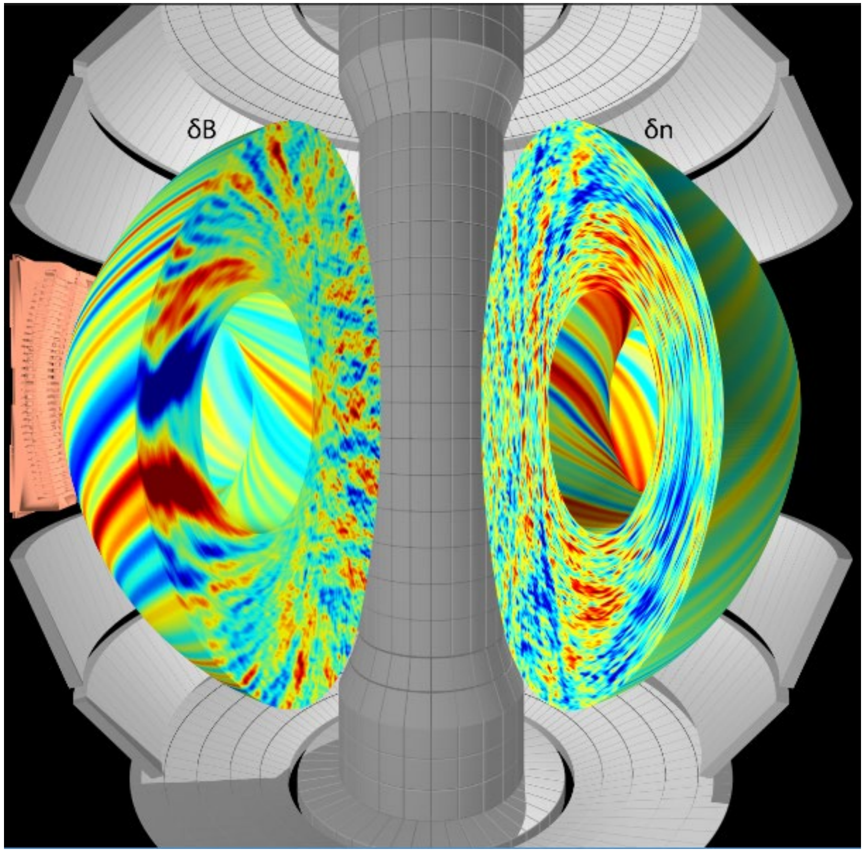
Toroidal magnetized plasma turbulence & transport



Walter Guttenfelder (PPPL)

PPPL Graduate seminar AST 558, April 21, 2023

Outline & references

Outline

- Intro & neutral fluid turbulence
- Magnetized plasmas & turbulence
- Drift waves & characteristics
- Ion Temperature Gradient (ITG) instability cartoon
- Miscellaneous (nonlinear saturation, zonal flows, E×B shear, validation)

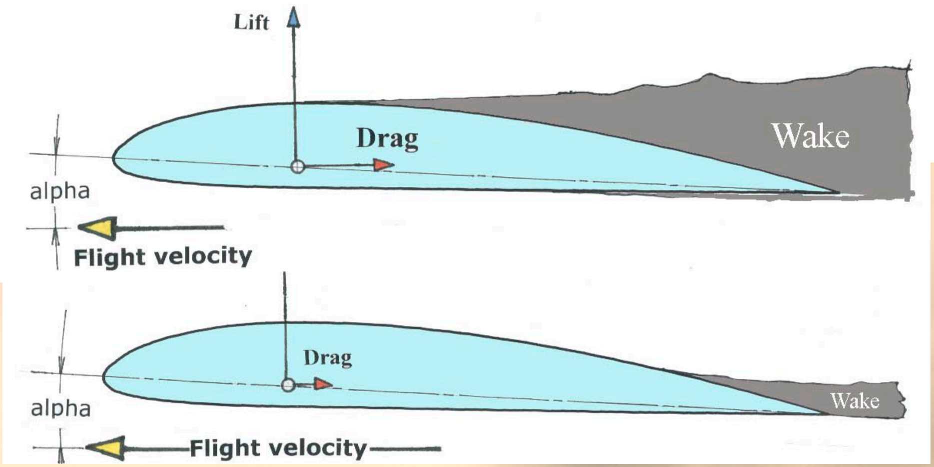
Various introductory & tutorial material on magnetized plasma turbulence and microinstabilities:

- Greg Hammett's webpage, w3.pppl.gov/~hammett
- SULI lectures, e.g. https://suli.pppl.gov/2020/index.html
- PPPL Graduate Summer School lectures, e.g. https://gss.pppl.gov/2020/
- Intro chapters of various Ph.D. theses (Dorland, Beer, Snyder) – all found on Hammett's webpage

Examples of turbulence

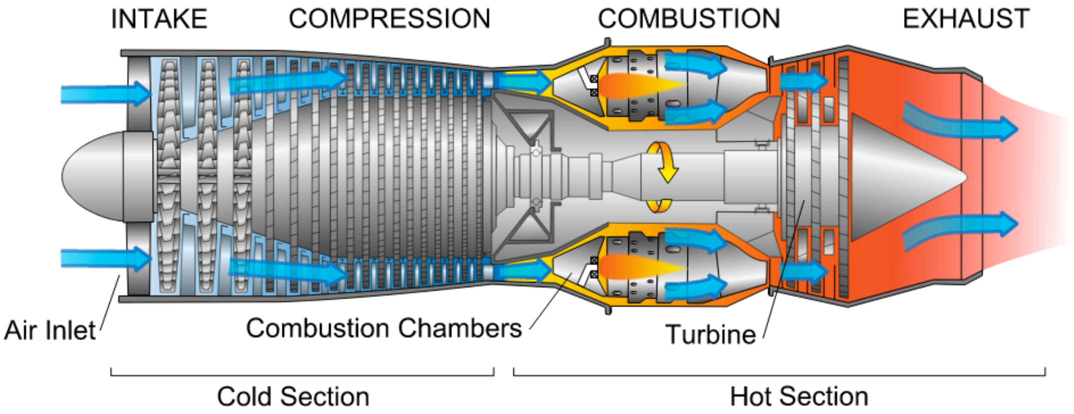
Turbulence is ubiquitous throughout planetary atmospheres

Turbulence is crucial to lift, drag & stall characteristics of airfoils



Increased turbulence on airfoil helps minimize boundary-layer separation and drag from adverse pressure gradient

Turbulence generators

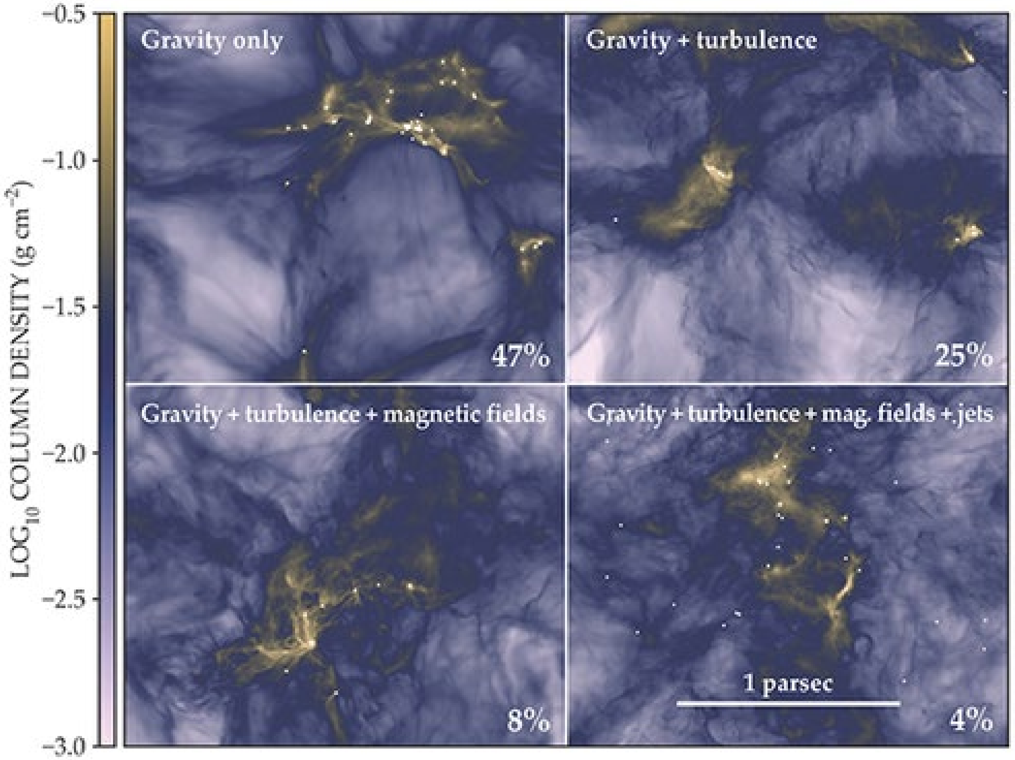



L/D~100 in non-premixed jet flames

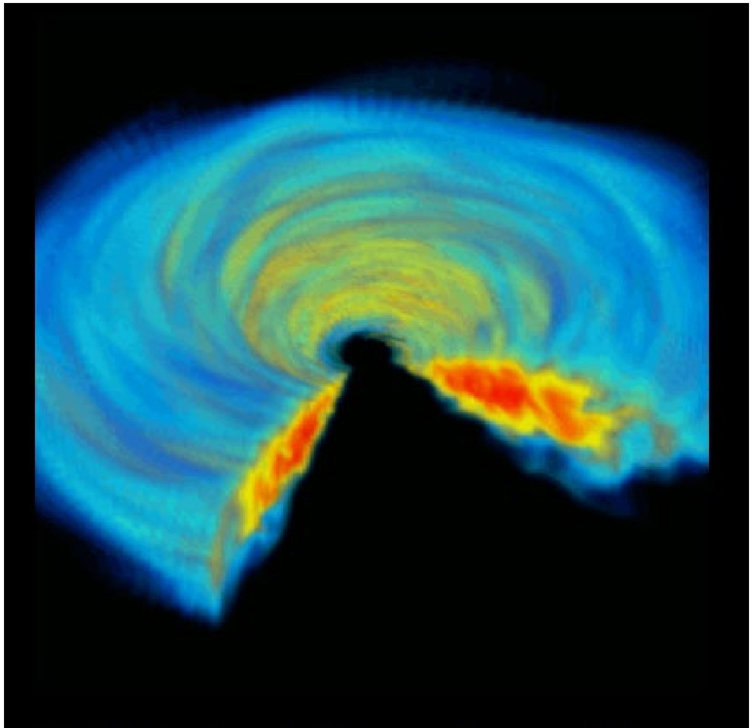

L/D much smaller in swirling burner

Turbulent mixing of fuel and air is critical for efficient & economical jet engines

Turbulence is important throughout astrophysics



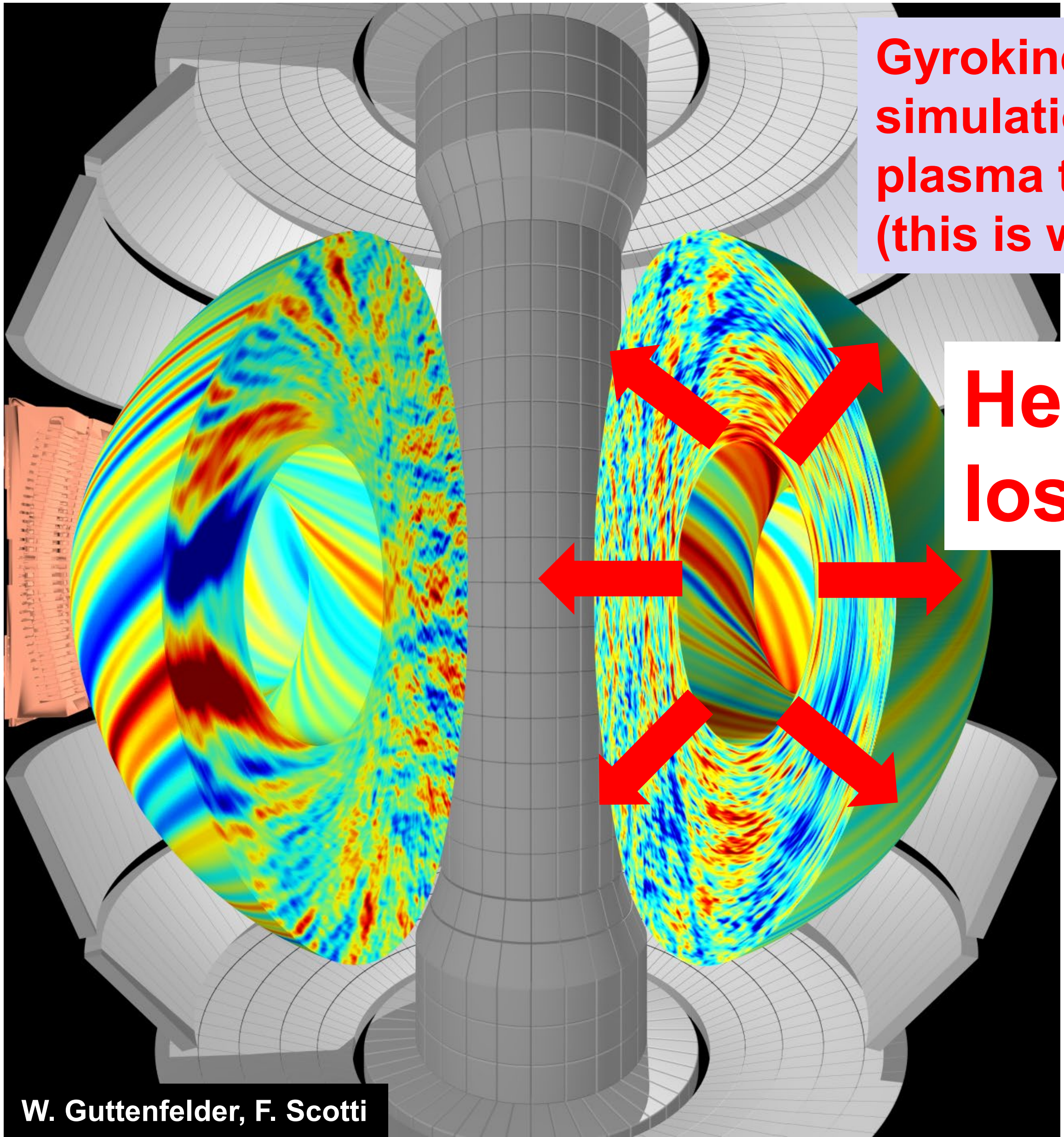
- Plays a role in star formation (C. Federrath, Physics Today, June 2018)



MHD simulation of accretion disk around a black hole

(Hawley, Balbus, & Stone 2001)

Plasma turbulence degrades energy confinement / insulation in magnetic fusion energy devices



Gyrokinetic simulation of plasma turbulence (this is what I do 🙂)

W. Guttenfelder, F. Scotti

Neutral fluids: Incompressible Navier-Stokes

- Momentum conservation law:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v} + \mathbf{f}_B$$

Unsteady flow	Convective acceleration	Pressure force	Viscosity	Body forces (g, J×B, qE)

- Assuming incompressible, from mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \rightarrow \nabla \cdot \mathbf{v} = \mathbf{0}$$

Use dimensionless Reynolds # to estimate laminar vs. turbulent dynamics

- Reynolds number: order-of-magnitude ratio of inertial to viscous forces

$$\frac{\mathbf{v} \cdot \nabla \mathbf{v}}{\nu \nabla^2 \mathbf{v}} \rightarrow \frac{V^2/L}{\nu V/L^2}$$

$$\boxed{\mathrm{Re} = \frac{VL}{\nu}}$$

Viscosities (m^2/s)

Air	**~1.5×10^{-5}**
Water	**~1.0×10^{-6}**

For L~1 m scale sizes and V~10 m/s, Re~10^6-10^7

- **Fully developed turbulence for Re > ~10^3**
- Flow according to Navier-Stokes becomes highly nonlinear, deterministic yet unpredictable → often treat statistically

Neutral fluid turbulence energy is distributed over a wide range of scale lengths (and times)

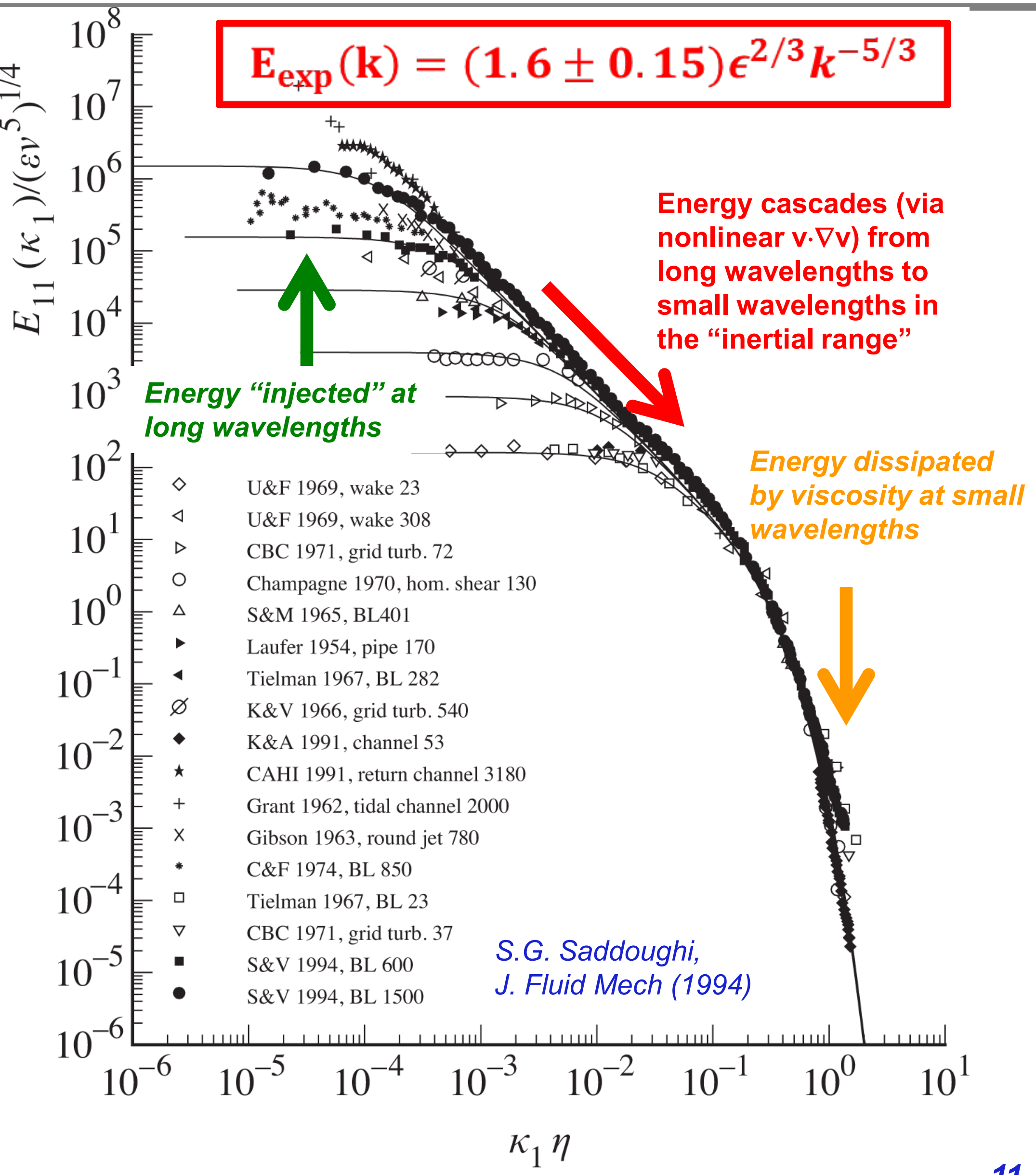
Distribution of fluctuation energy in Fourier wavenumber space exhibits Kolmogorov scaling: E(k) ~ $k^{-5/3}$

