ma01ag

Constructs matrices that represent the Beltrami linear system.

[called by: dforce.]

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1.1 gauge conditions

1. In the v-th annulus, bounded by the $(v-1)$-th and v-th interfaces, a general covariant representation of the magnetic vector-potential is written

$$\bar{\mathbf{A}} = \bar{A}_s \nabla s + \bar{A}_\theta \nabla\theta + \bar{A}_\zeta \nabla\zeta. \tag{1}$$

2. To this add $\nabla g(s,\theta,\zeta)$, where g satisfies

$$\begin{aligned} \partial_s g(s,\theta,\zeta) &= - \bar{A}_s(s,\theta,\zeta) \\ \partial_\theta g(-1,\theta,\zeta) &= - \bar{A}_\theta(-1,\theta,\zeta) \\ \partial_\zeta g(-1,0,\zeta) &= - \bar{A}_\zeta(-1,0,\zeta). \end{aligned} \tag{2}$$

3. Then $\mathbf{A} = \bar{\mathbf{A}} + \nabla g$ is given by $\mathbf{A} = A_\theta \nabla\theta + A_\zeta \nabla\zeta$ with

$$A_\theta(-1,\theta,\zeta) = 0 \tag{3}$$

$$A_\zeta(-1,0,\zeta) = 0 \tag{4}$$

4. This specifies the gauge: to see this, notice that no gauge term can be added without violating the conditions in Eq.(3) or Eq.(4).

5. Note that the gauge employed in each volume is distinct.

1.2 boundary conditions

1. The magnetic field is $\sqrt{g}\,\mathbf{B} = (\partial_\theta A_\zeta - \partial_\zeta A_\theta)\,\mathbf{e}_s - \partial_s A_\zeta\,\mathbf{e}_\theta + \partial_s A_\theta\,\mathbf{e}_\zeta$.

2. In the annular volumes, the condition that the field is tangential to the inner interface gives $\partial_\theta A_\zeta - \partial_\zeta A_\theta = 0$. With the above gauge condition on A_θ given in Eq.(3), this gives $\partial_\theta A_\zeta = 0$, which with Eq.(4) gives

$$A_\zeta(-1,\theta,\zeta) = 0. \tag{5}$$

3. The condition at the outer interface is that the field is $\sqrt{g}\,\mathbf{B}\cdot\nabla s = \partial_\theta A_\zeta - \partial_\zeta A_\theta = b$, where b is supplied by the user. For each of the plasma regions, $b = 0$. For the vacuum region, generally $b \neq 0$.

1.3 enclosed fluxes

1. In the plasma regions, the enclosed fluxes must be constrained.

2. The toroidal and poloidal fluxes enclosed in each volume are determined using

$$\int_S \mathbf{B}\cdot\mathbf{ds} = \int_{\partial S} \mathbf{A}\cdot\mathbf{dl}. \tag{6}$$

1.4 Fourier-Chebyshev representation

1. The components of the vector potential are

$$A_\theta(s,\theta,\zeta) = \sum_{i,l} A_{\theta,e,i,l}\, T_l(s)\cos\alpha_i + \sum_{i,l} A_{\theta,o,i,l}\, T_l(s)\sin\alpha_i, \tag{7}$$

$$A_\zeta(s,\theta,\zeta) = \sum_{i,l} A_{\zeta,e,i,l}\, T_l(s)\cos\alpha_i + \sum_{i,l} A_{\zeta,o,i,l}\, T_l(s)\sin\alpha_i, \tag{8}$$

where $\alpha_j \equiv m_j\theta - n_j\zeta$.

1.5 constrained energy functional

1. The constrained energy functional in each volume is

$$\begin{aligned}
\mathcal{F}_l \equiv & \int \mathbf{B}\cdot\mathbf{B}\,dv - \frac{\mu}{2}\left[\int \mathbf{A}\cdot\mathbf{B}\,dv - K_l\right] \\
+ & \sum_i a_i \left[\sum_l A_{\theta,e,i,l} T_l(-1) - 0\right] \\
+ & \sum_i b_i \left[\sum_l A_{\zeta,e,i,l} T_l(-1) - 0\right] \\
+ & \sum_i c_i \left[\sum_l A_{\theta,o,i,l} T_l(-1) - 0\right] \\
+ & \sum_i d_i \left[\sum_l A_{\zeta,o,i,l} T_l(-1) - 0\right] \\
+ & \sum_i e_i \left[\sum_l \left(-m_i A_{\zeta,e,i,l} - n_i A_{\theta,e,i,l}\right) T_l(+1) - b_{s,i}\right] \\
+ & \sum_i f_i \left[\sum_l \left(+m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l}\right) T_l(+1) - b_{c,i}\right] \\
+ & \alpha \left[\sum_l A_{\theta,e,1,l} T_l(+1) - \Delta\psi_t\right] \\
+ & \beta \left[\sum_l A_{\zeta,e,1,l} T_l(+1) - \Delta\psi_p\right]
\end{aligned} \tag{9}$$

where

i. a_i, b_i, c_i and d_i are Lagrange multipliers used to enforce the combined gauge and interface boundary condition on the inner interface,

ii. e_i and f_i are Lagrange multipliers used to enforce the interface boundary condition on the outer interface, namely $\sqrt{g}\mathbf{B}\cdot\nabla s = b$; and

iii. α and β are Lagrange multipliers used to enforce the constraints on the enclosed fluxes.

