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad n_j \quad \varphi'_{r,q} \quad \varphi_{l,p} \quad \langle s_j | \bar{g}_{s\theta} | c_k \rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad m_k \quad \varphi_{r,q} \quad \varphi'_{l,p} \quad \langle c_j | \bar{g}_{s\zeta} | s_k \rangle \\
- \quad & \int_{s_{i-1}}^{s_i} ds \quad \varphi'_{r,q} \quad \varphi'_{l,p} \quad \langle c_j | \bar{g}_{\theta\zeta} | c_k \rangle \\
+ \quad \tfrac{\mu}{2} \quad & \int_{s_{i-1}}^{s_i} ds \quad \varphi'_{r,q} \quad \varphi_{l,p} \quad \langle c_j | 1 | c_k \rangle \\
- \quad \tfrac{\mu}{2} \quad & \int_{s_{i-1}}^{s_i} ds \quad \varphi_{r,q} \quad \varphi'_{l,p} \quad \langle c_j | 1 | c_k \rangle
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,k,i-1+r,q}}\frac{\partial}{\partial A_{\zeta,j,i-1+l,p}}P_i \quad = \quad & \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad n_k \quad m_j \quad \varphi_{r,q} \quad \varphi_{l,p} \quad \langle s_j \,|\bar{g}_{ss}|\, s_k\rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad n_k \quad \quad \varphi_{r,q} \quad \varphi'_{l,p} \quad \langle c_j \,|\bar{g}_{s\theta}|\, s_k\rangle \\
- \quad & \int_{s_{i-1}}^{s_i} ds \quad \quad m_j \quad \varphi'_{r,q} \quad \varphi_{l,p} \quad \langle s_j \,|\bar{g}_{s\zeta}|\, c_k\rangle \\
- \quad & \int_{s_{i-1}}^{s_i} ds \quad \quad \quad \varphi'_{r,q} \quad \varphi'_{l,p} \quad \langle c_j \,|\bar{g}_{\theta\zeta}|\, c_k\rangle \\
+ \quad \tfrac{\mu}{2} & \int_{s_{i-1}}^{s_i} ds \quad \quad \quad \varphi_{r,q} \quad \varphi'_{l,p} \quad \langle c_j \,|1|\, c_k\rangle \\
- \quad \tfrac{\mu}{2} & \int_{s_{i-1}}^{s_i} ds \quad \quad \quad \varphi'_{r,q} \quad \varphi_{l,p} \quad \langle c_j \,|1|\, c_k\rangle
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,k,i-1+r,q}}\frac{\partial}{\partial A_{\zeta,j,i-1+l,p}}P_i \quad = \quad & \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad m_k \quad m_j \quad \varphi_{r,q} \quad \varphi_{l,p} \quad \langle s_j \,|\bar{g}_{ss}|\, s_k\rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad \quad m_j \quad \varphi'_{r,q} \quad \varphi_{l,p} \quad \langle s_j \,|\bar{g}_{s\theta}|\, c_k\rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad m_k \quad \quad \varphi_{r,q} \quad \varphi'_{l,p} \quad \langle c_j \,|\bar{g}_{s\theta}|\, s_k\rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad \quad \quad \varphi'_{r,q} \quad \varphi'_{l,p} \quad \langle c_j \,|\bar{g}_{\theta\theta}|\, c_k\rangle
\end{aligned}
\tag{26}
$$

0.1.8 matrix elements – annular regions

1. The required matrix elements take the form, where $\alpha = \theta, \zeta$ and $\beta = \theta, \zeta$,

$$
\begin{aligned}
\frac{\partial}{\partial A_{\beta,k,i-1,q}} \quad \frac{\partial}{\partial A_{\alpha,j,i,p}} \quad P \quad &= \quad \frac{\partial}{\partial A_{\beta,k,i-1,q}} \quad \frac{\partial}{\partial A_{\alpha,j,i,p}} \quad P_i \\
\frac{\partial}{\partial A_{\beta,k,i,q}} \quad \frac{\partial}{\partial A_{\alpha,j,i,p}} \quad P \quad &= \quad \frac{\partial}{\partial A_{\beta,k,i,q}} \quad \frac{\partial}{\partial A_{\alpha,j,i,p}} \quad P_i \quad + \quad \frac{\partial}{\partial A_{\beta,k,i,q}} \quad \frac{\partial}{\partial A_{\alpha,j,i,p}} \quad P_{i+1} \\
\frac{\partial}{\partial A_{\beta,k,i+1,q}} \quad \frac{\partial}{\partial A_{\alpha,j,i,p}} \quad P \quad &= \quad \quad \frac{\partial}{\partial A_{\beta,k,i+1,q}} \quad \frac{\partial}{\partial A_{\alpha,j,i,p}} \quad P_{i+1}
\end{aligned}
\tag{27}
$$