(k=2π/λ)

Assumes turbulence is homogeneous, isotropic, and large separation between energy injection (at low k) and viscous dissipation (at high k)

Ratio of forcing (injection) to Kolmogorov (dissipation) scale lengths increases with Re #

$L_{injection}/\ell_{dissipation} \sim Re^{3/4}$



$$\boxed{E_{exp}(k) = (1.6 \pm 0.15)\epsilon^{2/3}k^{-5/3}}$$

Energy cascades (via nonlinear v·∇v) from long wavelengths to small wavelengths in the "inertial range"

Energy "injected" at long wavelengths

Energy dissipated by viscosity at small wavelengths

S.G. Saddoughi, J. Fluid Mech (1994)

Concepts of turbulence to remember

- Turbulence is deterministic yet unpredictable (chaotic), appears random
  - Turbulence is not a property of the *fluid / plasma*, it's a feature of the *flow*
  - We often treat & diagnose statistically, but also rely on first-principles direct numerical simulation (DNS)

- Turbulence spans a wide range of spatial & temporal
  - Re >>> 1 for neutral fluids, $L_{\text{forcing}} / \ell_{\text{dissipation}} \sim (\text{Re})^{3/4}$
  - Or in the case of hot, low-collisionality plasma, a wide range of scales in 6D phase-space (**x**,**v**)

- Turbulence causes increased mixing, transport larger than collisional transport
  - **Transport** is the key application of why we care about turbulence (e.g. fusion gain ~ $nT\tau_E$, energy confinement time τ_E set by turbulence)

- It's cool! "Turbulence is the most important unsolved problem in classical physics" (~Feynman)

Plasma in a strong toroidal magnetic field

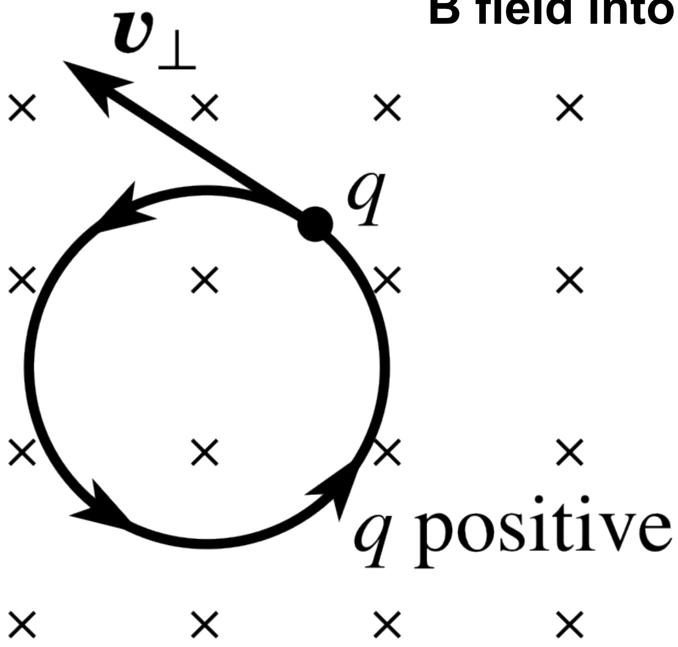
Charged particles experience Lorentz force in a magnetic field → gyro-orbits

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

- Magnetic force acts perpendicular to direction of particle

→Particles follow circular gyro-orbits

B field into plane



$$\Omega_c = \frac{eB}{m}$$

$f_c \sim 10^7$ / 10^{10} Hz

(for deuteron / electron, B=5 T)

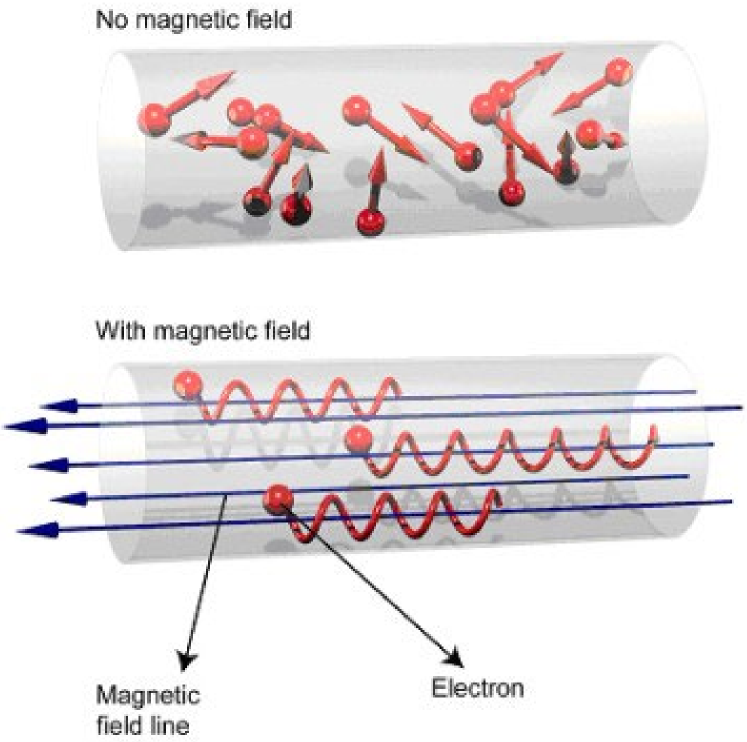
Magnetic field confines particles away from boundaries, leads to strong anisotropy

gyroradius: $\rho = \frac{v_T}{\Omega_c}$

$B \approx 5$ T

$T \approx 10$ keV

ρ_i ~ 3 mm ρ_e ~ 0.05 mm	<<	**1-2 meter device size**



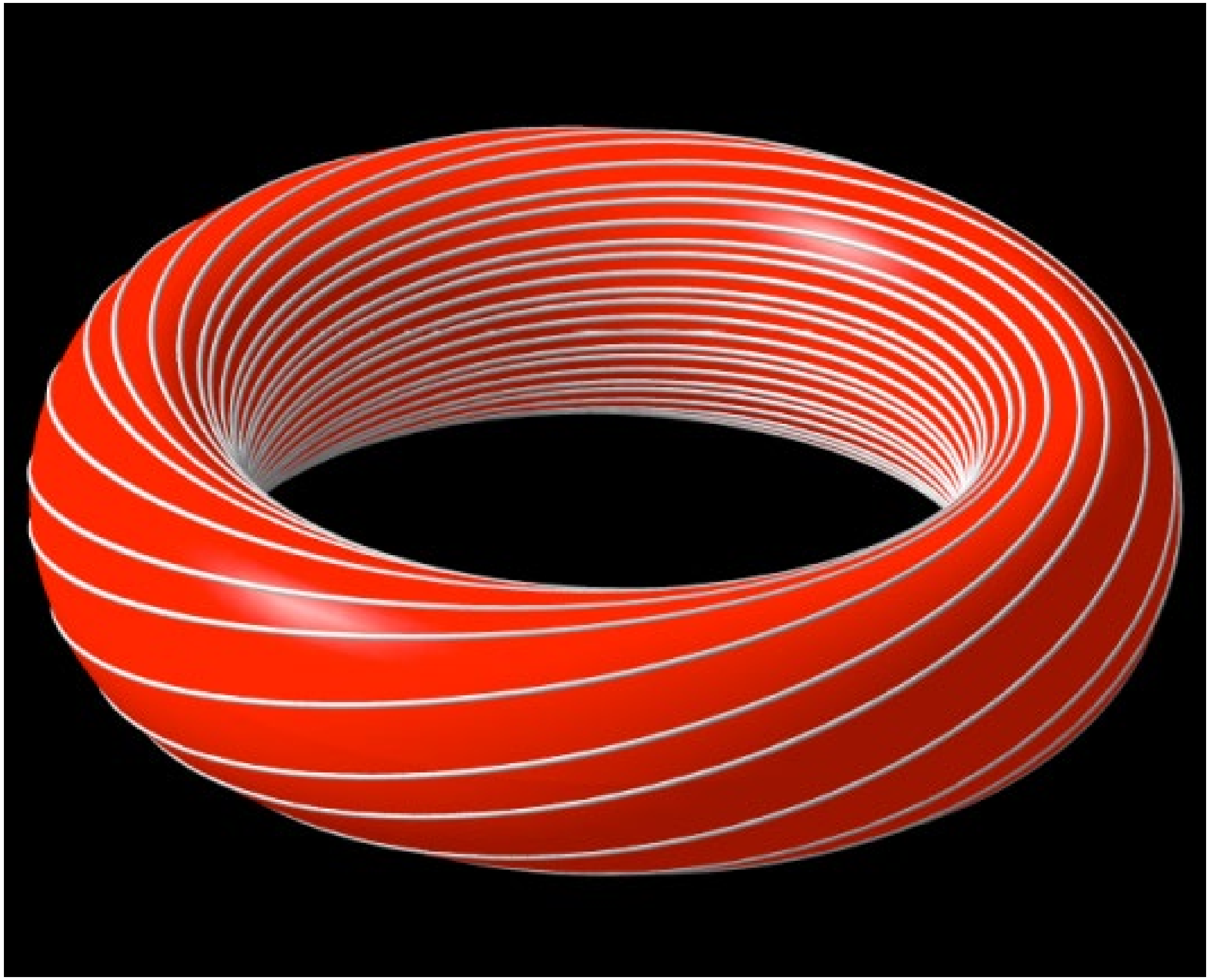
Low collision frequency $\nu \sim n/T^{3/2}$

$\lambda_{MFP} \sim$ km's >> device size

$\lambda_{MFP} / \rho_i \sim 10^6$

$\chi_{||}/\chi_{\perp} \sim (\lambda_{mfp}/\rho)^2 \sim 10^{12}$ → *strong anisotropy*

Particles easily lost from ends → bend into a torus

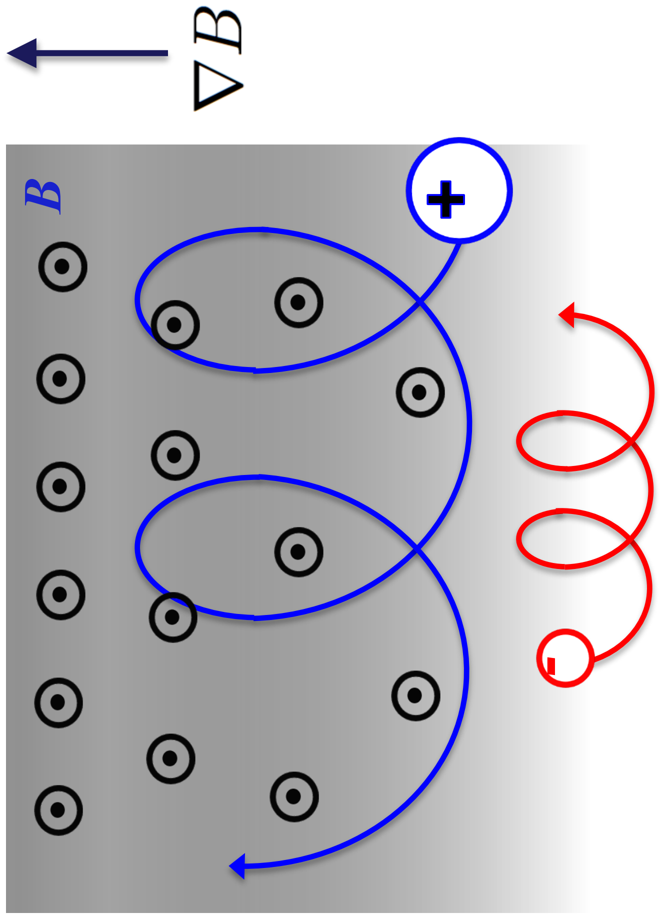


But toroidicity leads to vertical drifts from ∇B & curvature

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \qquad B \sim \frac{1}{R}$$

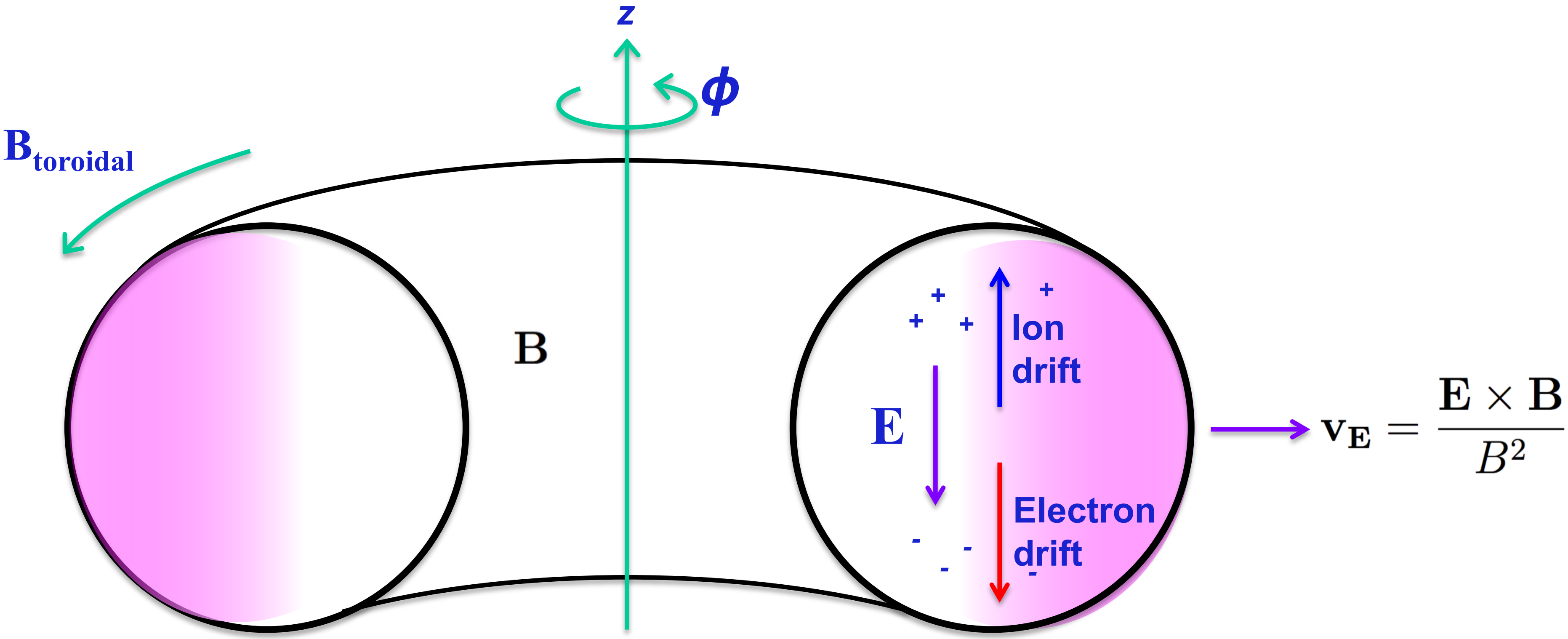
$$v_{\nabla B} \approx \left(\frac{\rho}{R}\right) v_T \approx \frac{T}{qBR}$$