2. In each plasma volume the boundary condition on the outer interface is $b = 0$.

3. In the vacuum volume (only for free-boundary), we may set $\mu = 0$.

1.6 energy and helicity integrands

1. The integrands are

$$
\begin{aligned}
\sqrt{g}\,\mathbf{B}\cdot\mathbf{B} \;=\;
&+ \quad (m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;(m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p})\;T_l\,T_p\,\bar{g}_{ss}\;\cos\alpha_i\;\cos\alpha_j\\
&- 2\;(m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;(m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p})\;T_l\,T_p\,\bar{g}_{ss}\;\cos\alpha_i\;\sin\alpha_j\\
&+ \quad (m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\;(m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p})\;T_l\,T_p\,\bar{g}_{ss}\;\sin\alpha_i\;\sin\alpha_j\\
&- 2\;(m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;A_{\zeta,e,j,p}\;T_l\,T_p'\,\bar{g}_{s\theta}\;\cos\alpha_i\;\cos\alpha_j\\
&- 2\;(m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;A_{\zeta,o,j,p}\;T_l\,T_p'\,\bar{g}_{s\theta}\;\cos\alpha_i\;\sin\alpha_j\\
&+ 2\;(m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\;A_{\zeta,e,j,p}\;T_l\,T_p'\,\bar{g}_{s\theta}\;\sin\alpha_i\;\cos\alpha_j\\
&+ 2\;(m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\;A_{\zeta,o,j,p}\;T_l\,T_p'\,\bar{g}_{s\theta}\;\sin\alpha_i\;\sin\alpha_j\\
&+ 2\;(m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;A_{\theta,e,j,p}\;T_l\,T_p'\,\bar{g}_{s\zeta}\;\cos\alpha_i\;\cos\alpha_j\\
&+ 2\;(m_i A_{\zeta,o,i,l} + n_i A_{\theta,o,i,l})\;A_{\theta,o,j,p}\;T_l\,T_p'\,\bar{g}_{s\zeta}\;\cos\alpha_i\;\sin\alpha_j\\
&- 2\;(m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\;A_{\theta,e,j,p}\;T_l\,T_p'\,\bar{g}_{s\zeta}\;\sin\alpha_i\;\cos\alpha_j\\
&- 2\;(m_i A_{\zeta,e,i,l} + n_i A_{\theta,e,i,l})\;A_{\theta,o,j,p}\;T_l\,T_p'\,\bar{g}_{s\zeta}\;\sin\alpha_i\;\sin\alpha_j\\
&+ \quad A_{\zeta,e,i,l}\;A_{\zeta,e,j,p}\;T_l'\,T_p'\,\bar{g}_{\theta\theta}\;\cos\alpha_i\;\cos\alpha_j\\
&+ 2\;A_{\zeta,e,i,l}\;A_{\zeta,o,j,p}\;T_l'\,T_p'\,\bar{g}_{\theta\theta}\;\cos\alpha_i\;\sin\alpha_j\\
&+ \quad A_{\zeta,o,i,l}\;A_{\zeta,o,j,p}\;T_l'\,T_p'\,\bar{g}_{\theta\theta}\;\sin\alpha_i\;\sin\alpha_j\\
&- 2\;A_{\zeta,e,i,l}\;A_{\theta,e,j,p}\;T_l'\,T_p'\,\bar{g}_{\theta\zeta}\;\cos\alpha_i\;\cos\alpha_j\\
&- 2\;A_{\zeta,e,i,l}\;A_{\theta,o,j,p}\;T_l'\,T_p'\,\bar{g}_{\theta\zeta}\;\cos\alpha_i\;\sin\alpha_j\\
&- 2\;A_{\zeta,o,i,l}\;A_{\theta,e,j,p}\;T_l'\,T_p'\,\bar{g}_{\theta\zeta}\;\sin\alpha_i\;\cos\alpha_j\\
&- 2\;A_{\zeta,o,i,l}\;A_{\theta,o,j,p}\;T_l'\,T_p'\,\bar{g}_{\theta\zeta}\;\sin\alpha_i\;\sin\alpha_j\\
&+ \quad A_{\theta,e,i,l}\;A_{\theta,e,j,p}\;T_l'\,T_p'\,\bar{g}_{\zeta\zeta}\;\cos\alpha_i\;\cos\alpha_j\\
&+ 2\;A_{\theta,e,i,l}\;A_{\theta,o,j,p}\;T_l'\,T_p'\,\bar{g}_{\zeta\zeta}\;\cos\alpha_i\;\sin\alpha_j\\
&+ \quad A_{\theta,o,i,l}\;A_{\theta,o,j,p}\;T_l'\,T_p'\,\bar{g}_{\zeta\zeta}\;\sin\alpha_i\;\sin\alpha_j
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\sqrt{g}\,\mathbf{A}\cdot\mathbf{B} \;=\;
&- \quad A_{\zeta,e,i,l}\;A_{\theta,e,j,p}\;T_l'\,T_p\;\cos\alpha_i\;\cos\alpha_j\\
&- \quad A_{\zeta,e,i,l}\;A_{\theta,o,j,p}\;T_l'\,T_p\;\cos\alpha_i\;\sin\alpha_j\\
&- \quad A_{\zeta,o,i,l}\;A_{\theta,e,j,p}\;T_l'\,T_p\;\sin\alpha_i\;\cos\alpha_j\\
&- \quad A_{\zeta,o,i,l}\;A_{\theta,o,j,p}\;T_l'\,T_p\;\sin\alpha_i\;\sin\alpha_j\\
&+ \quad A_{\theta,e,i,l}\;A_{\zeta,e,j,p}\;T_l'\,T_p\;\cos\alpha_i\;\cos\alpha_j\\
&+ \quad A_{\theta,e,i,l}\;A_{\zeta,o,j,p}\;T_l'\,T_p\;\cos\alpha_i\;\sin\alpha_j\\
&+ \quad A_{\theta,o,i,l}\;A_{\zeta,e,j,p}\;T_l'\,T_p\;\sin\alpha_i\;\cos\alpha_j\\
&+ \quad A_{\theta,o,i,l}\;A_{\zeta,o,j,p}\;T_l'\,T_p\;\sin\alpha_i\;\sin\alpha_j
\end{aligned}
\tag{11}
$$