0.1.9 gauge and regularity conditions near coordinate origin

1. Near the polar coordinate origin, where the polar coordinates satisfy $x = r\cos\theta$, $y = r\sin\theta$, we may exploit the regularity constraints and gauge freedom [1] to write $\mathbf{A} = A_\theta \nabla\theta + A_\zeta \nabla\zeta$ where

$$
A_\theta = \sum_{j=1}\sum_{p=0} a_{j,p} r^{m_j+2+2p} \cos(m_j\theta - n_j\zeta), \tag{28}
$$

$$
A_\zeta = \sum_{j=1}\sum_{p=0} b_{j,p} r^{m_j+0+2p} \cos(m_j\theta - n_j\zeta), \tag{29}
$$

   where the $a_{j,p}$ and $b_{j,p}$ are degrees of freedom, except $b_{j,0} = 0$ if $m_j = 0$.

2. Note that the coordinate origin is *not* required to coincide with the magnetic axis (the location of which is as yet not known apriori),

3. Note that the location of the coordinate axis is determined by `ex00aa`.

4. The factors s^{m_j+2} and s^{m_j+0} can become very small and cause finite-precision errors unless treated carefully. Thus, the discretization Eq.(12) and Eq.(13) are modified (in the entire innermost volume) and we write

$$
A_\theta(s,\theta,\zeta) = \sum_{j,l,p} s^{m_j/2}\, A_{\theta,j,i-1+l,p}\; \varphi_{l,p}(s)\; \cos(m_j\theta - n_j\zeta), \tag{30}
$$

$$
A_\zeta(s,\theta,\zeta) = \sum_{j,l,p} s^{m_j/2}\, A_{\zeta,j,i-1+l,p}\; \varphi_{l,p}(s)\; \cos(m_j\theta - n_j\zeta), \tag{31}
$$

[1] This was done using Mathematica, and further details will be provided on demand.

with the coordinate origin boundary condition

$A_{\theta,j,0,0} = 0$ for all j;

$A_{\zeta,j,0,0} = 0$ for $m_j = 0$.