τ_{loss} **~ 5 ms from vertical drifts (B~5 T, R~5 m, T~15 keV)**



$$\boxed{\rho_* = \frac{\rho}{R}}$$ **Key parameter in magnetized confinement**

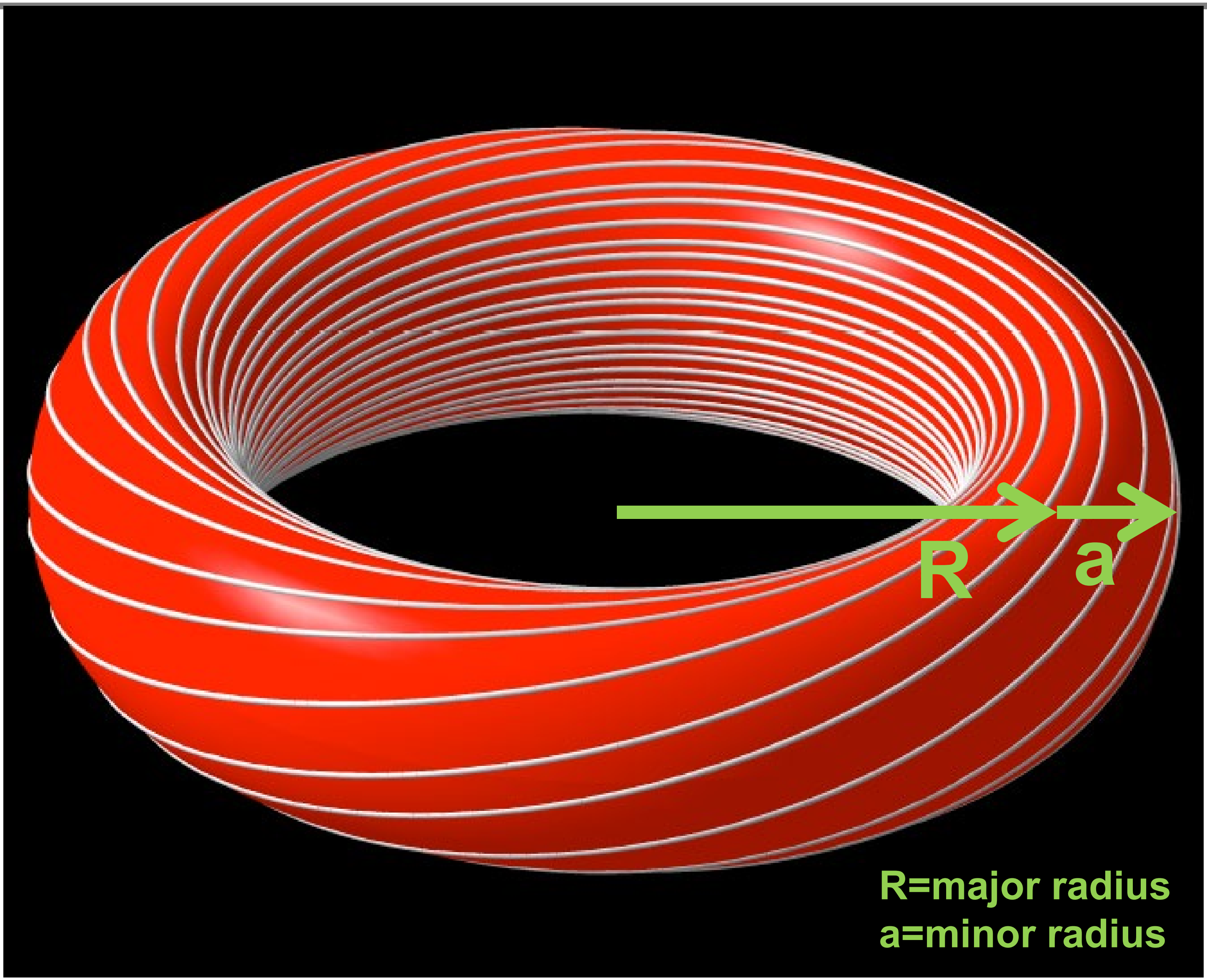
Even worse, charge separation leads to faster E×B drifts out to the walls



$$\mathbf{v_E} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

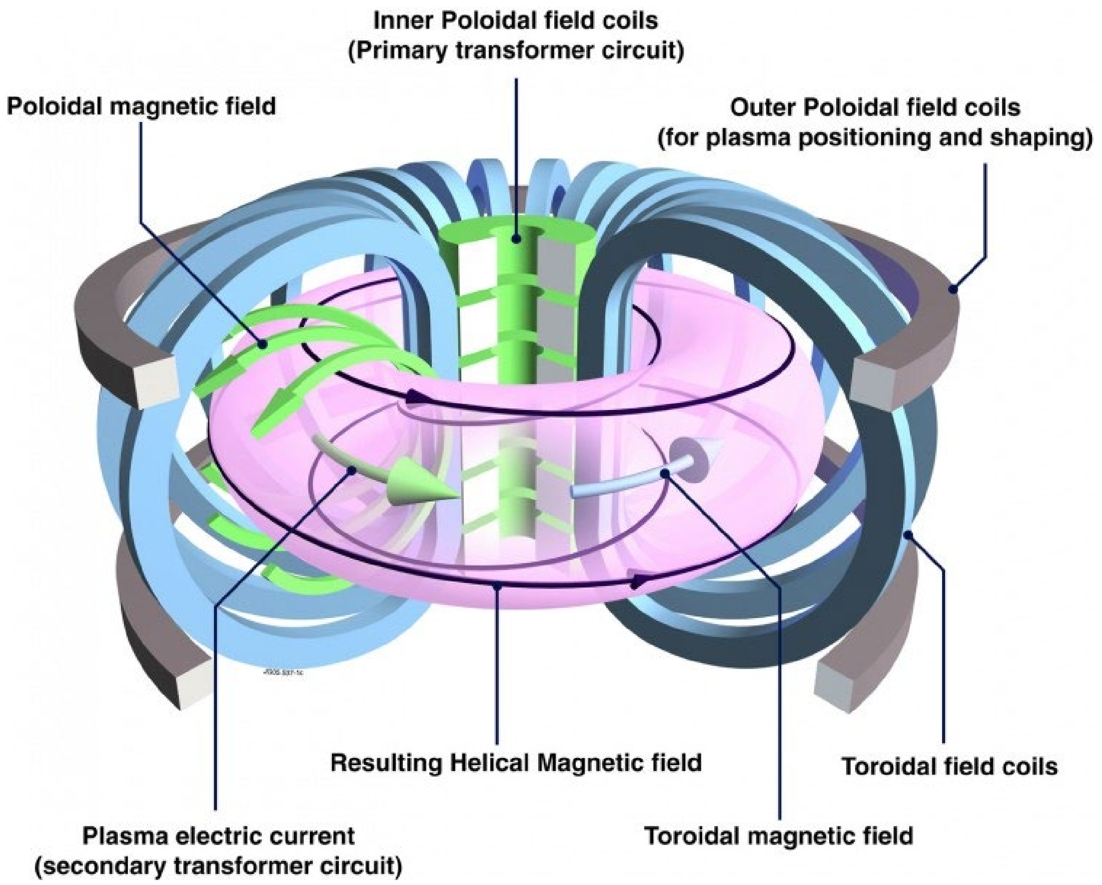
τ_{loss} ~ μs from E×B drifts (due to charge separation from vertical drifts)

Solution: need a helical magnetic field for confined (closed) particle orbits

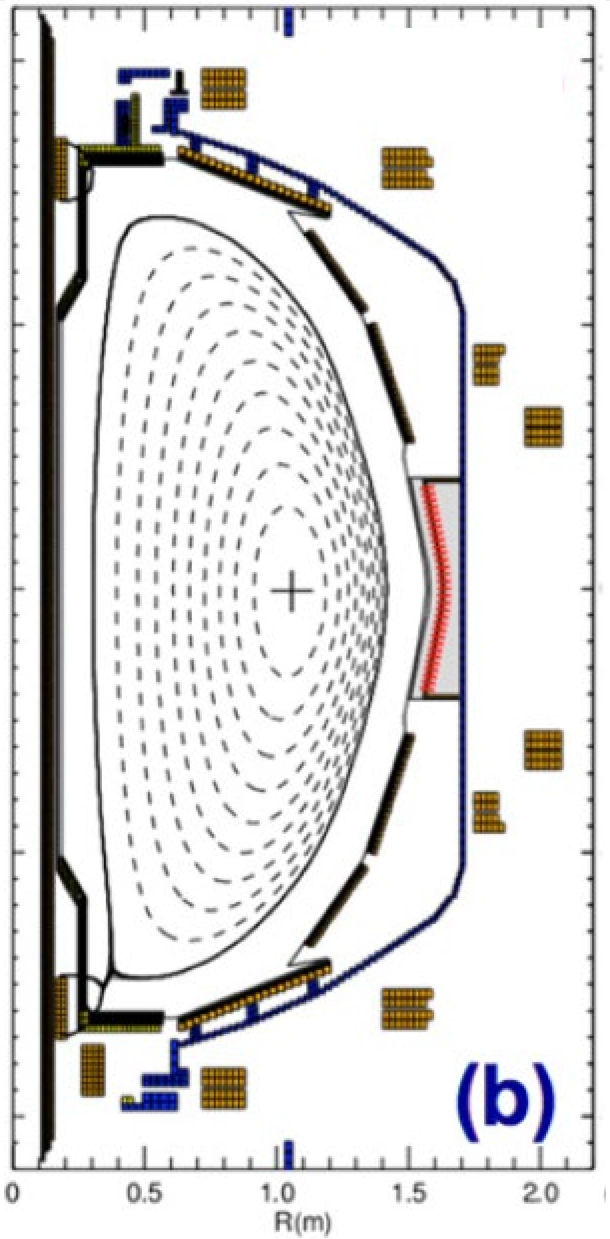


Tokamaks

- Toroidal, axisymmetric
- Helical field lines confine plasma
- Closed, nested flux surfaces in force balance: $J \times B = \nabla p$ (MHD equilibrium)



NSTX



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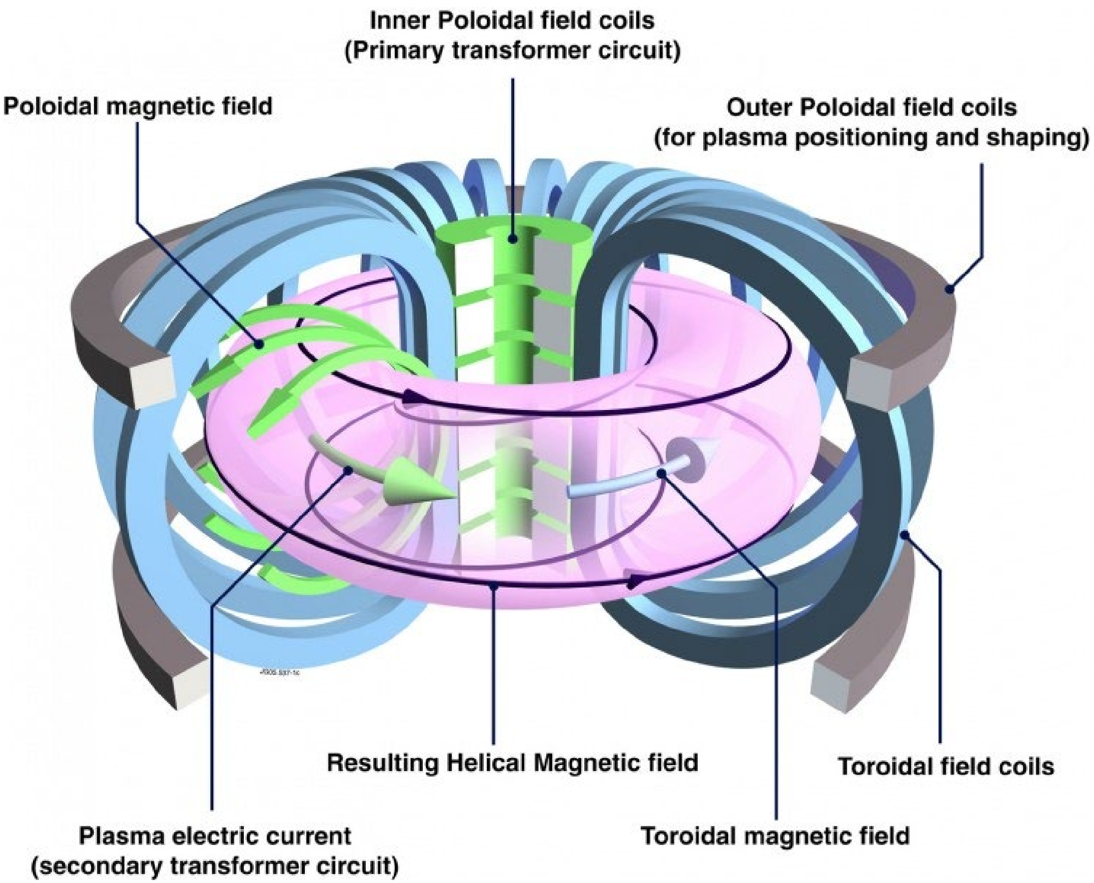

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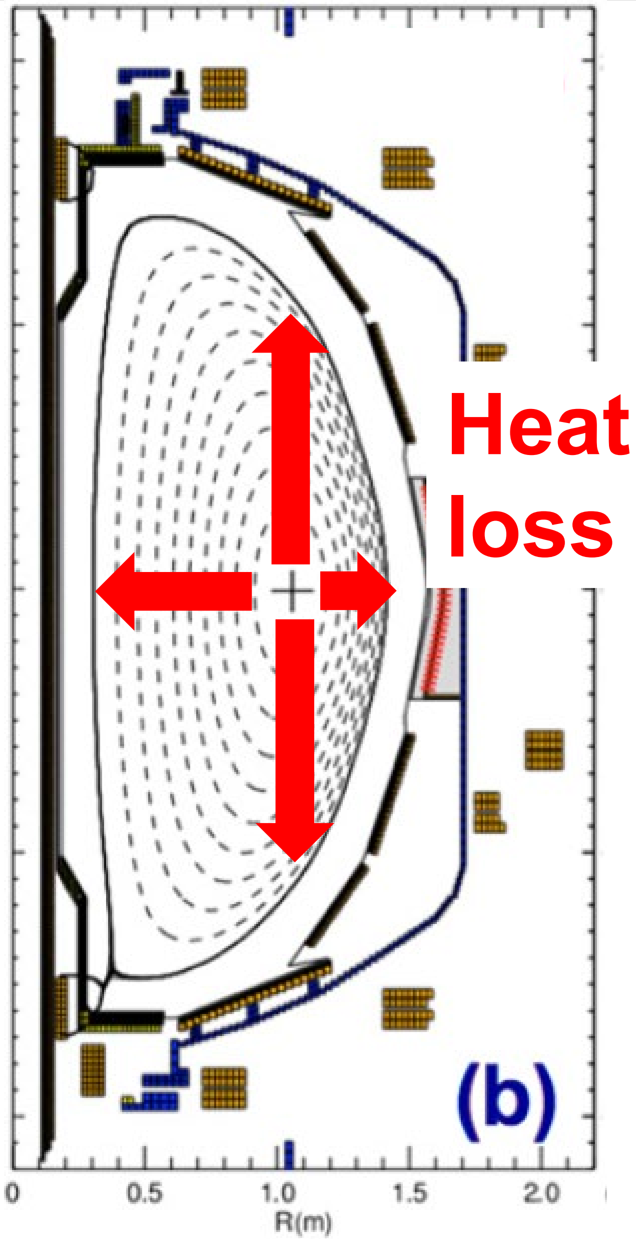
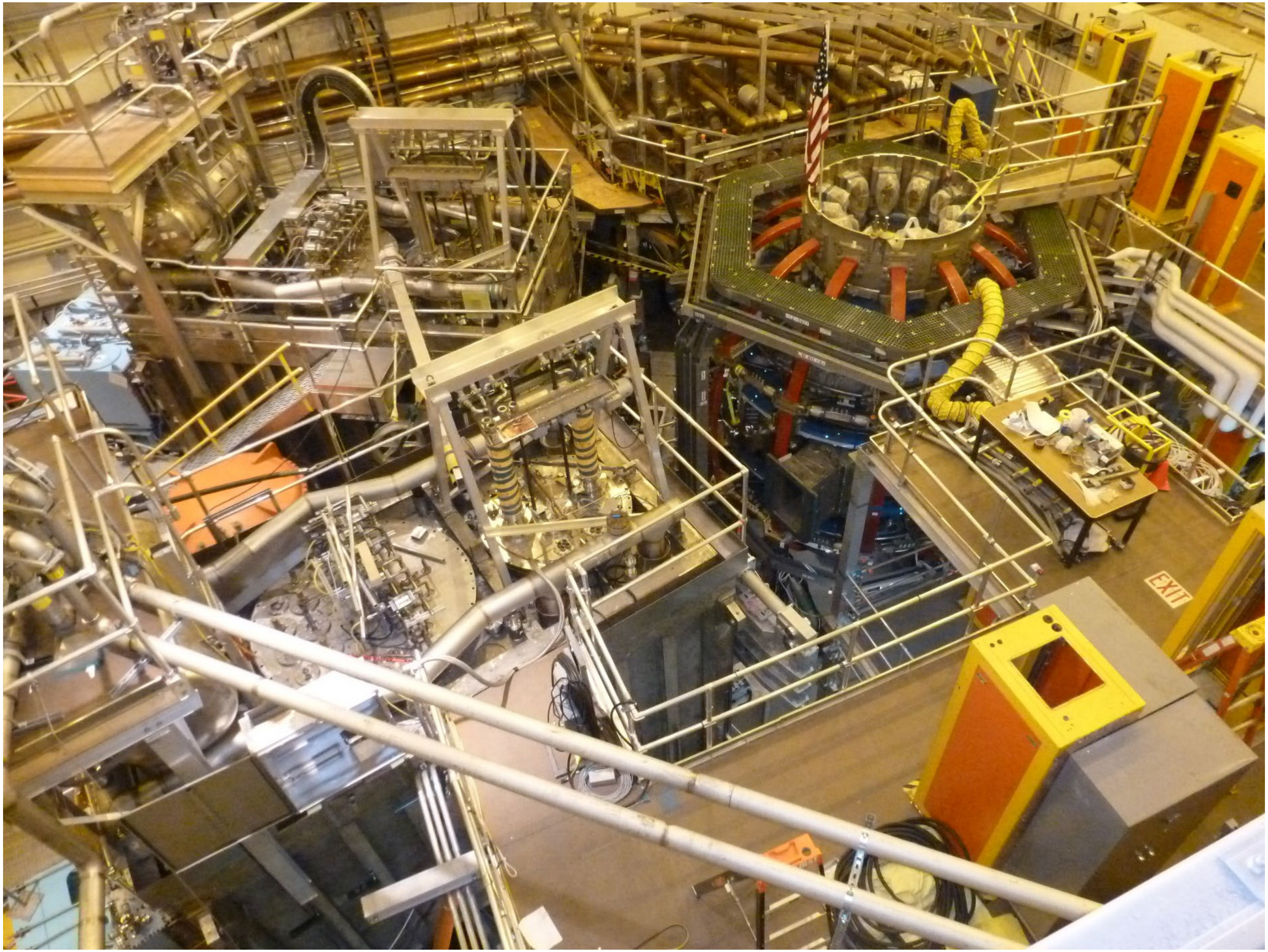


At Princeton Plasma Physics Lab (PPPL): National Spherical Torus Experiment-Upgrade (NSTX-U)