1.7 first derivatives of energy and helicity integrands with respect to $A_{\theta,e,i,l}$ and $A_{\theta,o,i,l}$

1. The first derivatives with respect to $A_{\theta,e,i,l}$ and $A_{\theta,o,i,l}$ are

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& - \; 2 \; n_i \; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \int ds \; T_p \, T_l \oiint d\theta d\zeta \; \bar{g}_{ss} \; \cos\alpha_j \; \sin\alpha_i \\
& + \; 2 \; n_i \; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \int ds \; T_l \, T_p \oiint d\theta d\zeta \; \bar{g}_{ss} \; \sin\alpha_i \; \sin\alpha_j \\
& + \; 2 \; n_i \; A_{\zeta,e,j,p} \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \sin\alpha_i \; \cos\alpha_j \\
& + \; 2 \; n_i \; A_{\zeta,o,j,p} \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \sin\alpha_i \; \sin\alpha_j \\
& + \; 2 \; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \int ds \; T_p \, T_l' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \cos\alpha_j \; \cos\alpha_i \\
& - \; 2 \; n_i \; A_{\theta,e,j,p} \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \sin\alpha_i \; \cos\alpha_j \\
& - \; 2 \; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \int ds \; T_p \, T_l' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \sin\alpha_j \; \cos\alpha_i \\
& - \; 2 \; n_i \; A_{\theta,o,j,p} \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \sin\alpha_i \; \sin\alpha_j \\
& - \; 2 \; A_{\zeta,e,j,p} \int ds \; T_p' \, T_l' \oiint d\theta d\zeta \; \bar{g}_{\theta\zeta} \; \cos\alpha_j \; \cos\alpha_i \\
& - \; 2 \; A_{\zeta,o,j,p} \int ds \; T_p' \, T_l' \oiint d\theta d\zeta \; \bar{g}_{\theta\zeta} \; \sin\alpha_j \; \cos\alpha_i \\
& + \; 2 \; A_{\theta,e,j,p} \int ds \; T_l' \, T_p' \oiint d\theta d\zeta \; \bar{g}_{\zeta\zeta} \; \cos\alpha_i \; \cos\alpha_j \\
& + \; 2 \; A_{\theta,o,j,p} \int ds \; T_l' \, T_p' \oiint d\theta d\zeta \; \bar{g}_{\zeta\zeta} \; \cos\alpha_i \; \sin\alpha_j
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& - \; A_{\zeta,e,j,p} \int ds \; T_p' \, T_l \oiint d\theta d\zeta \; \cos\alpha_j \; \cos\alpha_i \\
& - \; A_{\zeta,o,j,p} \int ds \; T_p' \, T_l \oiint d\theta d\zeta \; \sin\alpha_j \; \cos\alpha_i \\
& + \; A_{\zeta,e,j,p} \int ds \; T_l' \, T_p \oiint d\theta d\zeta \; \cos\alpha_i \; \cos\alpha_j \\
& + \; A_{\zeta,o,j,p} \int ds \; T_l' \, T_p \oiint d\theta d\zeta \; \cos\alpha_i \; \sin\alpha_j
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& + \; 2 \; n_i \; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \int ds \; T_l \, T_p \oiint d\theta d\zeta \; \bar{g}_{ss} \; \cos\alpha_i \; \cos\alpha_j \\
& - \; 2 \; n_i \; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \int ds \; T_l \, T_p \oiint d\theta d\zeta \; \bar{g}_{ss} \; \cos\alpha_i \; \sin\alpha_j \\
& - \; 2 \; n_i \; A_{\zeta,e,j,p} \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \cos\alpha_i \; \cos\alpha_j \\
& - \; 2 \; n_i \; A_{\zeta,o,j,p} \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \cos\alpha_i \; \sin\alpha_j \\
& + \; 2 \; n_i \; A_{\theta,e,j,p} \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \cos\alpha_i \; \cos\alpha_j \\
& + \; 2 \; n_i \; A_{\theta,o,j,p} \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \cos\alpha_i \; \sin\alpha_j \\
& + \; 2 \; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \int ds \; T_p \, T_l' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \cos\alpha_j \; \sin\alpha_i \\
& - \; 2 \; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \int ds \; T_p \, T_l' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \sin\alpha_j \; \sin\alpha_i \\
& - \; 2 \; A_{\zeta,e,j,p} \int ds \; T_p' \, T_l' \oiint d\theta d\zeta \; \bar{g}_{\theta\zeta} \; \cos\alpha_j \; \sin\alpha_i \\
& - \; 2 \; A_{\zeta,o,j,p} \int ds \; T_p' \, T_l' \oiint d\theta d\zeta \; \bar{g}_{\theta\zeta} \; \sin\alpha_j \; \sin\alpha_i \\
& + \; 2 \; A_{\theta,e,j,p} \int ds \; T_p' \, T_l' \oiint d\theta d\zeta \; \bar{g}_{\zeta\zeta} \; \cos\alpha_j \; \sin\alpha_i \\
& + \; 2 \; A_{\theta,o,j,p} \int ds \; T_l' \, T_p' \oiint d\theta d\zeta \; \bar{g}_{\zeta\zeta} \; \sin\alpha_i \; \sin\alpha_j
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& - \; A_{\zeta,e,j,p} \int ds \; T_p' \, T_l \oiint d\theta d\zeta \; \cos\alpha_j \; \sin\alpha_i \\
& - \; A_{\zeta,o,j,p} \int ds \; T_p' \, T_l \oiint d\theta d\zeta \; \sin\alpha_j \; \sin\alpha_i \\
& + \; A_{\zeta,e,j,p} \int ds \; T_l' \, T_p \oiint d\theta d\zeta \; \sin\alpha_i \; \cos\alpha_j \\
& + \; A_{\zeta,o,j,p} \int ds \; T_l' \, T_p \oiint d\theta d\zeta \; \sin\alpha_i \; \sin\alpha_j
\end{aligned}
\tag{15}
$$