5. Note that we have restricted attention to the case $s \approx r^2 \propto \psi_t$, i.e. `Lradial.eq.2`.

6. The second derivatives required are then

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,k,i-1+r,q}} \frac{\partial}{\partial A_{\theta,j,i-1+l,p}} P_i \quad = \quad & \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad n_k \quad n_j \quad (s^{m_k}\varphi_{r,q}) \quad (s^{m_j}\varphi_{l,p}) \quad \langle s_j \,|\bar{g}_{ss}|\, s_k \rangle \\
- \quad & \int_{s_{i-1}}^{s_i} ds \quad\quad n_j \quad (s^{m_k}\varphi_{r,q})' \quad (s^{m_j}\varphi_{l,p}) \quad \langle s_j \,|\bar{g}_{s\zeta}|\, c_k \rangle \\
- \quad & \int_{s_{i-1}}^{s_i} ds \quad n_k \quad\quad (s^{m_k}\varphi_{r,q}) \quad (s^{m_j}\varphi_{l,p})' \quad \langle c_j \,|\bar{g}_{s\zeta}|\, s_k \rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad\quad (s^{m_k}\varphi_{r,q})' \quad (s^{m_j}\varphi_{l,p})' \quad \langle c_j \,|\bar{g}_{\zeta\zeta}|\, c_k \rangle
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,k,i-1+r,q}} \frac{\partial}{\partial A_{\theta,j,i-1+l,p}} P_i \quad = \quad & \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad m_k \quad n_j \quad (s^{m_k}\varphi_{r,q}) \quad (s^{m_j}\varphi_{l,p}) \quad \langle s_j \,|\bar{g}_{ss}|\, s_k \rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad\quad n_j \quad (s^{m_k}\varphi_{r,q})' \quad (s^{m_j}\varphi_{l,p}) \quad \langle s_j \,|\bar{g}_{s\theta}|\, c_k \rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad m_k \quad\quad (s^{m_k}\varphi_{r,q}) \quad (s^{m_j}\varphi_{l,p})' \quad \langle c_j \,|\bar{g}_{s\zeta}|\, s_k \rangle \\
- \quad & \int_{s_{i-1}}^{s_i} ds \quad\quad (s^{m_k}\varphi_{r,q})' \quad (s^{m_j}\varphi_{l,p})' \quad \langle c_j \,|\bar{g}_{\theta\zeta}|\, c_k \rangle \\
+ \quad & \tfrac{\mu}{2} \int_{s_{i-1}}^{s_i} ds \quad\quad (s^{m_k}\varphi_{r,q})' \quad (s^{m_j}\varphi_{l,p}) \quad \langle c_j \,|1|\, c_k \rangle \\
- \quad & \tfrac{\mu}{2} \int_{s_{i-1}}^{s_i} ds \quad\quad (s^{m_k}\varphi_{r,q}) \quad (s^{m_j}\varphi_{l,p})' \quad \langle c_j \,|1|\, c_k \rangle
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,k,i-1+r,q}} \frac{\partial}{\partial A_{\zeta,j,i-1+l,p}} P_i \quad = \quad & \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad n_k \quad m_j \quad (s^{m_k}\varphi_{r,q}) \quad (s^{m_j}\varphi_{l,p}) \quad \langle s_j \,|\bar{g}_{ss}|\, s_k \rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad n_k \quad\quad (s^{m_k}\varphi_{r,q}) \quad (s^{m_j}\varphi_{l,p})' \quad \langle c_j \,|\bar{g}_{s\theta}|\, s_k \rangle \\
- \quad & \int_{s_{i-1}}^{s_i} ds \quad\quad m_j \quad (s^{m_k}\varphi_{r,q})' \quad (s^{m_j}\varphi_{l,p}) \quad \langle s_j \,|\bar{g}_{s\zeta}|\, c_k \rangle \\
- \quad & \int_{s_{i-1}}^{s_i} ds \quad\quad (s^{m_k}\varphi_{r,q})' \quad (s^{m_j}\varphi_{l,p})' \quad \langle c_j \,|\bar{g}_{\theta\zeta}|\, c_k \rangle \\
+ \quad & \tfrac{\mu}{2} \int_{s_{i-1}}^{s_i} ds \quad\quad (s^{m_k}\varphi_{r,q}) \quad (s^{m_j}\varphi_{l,p})' \quad \langle c_j \,|1|\, c_k \rangle \\
- \quad & \tfrac{\mu}{2} \int_{s_{i-1}}^{s_i} ds \quad\quad (s^{m_k}\varphi_{r,q})' \quad (s^{m_j}\varphi_{l,p}) \quad \langle c_j \,|1|\, c_k \rangle
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,k,i-1+r,q}} \frac{\partial}{\partial A_{\zeta,j,i-1+l,p}} P_i \quad = \quad & \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad m_k \quad m_j \quad (s^{m_k}\varphi_{r,q}) \quad (s^{m_j}\varphi_{l,p}) \quad \langle s_j \,|\bar{g}_{ss}|\, s_k \rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad\quad m_j \quad (s^{m_k}\varphi_{r,q})' \quad (s^{m_j}\varphi_{l,p}) \quad \langle s_j \,|\bar{g}_{s\theta}|\, c_k \rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad m_k \quad\quad (s^{m_k}\varphi_{r,q}) \quad (s^{m_j}\varphi_{l,p})' \quad \langle c_j \,|\bar{g}_{s\theta}|\, s_k \rangle \\
+ \quad & \int_{s_{i-1}}^{s_i} ds \quad\quad (s^{m_k}\varphi_{r,q})' \quad (s^{m_j}\varphi_{l,p})' \quad \langle c_j \,|\bar{g}_{\theta\theta}|\, c_k \rangle
\end{aligned}
\tag{35}
$$

7. The outer interface boundary condition, Eq.(11), becomes for $j > 1$,

$$A_{\theta,j,N,0} = m_j f_j / \rho^{m_j} \ , \quad \text{and} \quad A_{\zeta,j,N,0} = -n_j f_j / \rho^{m_j} \tag{36}$$

ma01af.h last modified on 2011-09-27 ;