Turbulence characteristics in tokamaks

40+ years of theory & simulation predicts turbulence in magnetized plasma should often be drift wave in nature

General predicted drift wave characteristics:

- Quasi-2D, elongated along the field lines ($\lambda_{||} >> \lambda_{\perp}$, $k_{||} << k_{\perp}$)
  - Particles can rapidly move along field lines to smooth out perturbations ($v_{||} \sim v_T$)
  - Perpendicular scales linked to local gyroradius, $\lambda_{\perp} \sim \rho$ or $k_{\perp}\rho \sim 1$

    ($v_{\perp drift} \sim \rho/R \cdot v_T$)

- Finite-frequency drifting waves, $\omega(k_\theta) \sim \omega_* \sim k_\theta V_* \sim (k_\theta \rho) v_T/L_n$
  - Driven by ∇n, ∇T ($1/L_n = -1/n \cdot \nabla n$)

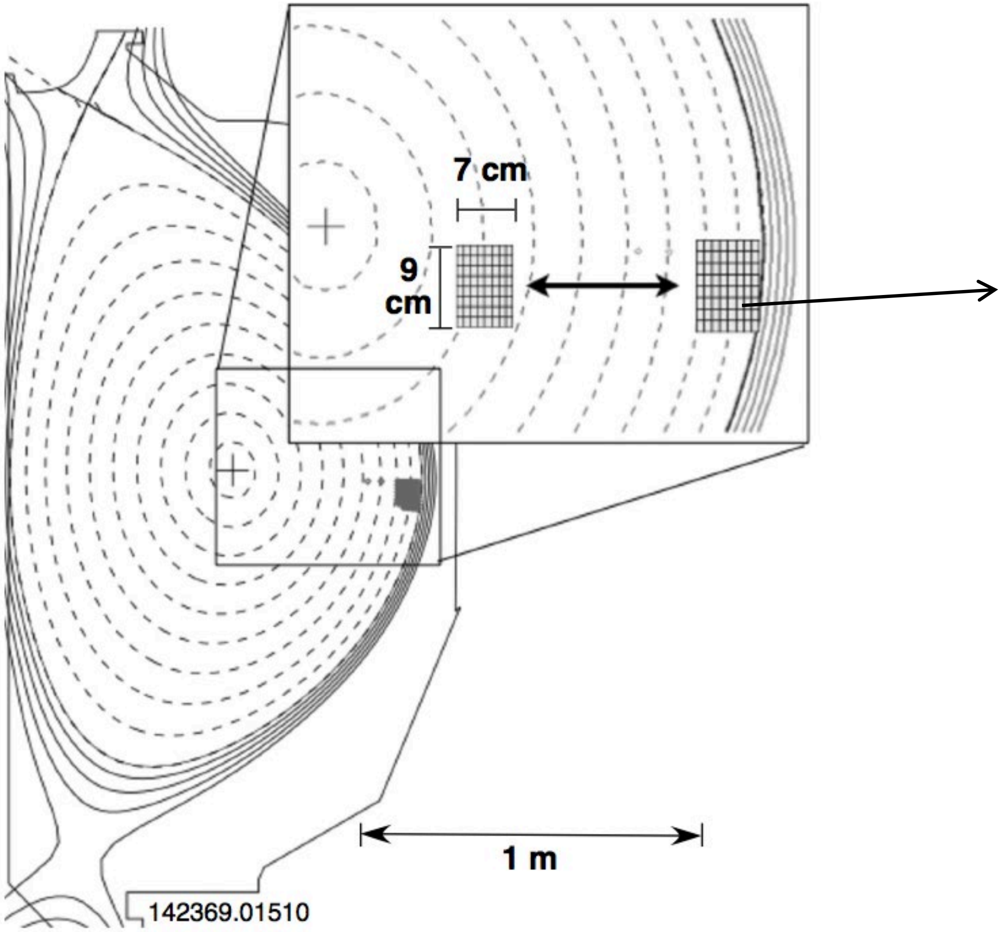
- In a tokamak expected to be “ballooning”, i.e. stronger on outboard side
  - Due to “bad curvature” / ”effective gravity” pointing outwards from symmetry axis

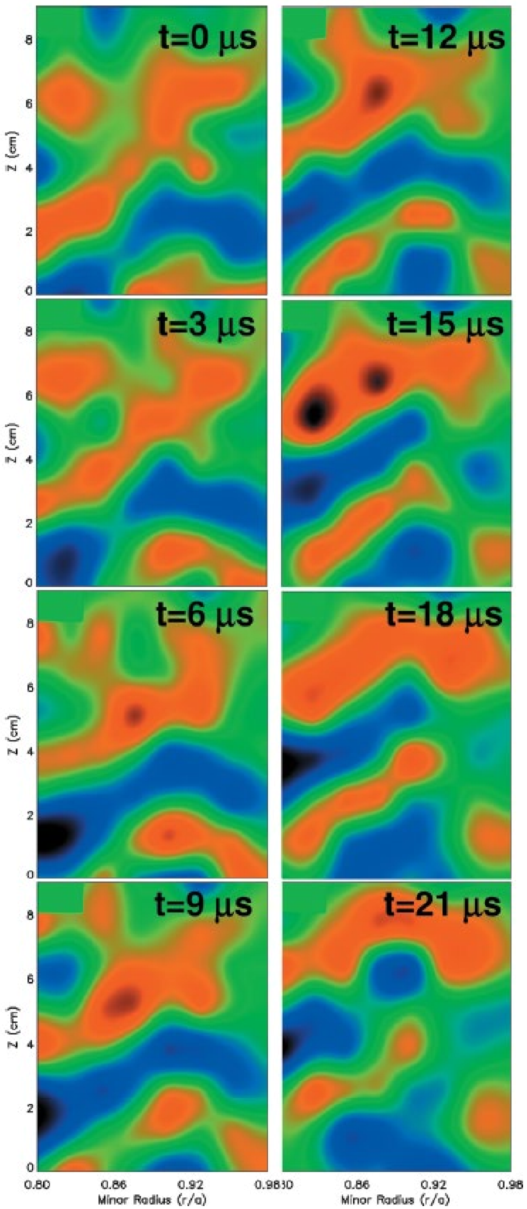
- Transport has gyrobohm scaling, $\chi_{GB} = \rho^2 v_T/R = (\boldsymbol{\rho/R}) \cdot T/B$
  - But other factors important like threshold and stiffness: $\chi_{turb} \sim \chi_{GB} \cdot F(\cdots) \cdot [R/L_T - R/L_{T,crit}]$

Spectroscopic imaging provides a 2D picture of turbulence in tokamaks: cm spatial scales, μs time scales, <1% amplitude

- Beam Emission Spectroscopy (UW-Madison) measures Doppler shifted D_α from neutral beam heating to infer plasma density

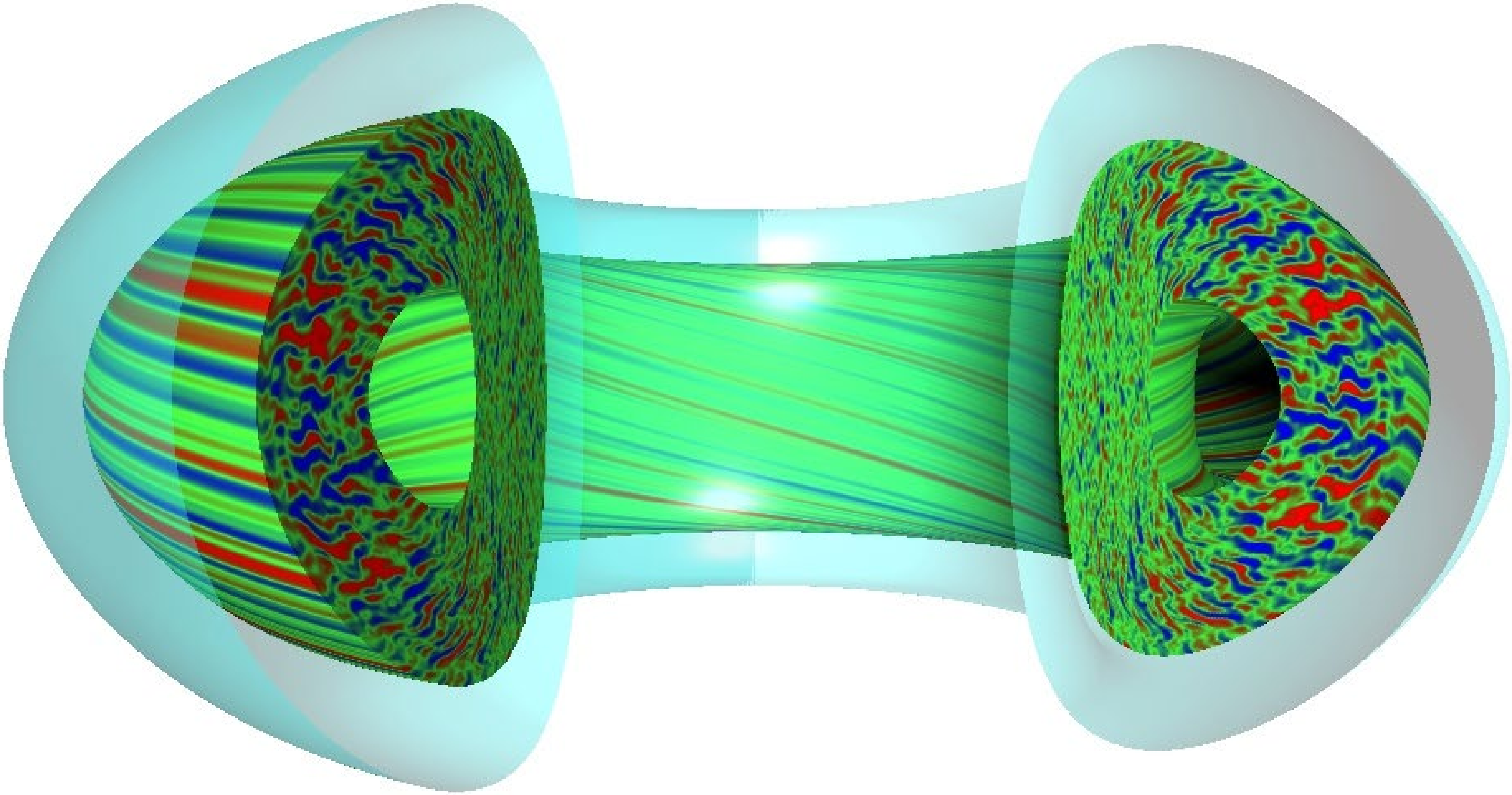
DIII-D tokamak (General Atomics)





Movies at: https://fusion.gat.com/global/BESMovies

Gyrokinetic simulations provide detailed prediction of expected turbulence characteristics



GYRO simulation (Candy, Waltz – General Atomics)

Gyrokinetic simulations provide detailed prediction of expected turbulence characteristics

Transport is of order the Gyrobohm diffusivity

- Although turbulence is advective, can estimate order of transport due to drift waves as a diffusive process, $\chi_{turb} \sim \langle \Delta x^2 \rangle / \langle \Delta t \rangle \sim (L_{\perp,corr})^2 / \tau_{corr}$

$L_{\perp,corr} \sim$ few ρ_s (~ cm's)

$\tau_{corr}^{-1} \sim c_s/R$ (~10^5 1/s)

$$\rho_s = c_s/\Omega_{ci}$$
$$c_s = \sqrt{T/m_d}$$

gyroBohm diffusivity

$$\boxed{\chi_{turb} \sim \chi_{GB} = \frac{L_\perp^2}{\tau_{corr}} = \frac{\rho_s^2 c_s}{R} = \frac{\rho_s}{R}\rho_s c_s = \frac{\rho_s}{R}\frac{T}{B}}$$

ρ_*

Bohm diffusivity $\approx \frac{1}{16}\frac{T}{B}$

$$\tau_E \sim \frac{R^2}{\chi} \sim \frac{R^3 B^2}{T^{3/2}}$$

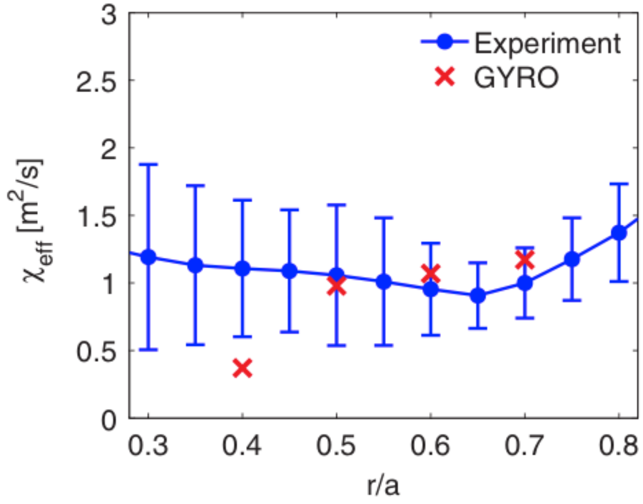
$\tau_E \sim$ (0.1) sec for current devices

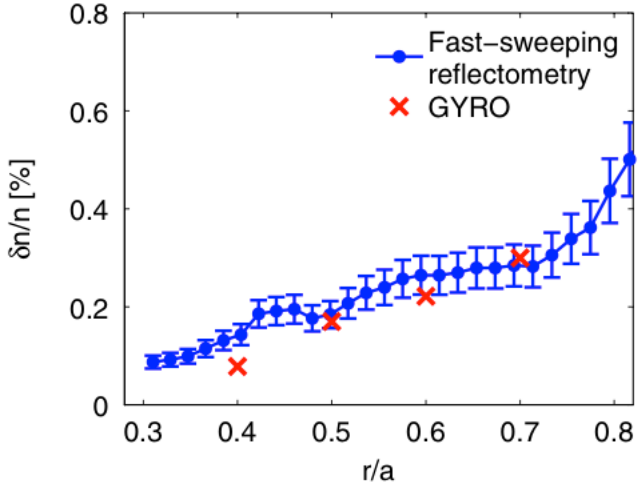
$\tau_E \sim$ (1+) sec for fusion gain (ITER)

- τ_E improves with field strength (B) and machine size (R)

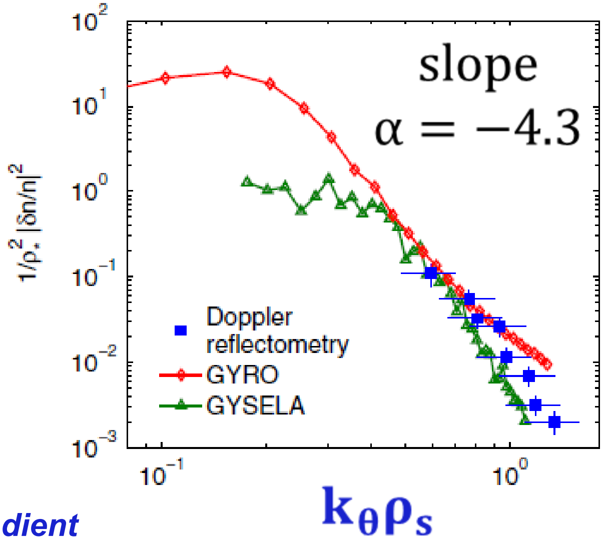
Transport, density fluctuation amplitude (from reflectometry) and spectral characteristics predicted by nonlinear gyrokinetic simulations

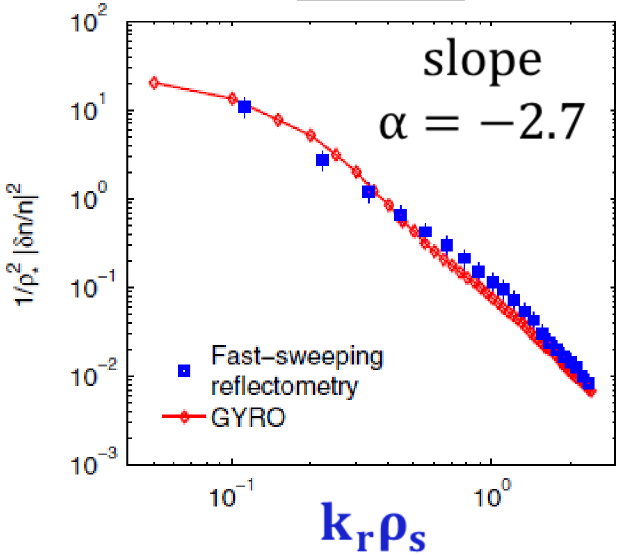
- Example is for ion temperature gradient (ITG) turbulence