1.8 first derivatives of energy and helicity integrands with respect to $A_{\zeta,e,i,l}$ and $A_{\zeta,o,i,l}$

1. The first derivatives with respect to $A_{\zeta,e,i,l}$ and $A_{\zeta,o,i,l}$ are

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& - \; 2 \; m_i \; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \; \int ds \; T_p \, T_l \; \oiint d\theta d\zeta \; \bar{g}_{ss} \; \cos\alpha_j \; \sin\alpha_i \\
& + \; 2 \; m_i \; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \; \int ds \; T_l \, T_p \; \oiint d\theta d\zeta \; \bar{g}_{ss} \; \sin\alpha_i \; \sin\alpha_j \\
& - \; 2 \; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \; \int ds \; T_p \, T_l' \; \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \cos\alpha_j \; \cos\alpha_i \\
& + \; 2 \; m_i \; A_{\zeta,e,j,p} \; \int ds \; T_l \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \sin\alpha_i \; \cos\alpha_j \\
& + \; 2 \; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \; \int ds \; T_p \, T_l' \; \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \sin\alpha_j \; \cos\alpha_i \\
& + \; 2 \; m_i \; A_{\zeta,o,j,p} \; \int ds \; T_l \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \sin\alpha_i \; \sin\alpha_j \\
& - \; 2 \; m_i \; A_{\theta,e,j,p} \; \int ds \; T_l \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \sin\alpha_i \; \cos\alpha_j \\
& - \; 2 \; m_i \; A_{\theta,o,j,p} \; \int ds \; T_l \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \sin\alpha_i \; \sin\alpha_j \\
& + \; 2 \; A_{\zeta,e,j,p} \; \int ds \; T_l' \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{\theta\theta} \; \cos\alpha_i \; \cos\alpha_j \\
& + \; 2 \; A_{\zeta,o,j,p} \; \int ds \; T_l' \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{\theta\theta} \; \cos\alpha_i \; \sin\alpha_j \\
& - \; 2 \; A_{\theta,e,j,p} \; \int ds \; T_l' \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{\theta\zeta} \; \cos\alpha_i \; \cos\alpha_j \\
& - \; 2 \; A_{\theta,o,j,p} \; \int ds \; T_l' \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{\theta\zeta} \; \cos\alpha_i \; \sin\alpha_j
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& - \; A_{\theta,e,j,p} \; \int ds \; T_l' \, T_p \; \oiint d\theta d\zeta \; \cos\alpha_i \; \cos\alpha_j \\
& - \; A_{\theta,o,j,p} \; \int ds \; T_l' \, T_p \; \oiint d\theta d\zeta \; \cos\alpha_i \; \sin\alpha_j \\
& + \; A_{\theta,e,j,p} \; \int ds \; T_p' \, T_l \; \oiint d\theta d\zeta \; \cos\alpha_j \; \cos\alpha_i \\
& + \; A_{\theta,o,j,p} \; \int ds \; T_p' \, T_l \; \oiint d\theta d\zeta \; \sin\alpha_j \; \cos\alpha_i
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& + \; 2 \; m_i \; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \; \int ds \; T_l \, T_p \; \oiint d\theta d\zeta \; \bar{g}_{ss} \; \cos\alpha_i \; \cos\alpha_j \\
& - \; 2 \; m_i \; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p}) \; \int ds \; T_l \, T_p \; \oiint d\theta d\zeta \; \bar{g}_{ss} \; \cos\alpha_i \; \sin\alpha_j \\
& - \; 2 \; m_i \; A_{\zeta,e,j,p} \; \int ds \; T_l \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \cos\alpha_i \; \cos\alpha_j \\
& - \; 2 \; m_i \; A_{\zeta,o,j,p} \; \int ds \; T_l \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \cos\alpha_i \; \sin\alpha_j \\
& - \; 2 \; (m_j A_{\zeta,o,j,p} + n_j A_{\theta,o,j,p}) \; \int ds \; T_p \, T_l' \; \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \cos\alpha_j \; \sin\alpha_i \\
& + \; 2 \; (m_j A_{\zeta,e,j,p} + n_j A_{\theta,e,j,p} \; \int ds \; T_p \, T_l' \; \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \sin\alpha_j \; \sin\alpha_i \\
& + \; 2 \; m_i \; A_{\theta,e,j,p} \; \int ds \; T_l \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \cos\alpha_i \; \cos\alpha_j \\
& + \; 2 \; m_i \; A_{\theta,o,j,p} \; \int ds \; T_l \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \cos\alpha_i \; \sin\alpha_j \\
& + \; 2 \; A_{\zeta,e,j,p} \; \int ds \; T_p' \, T_l' \; \oiint d\theta d\zeta \; \bar{g}_{\theta\theta} \; \cos\alpha_j \; \sin\alpha_i \\
& + \; 2 \; A_{\zeta,o,j,p} \; \int ds \; T_l' \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{\theta\theta} \; \sin\alpha_i \; \sin\alpha_j \\
& - \; 2 \; A_{\theta,e,j,p} \; \int ds \; T_l' \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{\theta\zeta} \; \sin\alpha_i \; \cos\alpha_j \\
& - \; 2 \; A_{\theta,o,j,p} \; \int ds \; T_l' \, T_p' \; \oiint d\theta d\zeta \; \bar{g}_{\theta\zeta} \; \sin\alpha_i \; \sin\alpha_j
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& - \; A_{\theta,e,j,p} \; \int ds \; T_l' \, T_p \; \oiint d\theta d\zeta \; \sin\alpha_i \; \cos\alpha_j \\
& - \; A_{\theta,o,j,p} \; \int ds \; T_l' \, T_p \; \oiint d\theta d\zeta \; \sin\alpha_i \; \sin\alpha_j \\
& + \; A_{\theta,e,j,p} \; \int ds \; T_p' \, T_l \; \oiint d\theta d\zeta \; \cos\alpha_j \; \sin\alpha_i \\
& + \; A_{\theta,o,j,p} \; \int ds \; T_p' \, T_l \; \oiint d\theta d\zeta \; \sin\alpha_j \; \sin\alpha_i
\end{aligned}
\tag{19}
$$

1.9 second derivatives of energy and helicity integrands

1. The second derivatives wrt $A_{\theta,e,i,l}$ (stellarator symmetric) are

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
+ \; 2 \; n_i \; n_j \; & \int ds \; T_l \, T_p \oiint d\theta d\zeta \; \bar{g}_{ss} \; \sin\alpha_i \; \sin\alpha_j \\
- \; 2 \; n_i \; & \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \sin\alpha_i \; \cos\alpha_j \\
- \; 2 \; n_j \; & \int ds \; T_p \, T_l' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \sin\alpha_j \; \cos\alpha_i \\
+ \; 2 \; & \int ds \; T_l' \, T_p' \oiint d\theta d\zeta \; \bar{g}_{\zeta\zeta} \; \cos\alpha_i \; \cos\alpha_j
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
- \; 2 \; n_i \; n_j \; & \int ds \; T_p \, T_l \oiint d\theta d\zeta \; \bar{g}_{ss} \; \cos\alpha_j \; \sin\alpha_i \; ! \\
+ \; 2 \; n_j \; & \int ds \; T_p \, T_l' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \cos\alpha_j \; \cos\alpha_i \\
- \; 2 \; n_i \; & \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \sin\alpha_i \; \sin\alpha_j \\
+ \; 2 \; & \int ds \; T_l' \, T_p' \oiint d\theta d\zeta \; \bar{g}_{\zeta\zeta} \; \cos\alpha_i \; \sin\alpha_j \; !
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
+ \; 2 \; n_i \; m_j \; & \int ds \; T_l \, T_p \oiint d\theta d\zeta \; \bar{g}_{ss} \; \sin\alpha_i \; \sin\alpha_j \\
+ \; 2 \; n_i \; & \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \sin\alpha_i \; \cos\alpha_j \\
- \; 2 \; m_j \; & \int ds \; T_p \, T_l' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \sin\alpha_j \; \cos\alpha_i \\
- \; 2 \; & \int ds \; T_p' \, T_l' \oiint d\theta d\zeta \; \bar{g}_{\theta\zeta} \; \cos\alpha_j \; \cos\alpha_i
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
- \; 2 \; n_i \; m_j \; & \int ds \; T_p \, T_l \oiint d\theta d\zeta \; \bar{g}_{ss} \; \cos\alpha_j \; \sin\alpha_i \; ! \\
+ \; 2 \; n_i \; & \int ds \; T_l \, T_p' \oiint d\theta d\zeta \; \bar{g}_{s\theta} \; \sin\alpha_i \; \sin\alpha_j \\
+ \; 2 \; m_j \; & \int ds \; T_p \, T_l' \oiint d\theta d\zeta \; \bar{g}_{s\zeta} \; \cos\alpha_j \; \cos\alpha_i \\
- \; 2 \; & \int ds \; T_p' \, T_l' \oiint d\theta d\zeta \; \bar{g}_{\theta\zeta} \; \sin\alpha_j \; \cos\alpha_i
\end{aligned}
\tag{23}
$$

$$
\frac{\partial}{\partial A_{\theta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad 0 \tag{24}
$$

$$
\frac{\partial}{\partial A_{\theta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad 0 \tag{25}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
- \; & \int ds \; T_p' \, T_l \oiint d\theta d\zeta \; \cos\alpha_j \; \cos\alpha_i \\
+ \; & \int ds \; T_l' \, T_p \oiint d\theta d\zeta \; \cos\alpha_i \; \cos\alpha_j
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,j,p}} \frac{\partial}{\partial A_{\theta,e,i,l}} \int dv\, \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
- \; & \int ds \; T_p' \, T_l \oiint d\theta d\zeta \; \sin\alpha_j \; \cos\alpha_i \\
+ \; & \int ds \; T_l' \, T_p \oiint d\theta d\zeta \; \cos\alpha_i \; \sin\alpha_j \; !
\end{aligned}
\tag{27}
$$