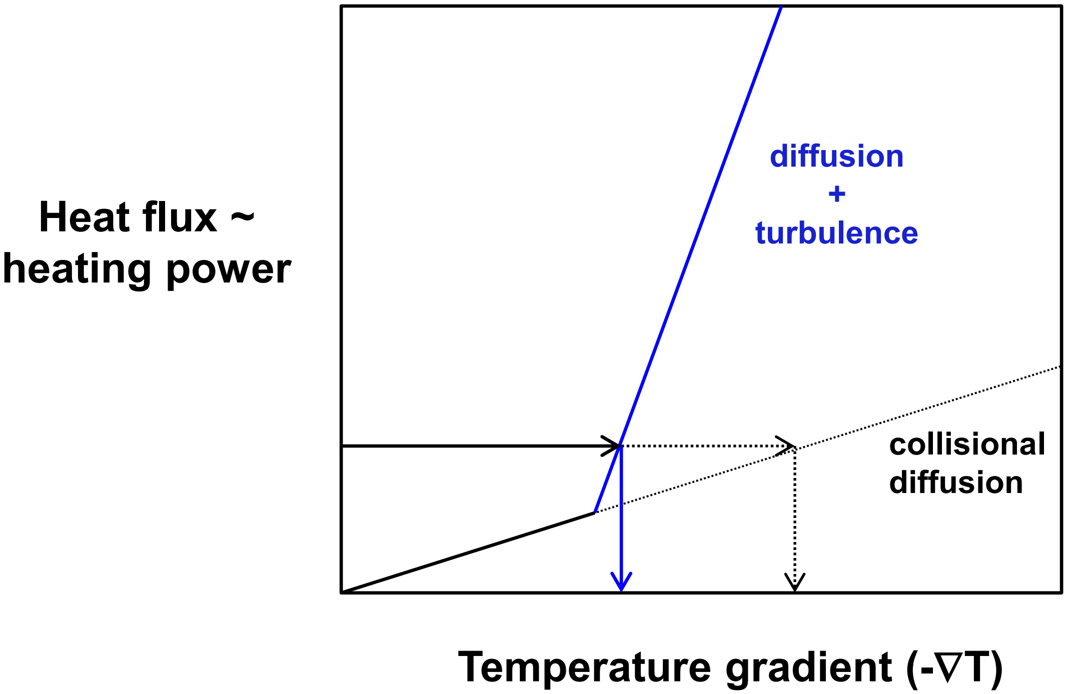


Casati, PRL (2009)
Tore Supra tokamak
Ion Temperature Gradient (ITG) turbulence





Tokamak turbulence has a threshold gradient for onset, transport tied to linear stability and nonlinear saturation



$$q_{turb} = -n\chi_{GB}[\nabla T - \nabla T_{crit}]F(\cdots)$$

$$q_{col} = -n\chi_{col}\nabla T$$

- GyroBohm scaling important, but linear threshold and scaling also matters

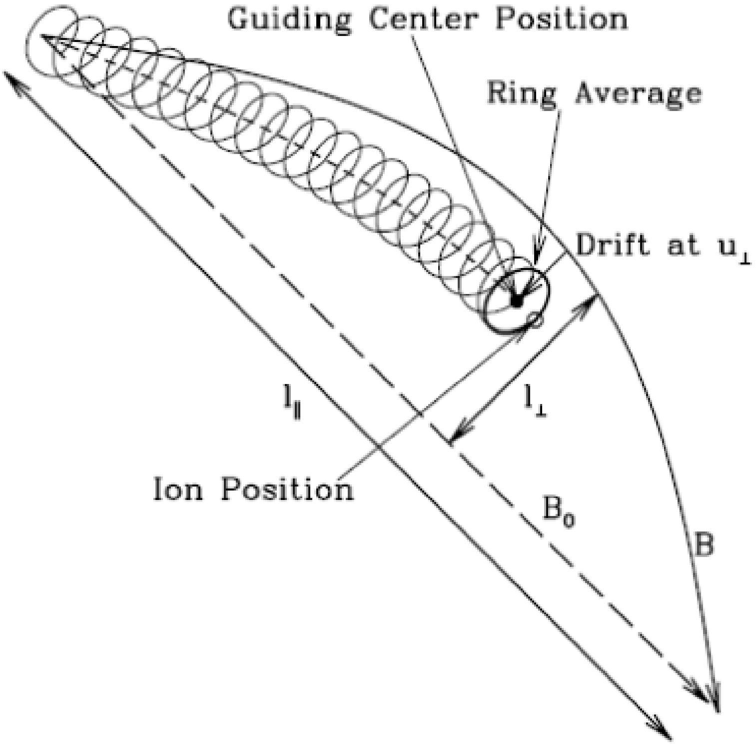
⇒ We must discuss linear drift wave and micro-stability in tokamaks as part of the turbulent transport problem (enter gyrokinetic theory)

Gyrokinetics in brief – evolving 5D gyro-averaged distribution function

$$\boxed{\frac{\omega}{\Omega} << 1}$$

$$f(\vec{x}, \vec{v}, t) \xrightarrow{\text{gyroaverage}} f(\vec{R}, v_{\parallel}, v_{\perp}, t)$$

$\sim \frac{10^5}{10^7}$ ***for ions***



- Average over fast gyro-motion → evolve a distribution of gyro-rings

(for each species)

Howes et al., Astro. J. (2006)

Gyrokinetics in brief – evolving 5D gyro-averaged distribution function

$$\frac{\omega}{\Omega},\frac{\rho}{L},\frac{\delta f}{f_0},\frac{k_{||}}{k_\perp} << 1 \qquad f(\vec{x},\vec{v},t)\xrightarrow{\text{gyroaverage}} f(\vec{R},v_\parallel,v_\perp,t) \qquad f = F_M + \delta f$$

$$\frac{\partial(\delta f)}{\partial t} + \underbrace{v_\parallel \hat{b}\cdot\nabla\delta f}_{\text{Fast parallel motion}} + \underbrace{\vec{v}_d\cdot\nabla\delta f}_{\text{Slow perpendicular toroidal drifts}} + \underbrace{\delta\vec{v}\cdot\nabla F_M}_{\text{Advection across equilibrium gradients } (\nabla T_0,\ \nabla n_0,\ \nabla V_0)} + \underbrace{\vec{v}_{E0}(r)\cdot\nabla\delta f}_{\text{Dopper shift due to sheared equilibrium } E_r(r)} + \underbrace{\delta\vec{v}\cdot\nabla\delta f}_{\text{Perpendicular non-linearity}} = \underbrace{C(\delta f)}_{\text{Collisions}}$$

$$\boxed{\begin{aligned}\vec{v}_\kappa &= m v_{||}^2\frac{\hat{b}\times\vec{\kappa}}{qB}\\ \vec{v}_{\nabla B} &= \frac{m v_\perp^2}{2}\frac{\hat{b}\times\nabla B/B}{qB}\end{aligned}}$$

$$\delta\mathbf{v}_a \doteq \frac{c}{B}\mathbf{b}\times\nabla\Psi_a$$

$$\Psi_a(\mathbf{R}) \doteq \left\langle \delta\phi(\mathbf{R}+\rho) - \frac{1}{c}(\mathbf{V}_0+\mathbf{v})\cdot\delta\mathbf{A}(\mathbf{R}+\rho)\right\rangle_{\mathbf{R}}$$

- Must also solve gyrokinetic Maxwell equations self-consistently to obtain δφ, δB

Drift waves

Can identify key terms in gyrofluid equations responsible for drift wave dynamics

- Start with toroidal GK equation in the δf limit (δf/F_M << 1)
- Take fluid moments ($\int d^3v\, \delta f\ [1, v, \frac{1}{2}v^2]$)
- Apply clever closures that “best” reproduce linear toroidal gyrokinetics (Hammett, Perkins, Beer, Dorland, Waltz, Staebler, …), e.g.:

ion continuity and energy (M. Beer thesis, 1995):

$$\frac{\partial \tilde{n}}{\partial t} + \mathbf{v}_E \cdot \nabla n_0 + \mathbf{v}_E \cdot \nabla \tilde{n} + n_0 \nabla \cdot \mathbf{v}_E + \frac{2}{m\Omega B^2}\mathbf{B} \times \nabla B \cdot \nabla \tilde{p} = 0. \qquad (1.5)$$

$$\frac{\partial \tilde{p}}{\partial t} + \mathbf{v}_E \cdot \nabla p_0 + \mathbf{v}_E \cdot \nabla \tilde{p} + p_0 \nabla \cdot \mathbf{v}_E + p_0 B \mathbf{v}_E \cdot \nabla \frac{1}{B} + \frac{p_0}{n_0 m\Omega B^2}\mathbf{B} \times \nabla B \cdot \nabla (6\tilde{p} - 3\tilde{n}T_0) = 0. \qquad (1.12)$$

Can identify key terms in gyrofluid equations responsible for drift wave dynamics

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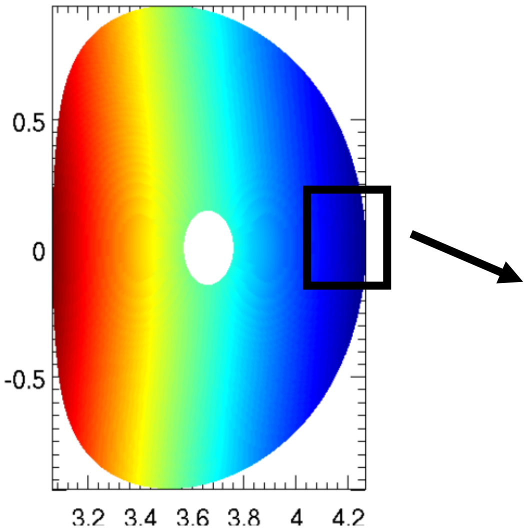
- Perturbed E×B drift + background gradients ($\delta v_E \cdot \nabla n_0$, $\delta v_E \cdot \nabla T_0$) are fundamental to drift wave dynamics, lead to finite frequency $\omega(k_\theta) \sim \omega_*$
- Toroidicity (curvature≈∇B/B≈1/R) enables “toroidal” drift instabilities like Ion Temperature Gradient (ITG) instability

Simple classic electron drift wave in a magnetic slab ($B=B_Z$)

- Assume cool ions ($v_{Ti} << \omega/k_{||}$), isothermal electrons, no temperature gradients, no toroidicity, electrostatic (β→0), no nonlinear term

$$\frac{\partial}{\partial t}\delta n_i + \delta v_E \cdot \nabla n_0(x) = 0 \qquad \textit{ion continuity}$$

Ignore |B| contours for now



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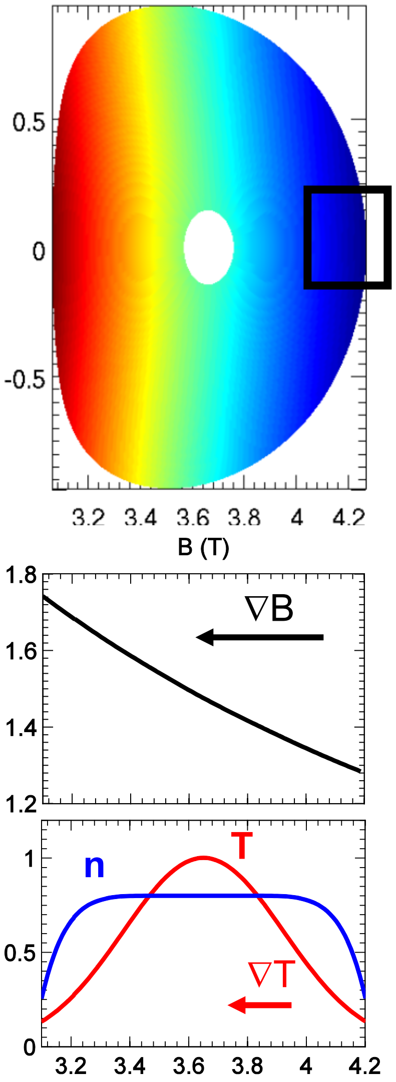
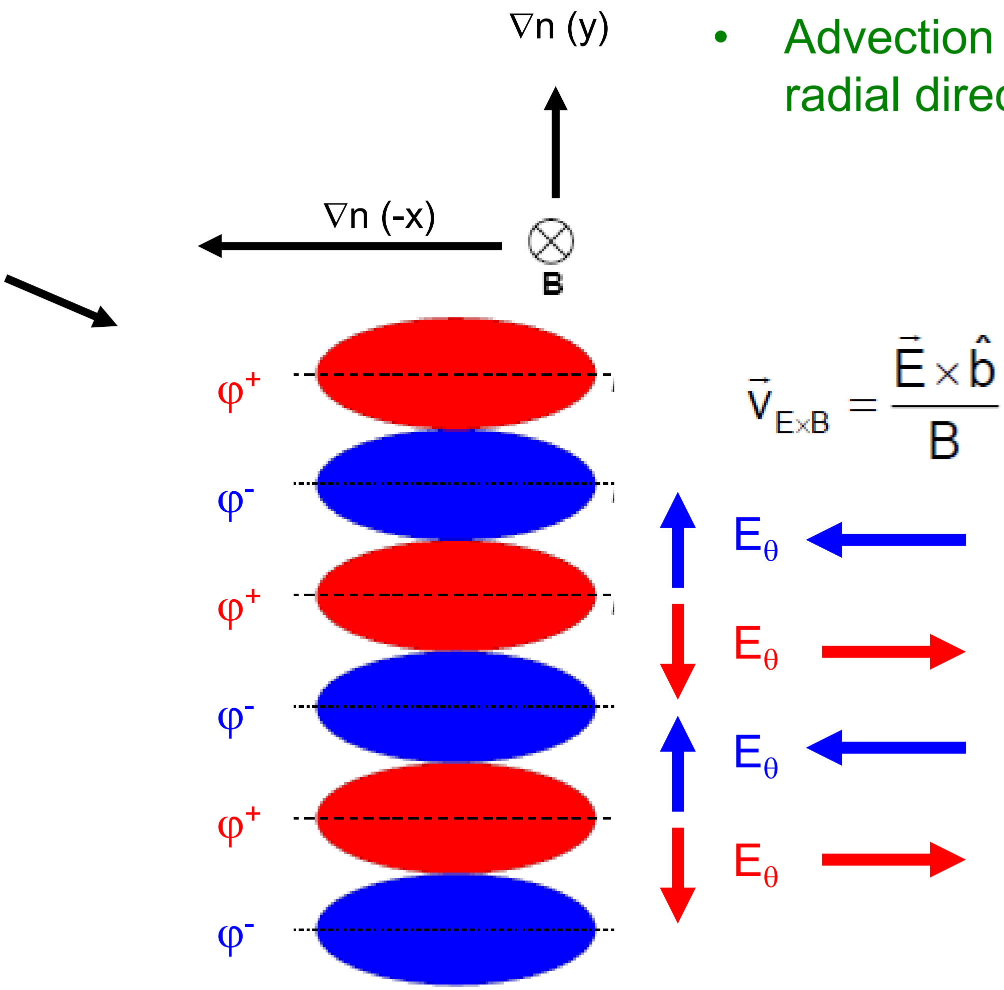
$$\frac{\partial}{\partial t}\delta n_i + \delta v_E \cdot \nabla n_0(x) = 0$$ ***ion continuity***

$$\delta v_E = \frac{\hat{b} \times \nabla \delta\phi}{B} = \frac{-ik_y \delta\phi}{B}\widehat{e_x}$$

$$\delta\phi \sim \exp\left(i\vec{k} \cdot \vec{x} - i\omega t\right)$$

Perturbed Potential Creates E×B Advection





- Advection occurs in the radial direction

$$\vec{v}_{E\times B} = \frac{\vec{E}\times\hat{b}}{B}$$

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- Assume cool ions ($v_{Ti} << \omega/k_{||}$), isothermal electrons, no temperature gradients, no toroidicity, electrostatic ($\beta \rightarrow 0$), no nonlinear term

$$\frac{\partial}{\partial t}\boldsymbol{\delta n_i} + \boldsymbol{\delta v_E} \cdot \boldsymbol{\nabla n_0(x)} = \mathbf{0}$$ ***ion continuity***

$$\delta v_E = \frac{\hat{b} \times \nabla\delta\phi}{B} = \frac{-ik_y\delta\phi}{B}\widehat{e_x}$$

$$\delta\phi \sim \exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right)$$

Gradient scale length (L_n)

$$\frac{dn_0}{dx} = -\frac{n_0}{L_n}$$

$$\delta v_E \cdot \nabla n_0(x) = \frac{-ik_y\delta\phi}{B}\frac{dn_0}{dx} = in_0\frac{k_y\delta\phi}{BL_n}$$

$$\delta v_E \cdot \nabla n_0(x) = in_0k_y\frac{T_e}{BL_n}\frac{\delta\phi}{T_e}$$

With some algebra we obtain a diamagnetic drift velocity & frequency

$$\delta v_E \cdot \nabla n_0(x) = i n_0 k_y \frac{T_e}{B L_n} \frac{\delta\phi}{T_e}$$

$$\frac{T_e}{B} = \rho_s c_s$$

$$\delta v_E \cdot \nabla n_0(x) = i n_0 k_y \frac{\rho_s}{L_n} c_s \frac{\delta\phi}{T_e} = i n_0 \omega_{*e} \frac{\delta\phi}{T_e}$$

$$\omega_{*e} = k_y V_{*e} \qquad V_{*e} = \underset{\uparrow}{\frac{\rho_s}{L_n}} c_s$$

ρ* like parameter

Electron diamagnetic drift velocity & frequency (a fluid drift, not a particle drift)

Simplified ion continuity equation

$$\frac{\partial}{\partial t}\delta n_i + \delta v_E \cdot \nabla n_0(x) = 0$$

$$-i\omega \frac{\delta n_i}{n_0} + i\omega_{*e} \frac{\delta \phi}{T_e} = 0$$

- Expect characteristic frequency ~ ω_{*e} ~ $(k_y \rho_s) \cdot c_s/L_n$

Need to add additional cartoon pictures here for classic electron drift wave

Dynamics Must Satisfy Quasi-neutrality; Rapid parallel electron motion gives Boltzmann distribution

- Quasi-neutrality (Poisson equation, $k_\perp^2 \lambda_D^2 << 1$) requires

$$-\nabla^2 \tilde{\varphi} = \frac{1}{\varepsilon_0} \sum_s eZ_s \int d^3 v f_s$$

$$\left(k_\perp^2 \lambda_D^2\right) \frac{\tilde{\varphi}}{T} = \frac{\tilde{n}_i - \tilde{n}_e}{n_0}$$

$$\boxed{\tilde{n}_i = \tilde{n}_e}$$

- For characteristic drift wave frequency, parallel electron motion is very rapid -- from parallel electron momentum eq, assuming isothermal T_e:

$$\omega < k_{||} v_{Te} \;\rightarrow\; 0 = -T_e \nabla_{||} \tilde{n}_e + n_e e \nabla_{||} \tilde{\phi}$$

$\Rightarrow$ Electrons (approximately) maintain a Boltzmann distribution

$$\left(n_0 + \tilde{n}_e\right) = n_0 \exp\left(e\tilde{\varphi} / T_e\right)$$

$$\tilde{n}_e \approx n_0 e \tilde{\varphi} / T_e \Rightarrow \underline{\tilde{n}_e \approx \tilde{\varphi}}$$

Ion continuity + quasi-neutrality + Boltzmann electron = electron drift wave (linear, slab, cold ions)

$$-i\omega\frac{\delta n_i}{n_0}+i\omega_{*e}\frac{\delta\phi}{T_e}=0$$

$$\boxed{\omega=\omega_{*e}=k_y V_{*e}}$$

$$\frac{\delta n_i}{n_0}=\frac{\delta n_e}{n_0}=\frac{\delta\phi}{T_e}$$

- Density and potential wave perturbations propagating perpendicular to B_Z and ∇n_0
  - $\delta v_E \cdot \nabla n_0$ gives δn 90° out-of-phase with initial δn perturbation
- Simple linear dispersion relation (will change with polarization drift / finite Larmor radius effects, toroidicity, other gradients)
- **No mechanism to drive instability (collisions, temperature gradient, toroidicity / trapped particles, …)**

Linear stability analysis of toroidal Ion Temperature Gradient (ITG) micro-instability

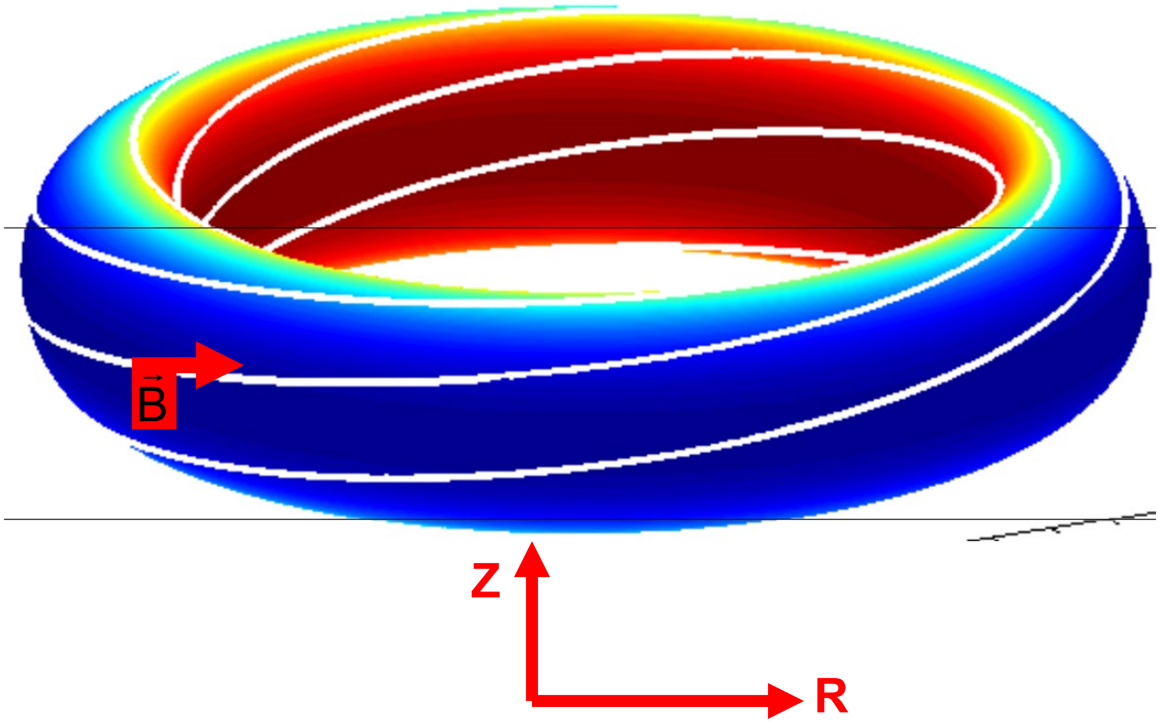
(expected to dominate in many tokamak plasmas)

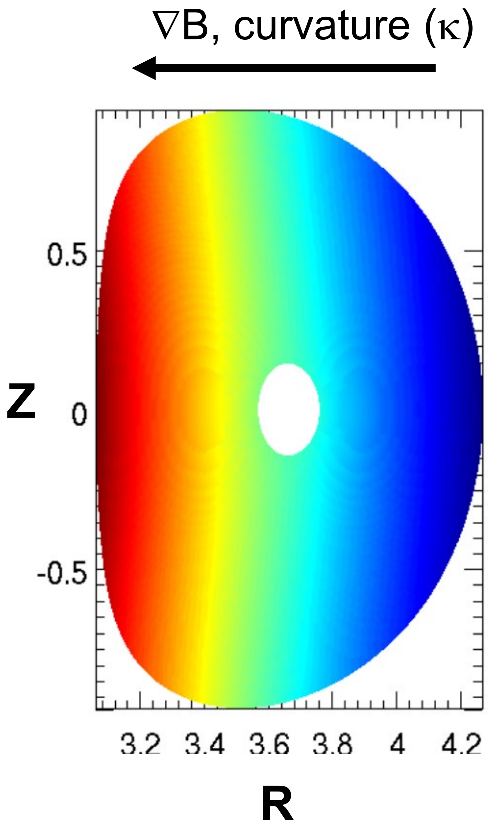
Toroidicity Leads To Inhomogeneity in |B|, gives ∇B and curvature (κ) drifts

$$\vec{v}_{\kappa} = m v_{\parallel}^{2} \frac{\hat{b}\times\vec{\kappa}}{qB} \sim T_{\parallel}$$

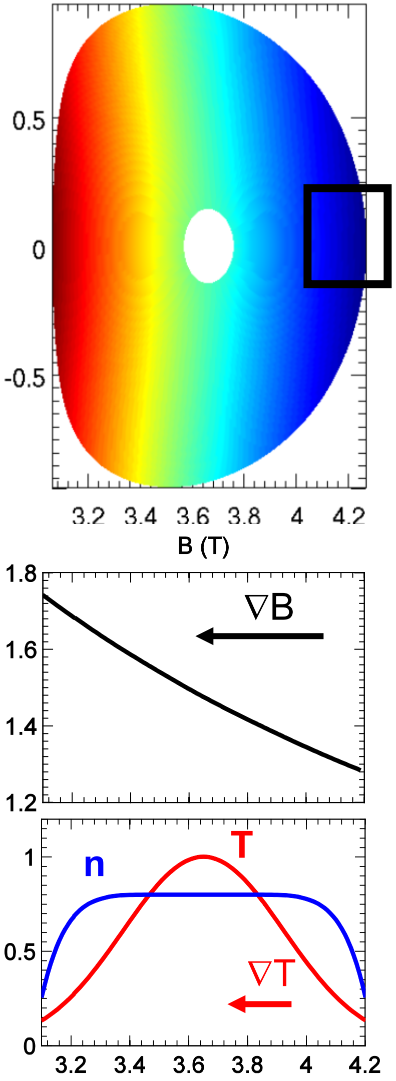
$$\vec{v}_{\nabla B} = \frac{m v_{\perp}^{2}}{2} \frac{\hat{b}\times\nabla B / B}{qB} \sim T_{\perp}$$

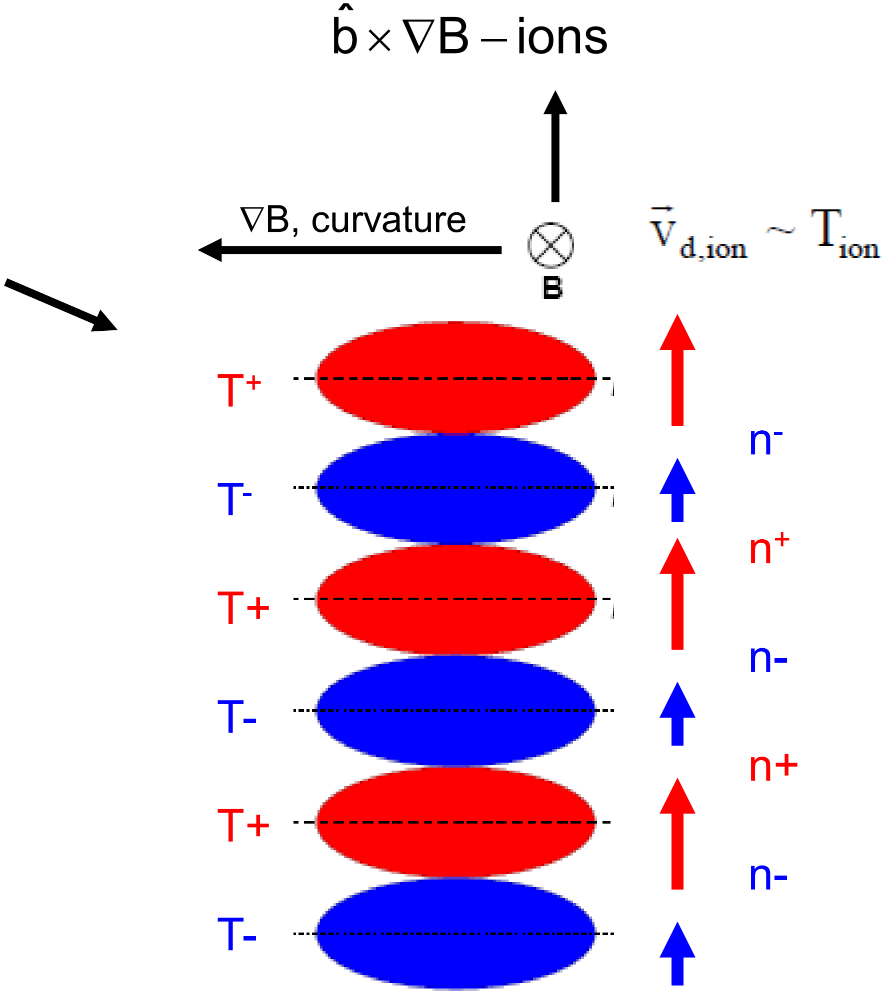
- What happens when there are small perturbations in $T_{\parallel}$, $T_{\perp}$? ⇒ Linear stability analysis…





Temperature perturbation (δT) leads to compression ($\nabla \cdot v_{di}$), density perturbation – 90° out-of-phase with δT





- Fourier decompose perturbations in space ($k_\theta \rho_i \leq 1$)
- Assume small δT perturbation

Dynamics Must Satisfy Quasi-neutrality; Rapid parallel electron motion gives Boltzmann distribution

- Poisson equation + long-wavelengths ($k_\perp^2\lambda_D^2$<<1) → quasi-neutrality

$$-\nabla^2\tilde{\varphi} = \frac{1}{\varepsilon_0}\sum_s eZ_s \int d^3v f_s$$

$$\boxed{\tilde{n}_i = \tilde{n}_e}$$

$$\left(k_\perp^2\lambda_D^2\right)\frac{\tilde{\varphi}}{T} = \frac{\tilde{n}_i - \tilde{n}_e}{n_0}$$

- For characteristic drift wave frequency, parallel electron motion is very rapid (from parallel electron momentum eq, assuming isothermal T_e:)

$$\omega < k_{||}v_{Te} \;\rightarrow\; 0 = -T_e\nabla_{||}\tilde{n}_e + n_e e\nabla_{||}\tilde{\phi}$$

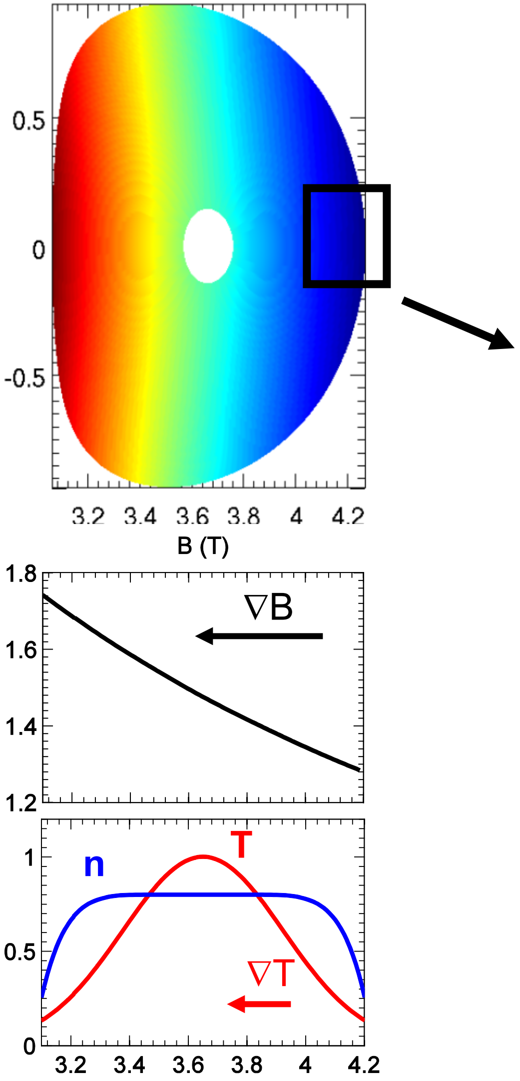
$\Rightarrow$ Electrons (approximately) maintain a Boltzmann distribution

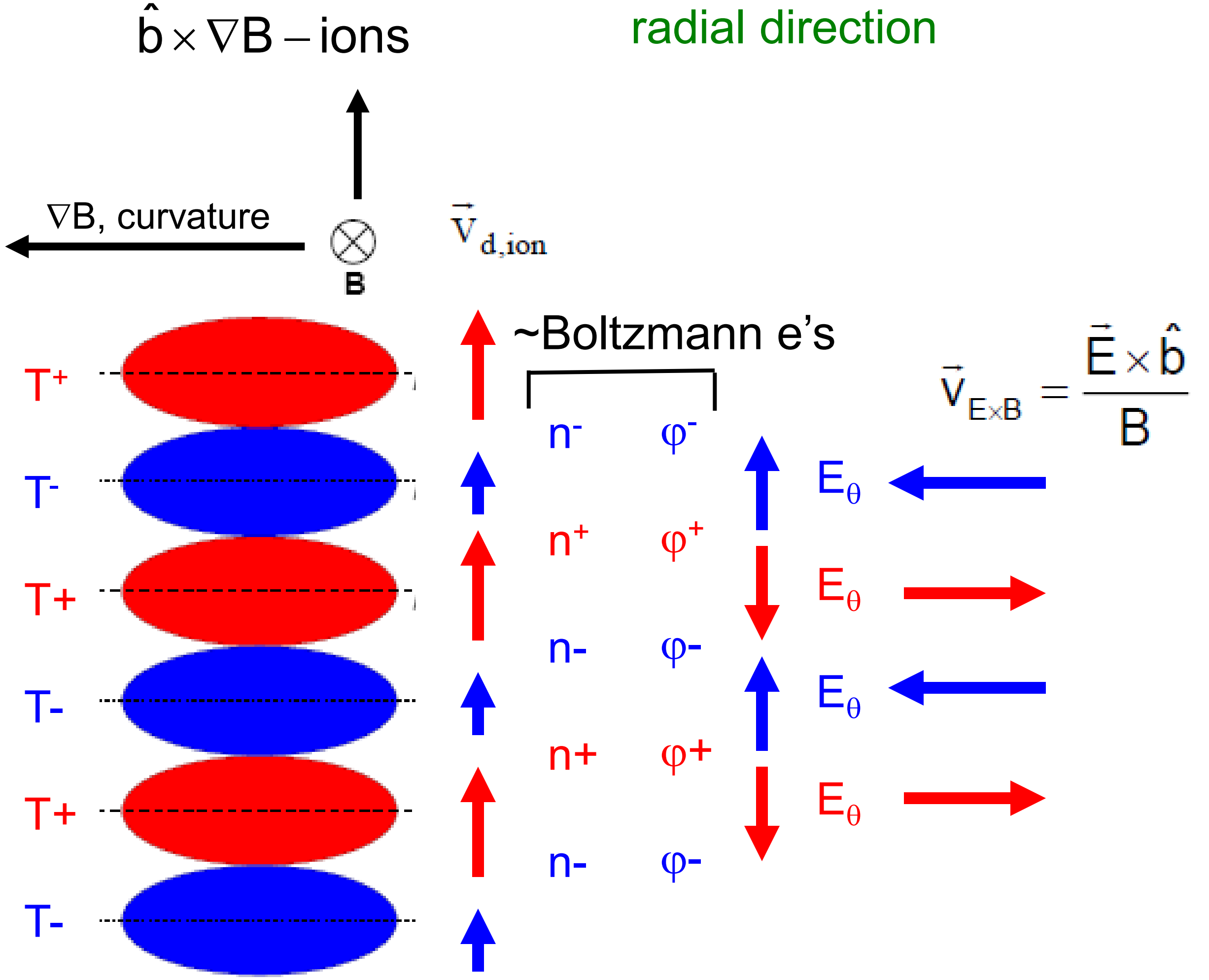
$$\left(n_0 + \tilde{n}_e\right) = n_0 \exp\left(e\tilde{\varphi}/T_e\right)$$

$$\tilde{n}_e \approx n_0 e\tilde{\varphi}/T_e \Rightarrow \underline{\tilde{n}_e \approx \tilde{\varphi}}$$