1.10 second derivatives of energy and helicity integrands

1. The second derivatives wrt $A_{\theta,o,i,l}$ (non-stellarator symmetric) are

$$\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
- \ 2 \ n_i \ n_j \ \int ds \ \ T_l\, T_p \ \oiint d\theta d\zeta \ \ \bar{g}_{ss} \ \cos\alpha_i \ \sin\alpha_j \ ! \\
+ \ 2 \ n_i \ \ \int ds \ \ T_l\, T_p' \ \oiint d\theta d\zeta \ \ \bar{g}_{s\zeta} \ \cos\alpha_i \ \cos\alpha_j \\
- \ 2 \ \ n_j \ \int ds \ \ T_p\, T_l' \ \oiint d\theta d\zeta \ \ \bar{g}_{s\zeta} \ \sin\alpha_j \ \sin\alpha_i \\
+ \ 2 \ \ \int ds \ \ T_p'\, T_l' \ \oiint d\theta d\zeta \ \ \bar{g}_{\zeta\zeta} \ \cos\alpha_j \ \sin\alpha_i \ !
\end{aligned} \tag{28}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
+ \ 2 \ n_i \ n_j \ \int ds \ \ T_l\, T_p \ \oiint d\theta d\zeta \ \ \bar{g}_{ss} \ \cos\alpha_i \ \cos\alpha_j \\
+ \ 2 \ n_i \ \ \int ds \ \ T_l\, T_p' \ \oiint d\theta d\zeta \ \ \bar{g}_{s\zeta} \ \cos\alpha_i \ \sin\alpha_j \ ! \\
+ \ 2 \ \ n_j \ \int ds \ \ T_p\, T_l' \ \oiint d\theta d\zeta \ \ \bar{g}_{s\zeta} \ \cos\alpha_j \ \sin\alpha_i \ ! \\
+ \ 2 \ \ \int ds \ \ T_l'\, T_p' \ \oiint d\theta d\zeta \ \ \bar{g}_{\zeta\zeta} \ \sin\alpha_i \ \sin\alpha_j
\end{aligned} \tag{29}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
- \ 2 \ n_i \ m_j \ \int ds \ \ T_l\, T_p \ \oiint d\theta d\zeta \ \ \bar{g}_{ss} \ \cos\alpha_i \ \sin\alpha_j \ ! \\
- \ 2 \ n_i \ \ \int ds \ \ T_l\, T_p' \ \oiint d\theta d\zeta \ \ \bar{g}_{s\theta} \ \cos\alpha_i \ \cos\alpha_j \\
- \ 2 \ \ m_j \ \int ds \ \ T_p\, T_l' \ \oiint d\theta d\zeta \ \ \bar{g}_{s\zeta} \ \sin\alpha_j \ \sin\alpha_i \\
- \ 2 \ \ \int ds \ \ T_p'\, T_l' \ \oiint d\theta d\zeta \ \ \bar{g}_{\theta\zeta} \ \cos\alpha_j \ \sin\alpha_i \ !
\end{aligned} \tag{30}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
+ \ 2 \ n_i \ m_j \ \int ds \ \ T_l\, T_p \ \oiint d\theta d\zeta \ \ \bar{g}_{ss} \ \cos\alpha_i \ \cos\alpha_j \\
- \ 2 \ n_i \ \ \int ds \ \ T_l\, T_p' \ \oiint d\theta d\zeta \ \ \bar{g}_{s\theta} \ \cos\alpha_i \ \sin\alpha_j \ ! \\
+ \ 2 \ \ m_j \ \int ds \ \ T_p\, T_l' \ \oiint d\theta d\zeta \ \ \bar{g}_{s\zeta} \ \cos\alpha_j \ \sin\alpha_i \ ! \\
- \ 2 \ \ \int ds \ \ T_p'\, T_l' \ \oiint d\theta d\zeta \ \ \bar{g}_{\theta\zeta} \ \sin\alpha_j \ \sin\alpha_i
\end{aligned} \tag{31}$$

$$\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad 0 \tag{32}$$

$$\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad 0 \tag{33}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
- \ \int ds \ \ T_p'\, T_l \ \oiint d\theta d\zeta \ \ \cos\alpha_j \ \sin\alpha_i \ ! \\
+ \ \int ds \ \ T_l'\, T_p \ \oiint d\theta d\zeta \ \ \sin\alpha_i \ \cos\alpha_j
\end{aligned} \tag{34}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\theta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
- \ \int ds \ \ T_p'\, T_l \ \oiint d\theta d\zeta \ \ \sin\alpha_j \ \sin\alpha_i \\
+ \ \int ds \ \ T_l'\, T_p \ \oiint d\theta d\zeta \ \ \sin\alpha_i \ \sin\alpha_j
\end{aligned} \tag{35}$$