Perturbed Potential Creates E×B Advection

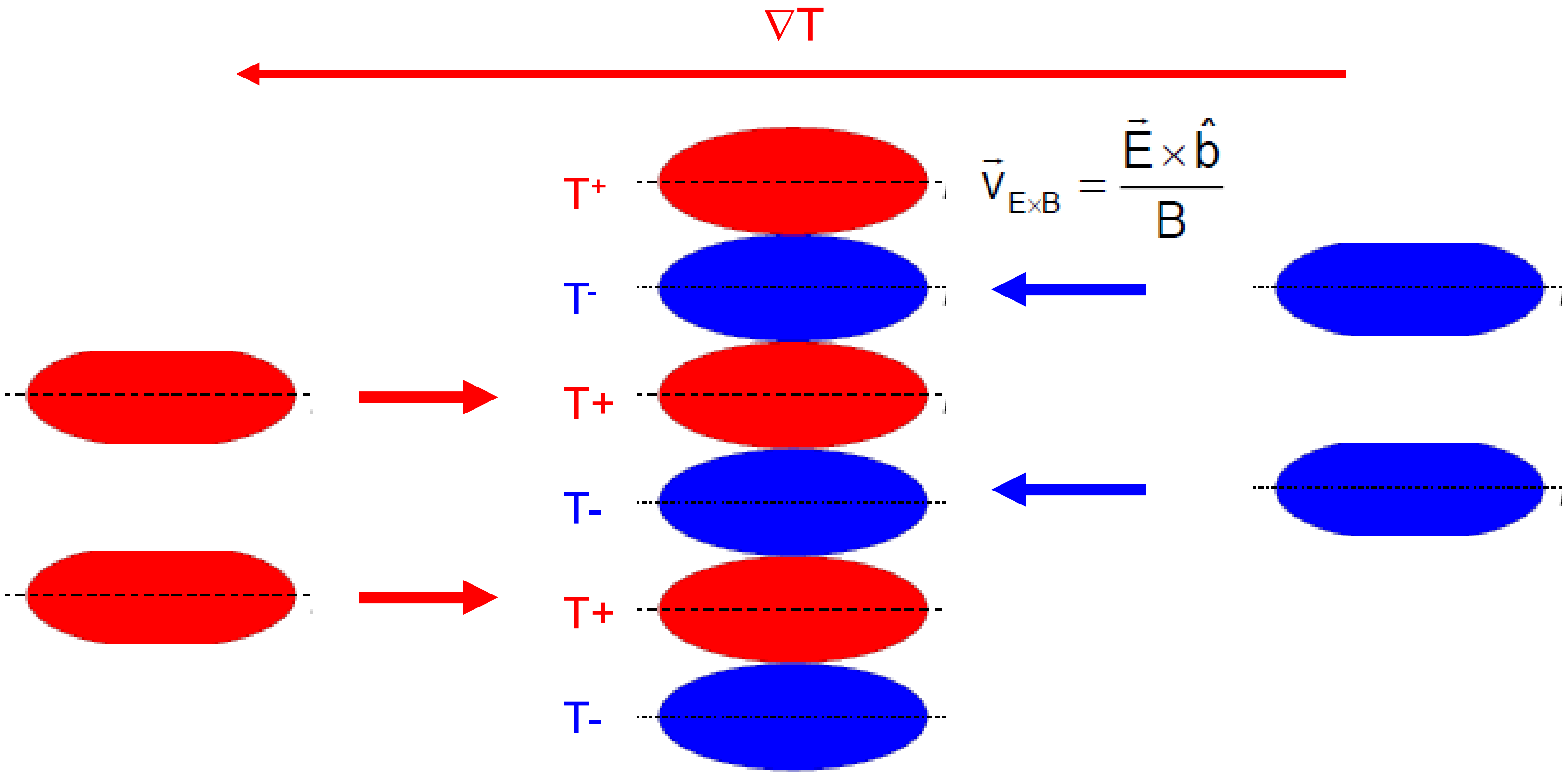
- Advection occurs in the radial direction





$$\vec{V}_{E\times B} = \frac{\vec{E} \times \hat{b}}{B}$$

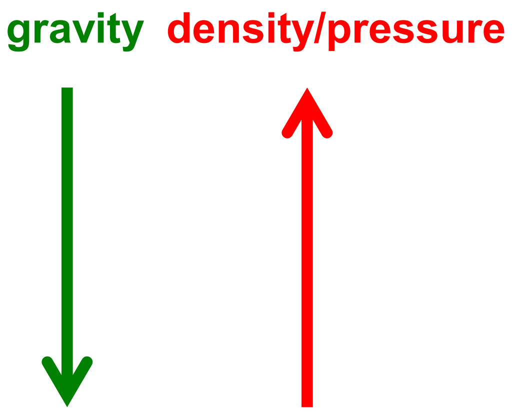
Background Temperature Gradient Reinforces Perturbation ⇒ Instability



This simple cartoon gives a purely growing "interchange" like mode (coarse derivation in backup slides). The complete derivation (all drifts, gradients) will give a real frequency dispersion, i.e. $\omega_r=\omega_r(k_\theta)$

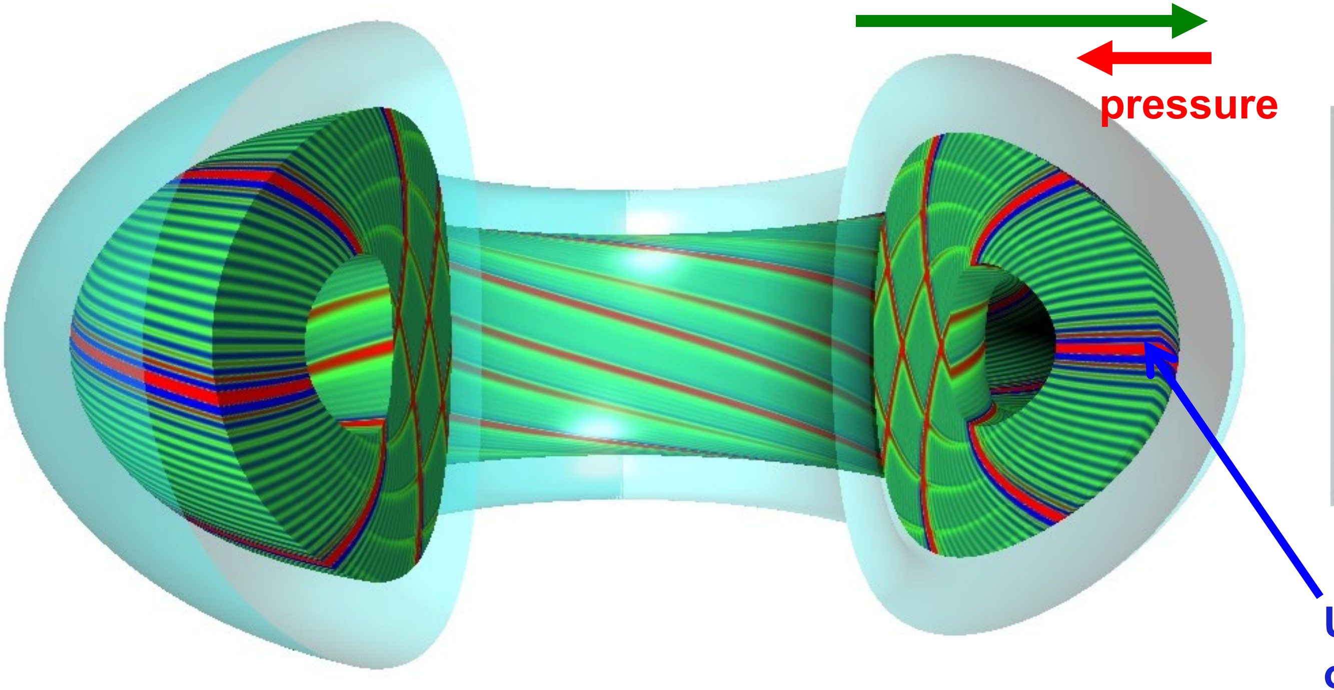
Analogy for turbulence in tokamaks – Rayleigh-Taylor instability

- Higher density on top of lower density, with gravity acting downwards



gravity **density/pressure**

Inertial force in toroidal field acts like an effective gravity

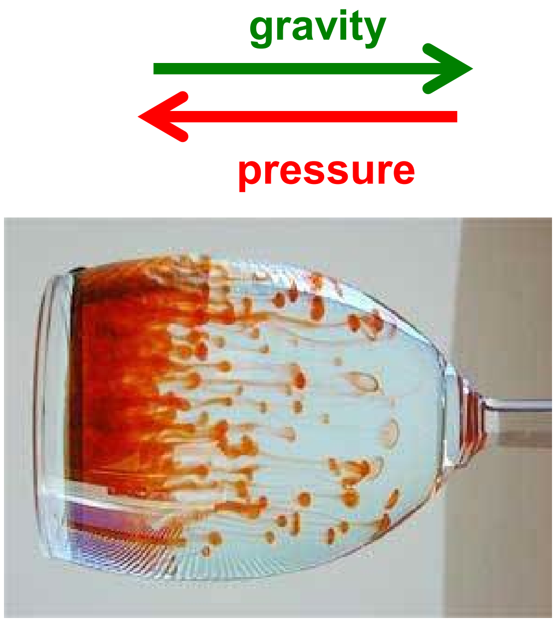


centrifugal force

→effective gravity

pressure

Unstable in the outer region

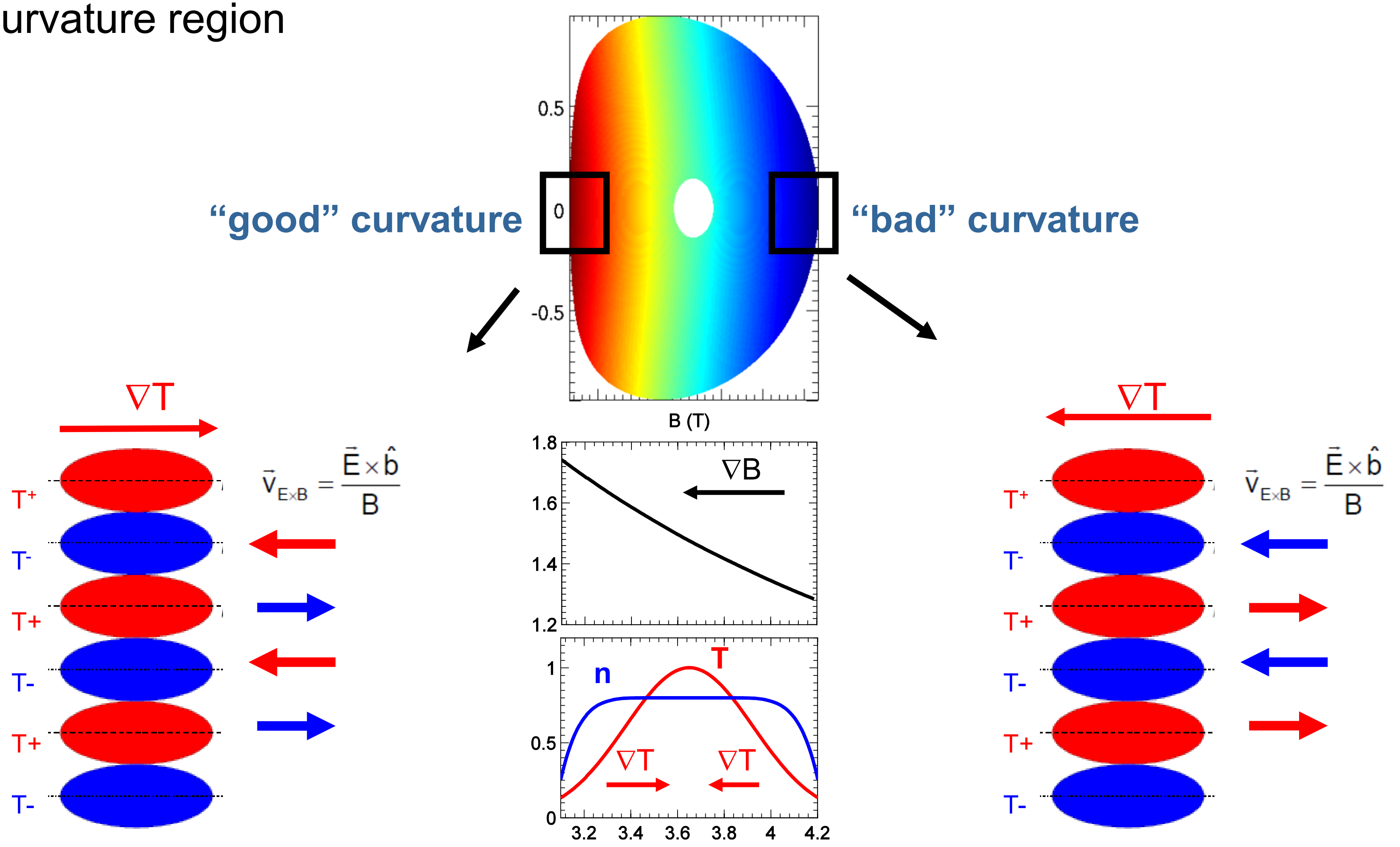


gravity

pressure

Same Dynamics Occur On Inboard Side But Now Temperature Gradient Is Stabilizing

- Advection with ∇T counteracts perturbations on inboard side – “good” curvature region



Fast Parallel Motion Along Helical Field Line Connects Good & Bad Curvature Regions

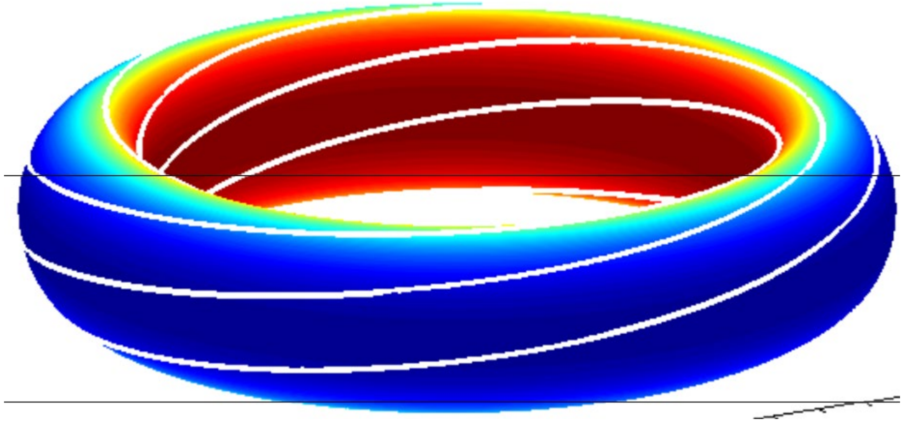
- Approximate growth rate on outboard side

  effective gravity: $g_{eff} = v_{th}^2/R$

  gradient scale length: $1/L_T = -1/T \cdot \nabla T$

$$\gamma_{instability} \sim \left(\frac{g_{eff}}{L}\right)^{1/2} \sim \frac{v_{th}}{\sqrt{RL_T}}$$

- Parallel transit time along helical field line with “safety factor” q



$$q = \frac{\#\ \text{toroidal transits}}{\#\ \text{poloidal transits}}$$

$$\gamma_{parallel} \sim \frac{v_{th}}{qR}$$

- Expect instability if $\gamma_{instability} > \gamma_{parallel}$, or $\left(\frac{R}{L_T}\right)_{threshold} \approx \frac{1}{q^2}$

Critical gradient for ITG determined from theory + linear gyrokinetic simulations

$$\left(\frac{\mathrm{R}}{\mathrm{L_T}}\right)_{\mathrm{crit}}^{\mathrm{ITG}} = \mathrm{Max}\left[\left(1+\frac{\mathrm{T_i}}{\mathrm{T_e}}\right)\left(1.3+1.9\frac{\mathrm{s}}{\mathrm{q}}\right)(\ldots), \frac{\mathrm{R}}{\mathrm{L_n}}\right]$$

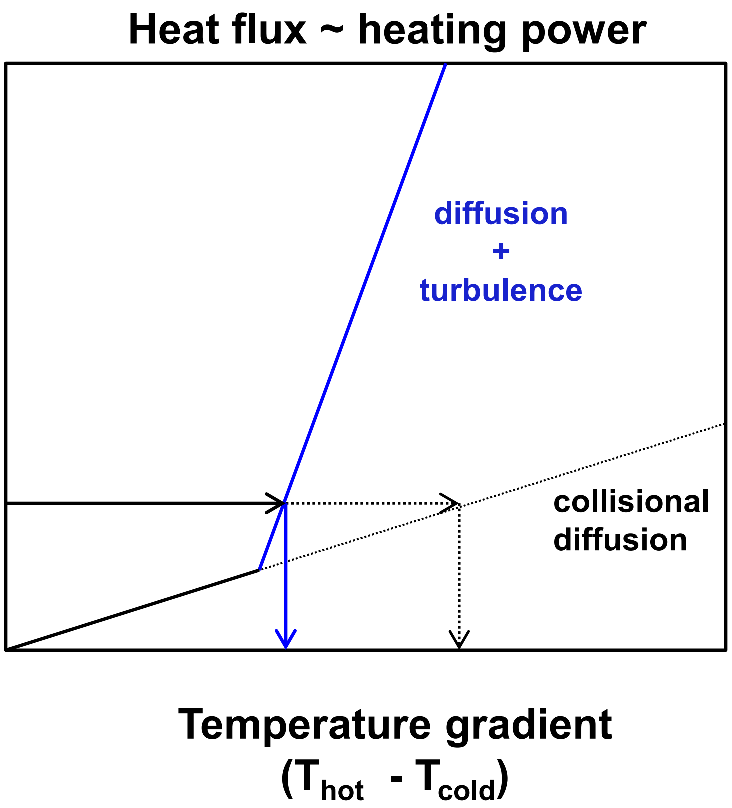
Jenko (2001)
Hahm (1989)
Romanelli (1989)

- $R/L_T = -R/T \cdot \nabla T$ is the normalized temperature gradient
- Natural way to normalize gradients for toroidal drift waves, i.e. ratio of diamagnetic-to-toroidal drift frequencies:

$$\omega_{*T} = k_y(B\times\nabla p) / nqB^2 \rightarrow (k_\theta\rho_i)v_T/L_T$$
$$\omega_D = k_y(B\times m v_\perp^2 \nabla B/2B) / qB^2 \rightarrow (k_\theta\rho_i)v_T/R$$
$$\rightarrow \omega_{*T}/\omega_D = R/L_T$$

- Another key stability parameter that often arises is $\eta = L_n/L_T = (R/L_T) / (R/L_n)$, i.e. a sufficiently large density gradient can set the temperature gradient threshold
  - E.g. can be important in the pedestal

Threshold-like behavior analogous to Rayleigh-Benard instability



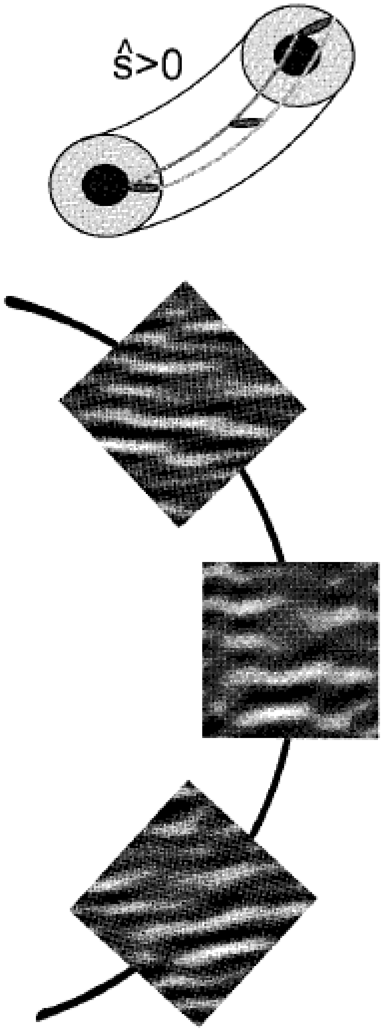
Analogous to convective transport when heating a fluid from below … boiling water (before the boiling)

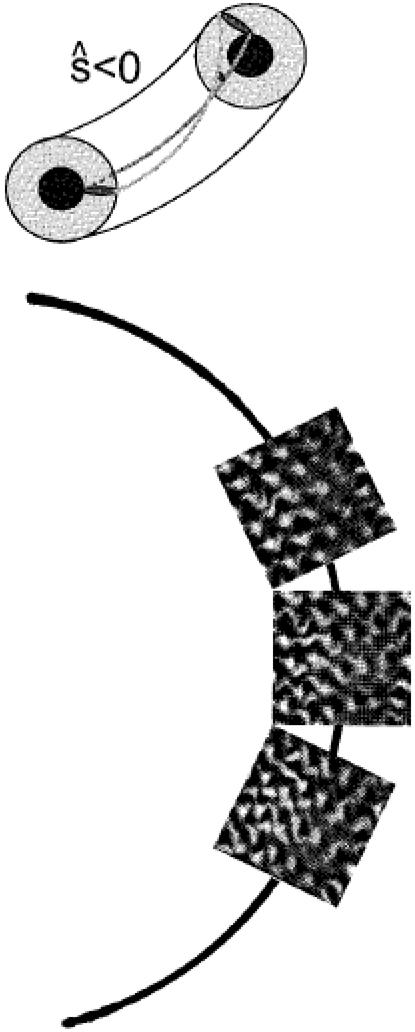
Rayleigh, Benard, early 1900's

"Hydrodynamic and hydromagnetic stability", S. Chandrasekhar (1961)
Discussed in E×B shear suppression review papers by K. Burrell (2020, 1999, 1997)

With physical understanding, can try to manipulate/optimize microstability

- E.g., magnetic shear influences stability by twisting radially-elongated instability to better align (or misalign) with bad curvature drive





Antoneson

What happens nonlinearly?