1.11 second derivatives of energy and helicity integrands

1. The second derivatives wrt $A_{\zeta,e,i,l}$ (stellarator symmetric) are

$$\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& + \ 2 \quad m_i \quad n_j \quad \int ds \quad T_l\, T_p \ \oiint d\theta d\zeta \ \bar{g}_{ss} \ \sin\alpha_i \ \sin\alpha_j \\
& + \ 2 \quad\quad\ \ n_j \quad \int ds \quad T_p\, T_l' \ \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \sin\alpha_j \ \cos\alpha_i \\
& - \ 2 \quad m_i \quad\quad\ \ \int ds \quad T_l\, T_p' \ \oiint d\theta d\zeta \ \bar{g}_{s\zeta} \ \sin\alpha_i \ \cos\alpha_j \\
& - \ 2 \quad\quad\quad\quad\ \ \int ds \quad T_l'\, T_p' \ \oiint d\theta d\zeta \ \bar{g}_{\theta\zeta} \ \cos\alpha_i \ \cos\alpha_j
\end{aligned} \tag{36}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& - \ 2 \quad m_i \quad n_j \quad \int ds \quad T_p\, T_l \ \oiint d\theta d\zeta \ \bar{g}_{ss} \ \cos\alpha_j \ \sin\alpha_i \ ! \\
& - \ 2 \quad\quad\ \ n_j \quad \int ds \quad T_p\, T_l' \ \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \cos\alpha_j \ \cos\alpha_i \\
& - \ 2 \quad m_i \quad\quad\ \ \int ds \quad T_l\, T_p' \ \oiint d\theta d\zeta \ \bar{g}_{s\zeta} \ \sin\alpha_i \ \sin\alpha_j \\
& - \ 2 \quad\quad\quad\quad\ \ \int ds \quad T_l'\, T_p' \ \oiint d\theta d\zeta \ \bar{g}_{\theta\zeta} \ \cos\alpha_i \ \sin\alpha_j \ !
\end{aligned} \tag{37}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& + \ 2 \quad m_i \quad m_j \quad \int ds \quad T_l\, T_p \ \oiint d\theta d\zeta \ \bar{g}_{ss} \ \sin\alpha_i \ \sin\alpha_j \\
& + \ 2 \quad m_i \quad\quad\ \ \int ds \quad T_l\, T_p' \ \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \sin\alpha_i \ \cos\alpha_j \\
& + \ 2 \quad\quad\ \ m_j \quad \int ds \quad T_p\, T_l' \ \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \sin\alpha_j \ \cos\alpha_i \\
& + \ 2 \quad\quad\quad\quad\ \ \int ds \quad T_l'\, T_p' \ \oiint d\theta d\zeta \ \bar{g}_{\theta\theta} \ \cos\alpha_i \ \cos\alpha_j
\end{aligned} \tag{38}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
& - \ 2 \quad m_i \quad m_j \quad \int ds \quad T_p\, T_l \ \oiint d\theta d\zeta \ \bar{g}_{ss} \ \cos\alpha_j \ \sin\alpha_i \ ! \\
& - \ 2 \quad\quad\ \ m_j \quad \int ds \quad T_p\, T_l' \ \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \cos\alpha_j \ \cos\alpha_i \\
& + \ 2 \quad m_i \quad\quad\ \ \int ds \quad T_l\, T_p' \ \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \sin\alpha_i \ \sin\alpha_j \\
& + \ 2 \quad\quad\quad\quad\ \ \int ds \quad T_l'\, T_p' \ \oiint d\theta d\zeta \ \bar{g}_{\theta\theta} \ \cos\alpha_i \ \sin\alpha_j \ !
\end{aligned} \tag{39}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& - \ \int ds \quad T_l'\, T_p \ \oiint d\theta d\zeta \ \cos\alpha_i \ \cos\alpha_j \\
& + \ \int ds \quad T_p'\, T_l \ \oiint d\theta d\zeta \ \cos\alpha_j \ \cos\alpha_i
\end{aligned} \tag{40}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
& - \ \int ds \quad T_l'\, T_p \ \oiint d\theta d\zeta \ \cos\alpha_i \ \sin\alpha_j \ ! \\
& + \ \int ds \quad T_p'\, T_l \ \oiint d\theta d\zeta \ \sin\alpha_j \ \cos\alpha_i
\end{aligned} \tag{41}$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad 0 \tag{42}$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,e,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad 0 \tag{43}$$

1.12 second derivatives of energy and helicity integrands

1. The second derivatives wrt $A_{\zeta,o,i,l}$ (non-stellarator symmetric) are

$$\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
- \ 2 \ m_i \ n_j \ \int ds \ & T_l\, T_p \oiint d\theta d\zeta \ \bar{g}_{ss} \ \cos\alpha_i \ \sin\alpha_j \ ! \\
+ \ 2 \ n_j \ \int ds \ & T_p\, T_l' \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \sin\alpha_j \ \sin\alpha_i \\
+ \ 2 \ m_i \ \int ds \ & T_l\, T_p' \oiint d\theta d\zeta \ \bar{g}_{s\zeta} \ \cos\alpha_i \ \cos\alpha_j \\
- \ 2 \ \int ds \ & T_l'\, T_p' \oiint d\theta d\zeta \ \bar{g}_{\theta\zeta} \ \sin\alpha_i \ \cos\alpha_j
\end{aligned} \tag{44}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
+ \ 2 \ m_i \ n_j \ \int ds \ & T_l\, T_p \oiint d\theta d\zeta \ \bar{g}_{ss} \ \cos\alpha_i \ \cos\alpha_j \\
- \ 2 \ n_j \ \int ds \ & T_p\, T_l' \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \cos\alpha_j \ \sin\alpha_i \ ! \\
+ \ 2 \ m_i \ \int ds \ & T_l\, T_p' \oiint d\theta d\zeta \ \bar{g}_{s\zeta} \ \cos\alpha_i \ \sin\alpha_j \ ! \\
- \ 2 \ \int ds \ & T_l'\, T_p' \oiint d\theta d\zeta \ \bar{g}_{\theta\zeta} \ \sin\alpha_i \ \sin\alpha_j
\end{aligned} \tag{45}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
- \ 2 \ m_i \ m_j \ \int ds \ & T_l\, T_p \oiint d\theta d\zeta \ \bar{g}_{ss} \ \cos\alpha_i \ \sin\alpha_j \ ! \\
- \ 2 \ m_i \ \int ds \ & T_l\, T_p' \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \cos\alpha_i \ \cos\alpha_j \\
+ \ 2 \ m_j \ \int ds \ & T_p\, T_l' \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \sin\alpha_j \ \sin\alpha_i \\
+ \ 2 \ \int ds \ & T_p'\, T_l' \oiint d\theta d\zeta \ \bar{g}_{\theta\theta} \ \cos\alpha_j \ \sin\alpha_i \ !
\end{aligned} \tag{46}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{B}\cdot\mathbf{B} \quad = \quad & \\
+ \ 2 \ m_i \ m_j \ \int ds \ & T_l\, T_p \oiint d\theta d\zeta \ \bar{g}_{ss} \ \cos\alpha_i \ \cos\alpha_j \\
- \ 2 \ m_i \ \int ds \ & T_l\, T_p' \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \cos\alpha_i \ \sin\alpha_j \ ! \\
- \ 2 \ m_j \ \int ds \ & T_p\, T_l' \oiint d\theta d\zeta \ \bar{g}_{s\theta} \ \cos\alpha_j \ \sin\alpha_i \ ! \\
+ \ 2 \ \int ds \ & T_l'\, T_p' \oiint d\theta d\zeta \ \bar{g}_{\theta\theta} \ \sin\alpha_i \ \sin\alpha_j
\end{aligned} \tag{47}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\theta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
- \ \int ds \ & T_l'\, T_p \oiint d\theta d\zeta \ \sin\alpha_i \ \cos\alpha_j \\
+ \ \int ds \ & T_p'\, T_l \oiint d\theta d\zeta \ \cos\alpha_j \ \sin\alpha_i \ !
\end{aligned} \tag{48}$$

$$\begin{aligned}
\frac{\partial}{\partial A_{\theta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad & \\
- \ \int ds \ & T_l'\, T_p \oiint d\theta d\zeta \ \sin\alpha_i \ \sin\alpha_j \\
+ \ \int ds \ & T_p'\, T_l \oiint d\theta d\zeta \ \sin\alpha_j \ \sin\alpha_i
\end{aligned} \tag{49}$$

$$\frac{\partial}{\partial A_{\zeta,e,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad 0 \tag{50}$$

$$\frac{\partial}{\partial A_{\zeta,o,j,p}}\frac{\partial}{\partial A_{\zeta,o,i,l}}\int dv\ \mathbf{A}\cdot\mathbf{B} \quad = \quad 0 \tag{51}$$

1.13 matrix elements

1. The energy, $W \equiv \int dv\ \mathbf{B}\cdot\mathbf{B}$, and helicity, $K \equiv \int dv\ \mathbf{A}\cdot\mathbf{B}$, functionals may be written

$$W = \frac{1}{2}\, a_i\ A_{i,j}\ a_j + a_i\ B_{i,j}\ \psi_j + \frac{1}{2}\ \psi_i\ C_{i,j}\ \psi_j \tag{52}$$

$$K = \frac{1}{2}\, a_i\ D_{i,j}\ a_j + a_i\ E_{i,j}\ \psi_j + \frac{1}{2}\ \psi_i\ F_{i,j}\ \psi_j \tag{53}$$

where $\mathbf{a} \equiv \{A_{\theta,e,i,l}, A_{\zeta,e,i,l}, A_{\theta,o,i,l}, A_{\zeta,o,i,l}, f_{e,i}, f_{o,i}\}$ contains the independent degrees of freedom and $\boldsymbol{\psi} \equiv \{\Delta\psi_t, \Delta\psi_p\}$.

2. The matrix elements are computed via

$$\texttt{MA(i,j)} \equiv A_{i,j} = \frac{\partial^2 W}{\partial a_i \partial a_j} \tag{54}$$

$$\texttt{MB(i,j)} \equiv B_{i,j} = \frac{\partial^2 W}{\partial a_i \partial \psi_j} \tag{55}$$

$$\texttt{MC(i,j)} \equiv C_{i,j} = \frac{\partial^2 W}{\partial \psi_i \partial \psi_j} \tag{56}$$

3. The energy functionals can also be represented as

$$\begin{array}{rcccccccccccc} W & = & \frac{1}{2} & \mathbf{a}^T & \cdot & A[\mathbf{x}] & \cdot & \mathbf{a} & + & B[\mathbf{x},\boldsymbol{\psi}] & \cdot & \mathbf{a} & + & C[\mathbf{x},\boldsymbol{\psi}], \\ K & = & \frac{1}{2} & \mathbf{a}^T & \cdot & D & \cdot & \mathbf{a} & + & E[\boldsymbol{\psi}] & \cdot & \mathbf{a} & + & F[\boldsymbol{\psi}]. \end{array} \tag{57}$$