Saturated spectrum shape governed by nonlinear (2D perpendicular) three-wave interactions

- Linearly unstable modes grow: $\delta\phi(k) \sim \exp\left[ik\cdot x + i\omega(k)t + \gamma(k)t\right]$

- At large amplitude, interact via nonlinear advection, $\boldsymbol{\delta v_E \cdot \nabla \delta f}$
  i.e. "three-wave" coupling in (2D perpendicular) wavenumber space

$$\frac{\partial}{\partial t}\delta f \sim \delta v_E \cdot \nabla \delta f$$

$$\frac{\partial}{\partial t}\delta f_{k_{\perp 3}} \sim \sum_{\substack{k_{\perp 1}, k_{\perp 2} \\ k_{\perp 3} = k_{\perp 1} + k_{\perp 2}}} \left(b \times k_{\perp 1}\delta\phi_{k_{\perp 1}}\right)\cdot k_{\perp 2}\delta f_{k_{\perp 2}}$$

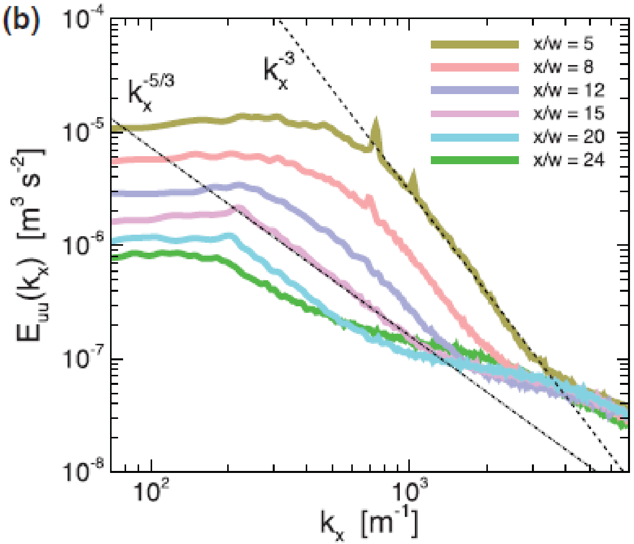
- Energy gets distributed across k space (& velocity space) until damped by stable modes (& collisions) → saturation
  - Local (in k) 2D cascades
  - Non-local (in k) interactions drive "zonal flows" that also mediate turbulence

Energy cascade in 2D turbulence is different than 3D

- Change in non-linear conservation properties → energy and vorticity is conserved
  - ***Inverse*** energy cascade E(k) ~ $k^{-5/3}$
  - Forward enstrophy [ω^2~$(\nabla\times v)^2$] cascade E(k)~k^{-3}
  - Non-local wavenumber interactions can couple over larger range in k-space (e.g. to zonal flows)

Quasi-2D turbulence exists in many places

- Geophysical flows like ocean currents, tropical cyclones, polar vortex, chemical mixing in polar stratosphere (→ ozone hole)
- Soap films →

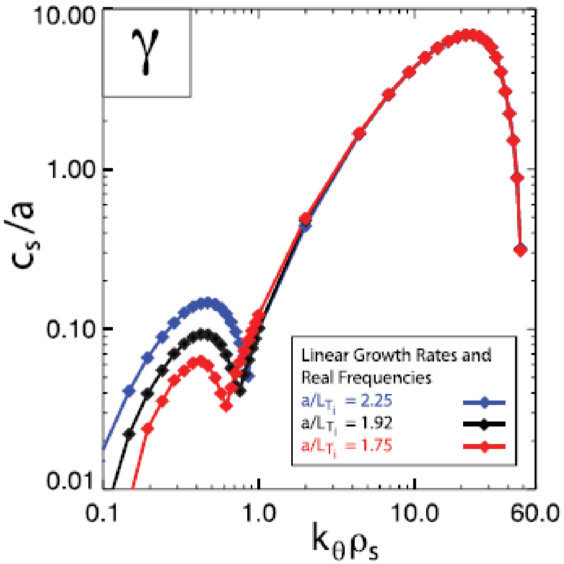


Liu et al., PRL (2016)

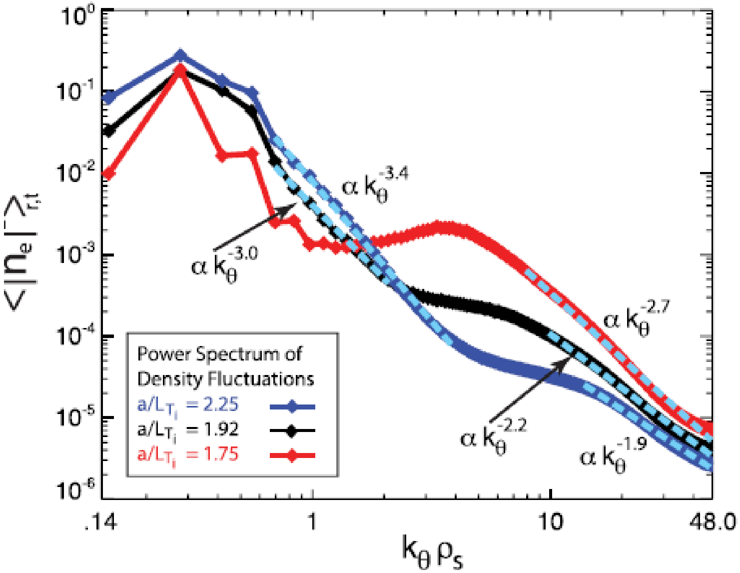
Energy drive can occur across large range of scales, but turbulent spectra still exhibit decay

- 2D energy & enstrophy cascades remain important → nonlinear spectra often downshifted in k_θ (w.r.t. linear growth rates)
- Damping can occur at all scales through kinetic effects (Terry, Hatch, …), very distinct from neutral fluid turbulence

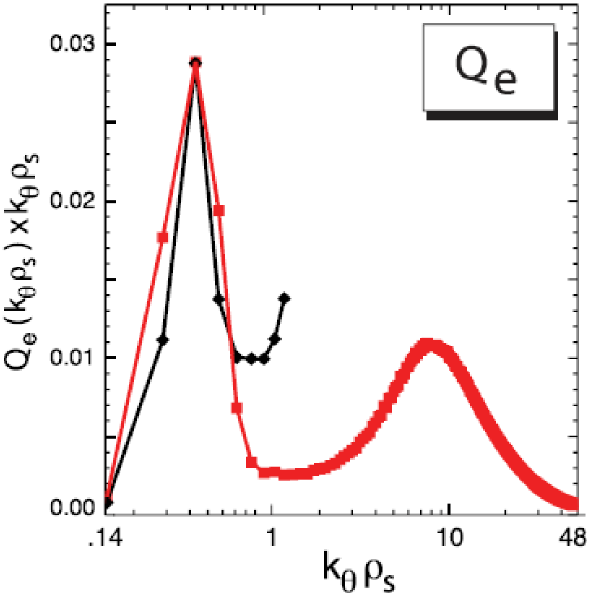
Linear growth rates



Nonlinear density power spectra



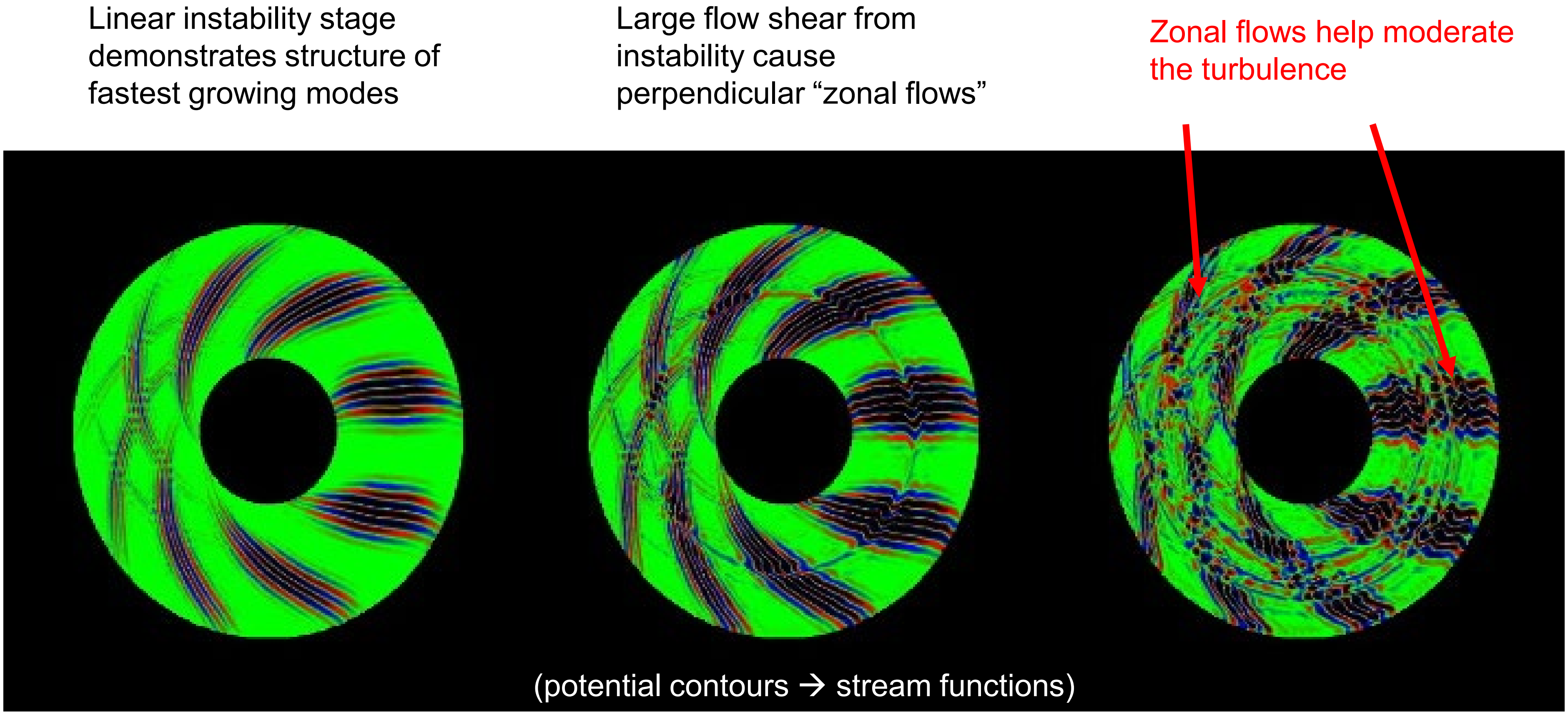
Nonlinear heat flux spectra



Howard, PoP & NF (2016)

Nonlinearly-generated "zonal flows" also impact saturation

- Potential perturbations uniform on flux surfaces (k_y=0) → marginally stable, do not cause transport
- Turbulence can condense to system size → ZF driven largely by non-local (in k) NL interactions (k >> k_{ZF})


Linear instability stage demonstrates structure of fastest growing modes

Large flow shear from instability cause perpendicular "zonal flows"

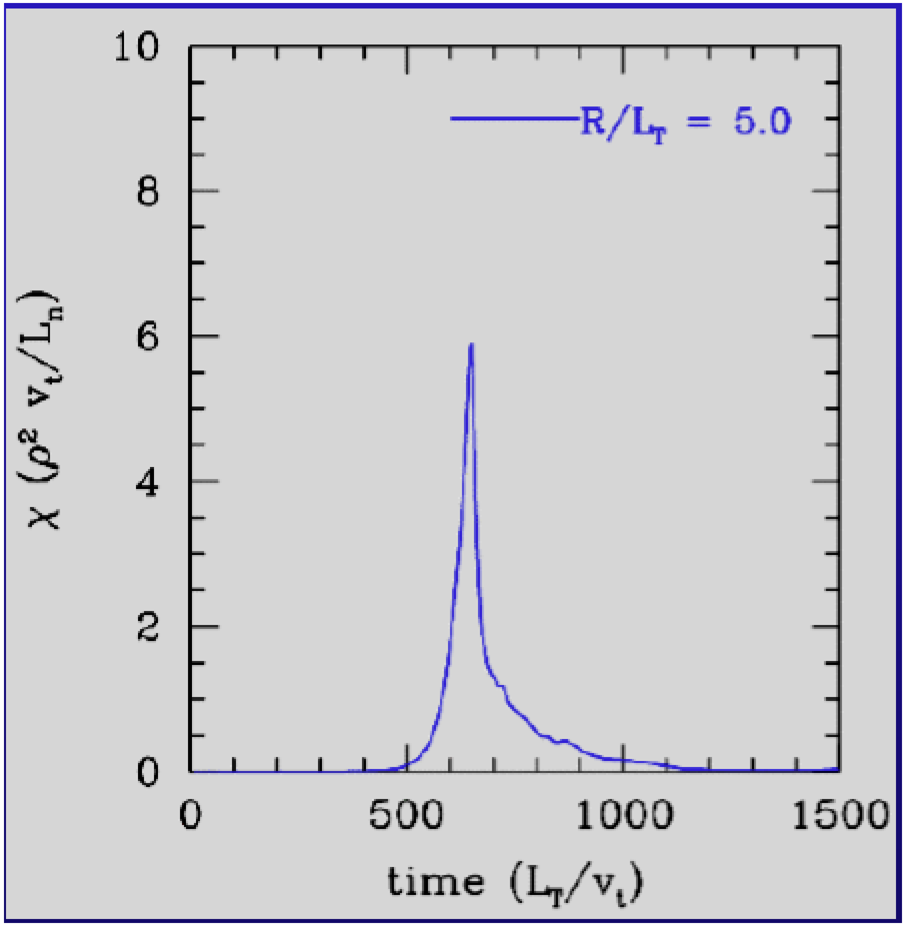
Zonal flows help moderate the turbulence

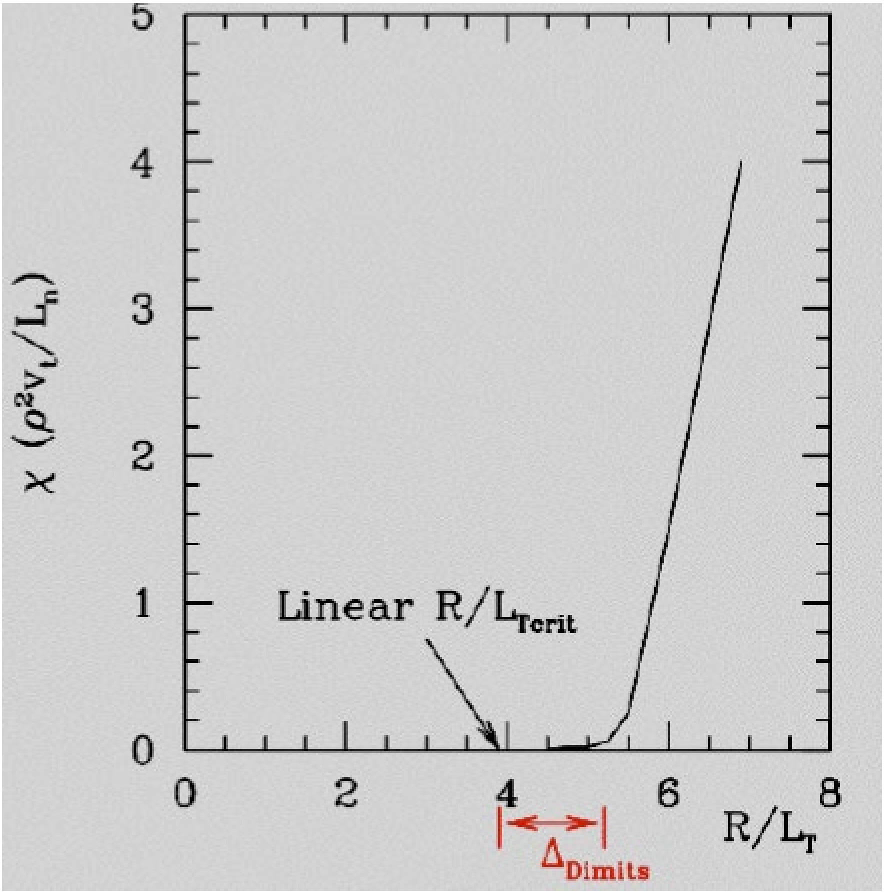
(potential contours → stream functions)

Rayleigh-Taylor like instability driving Kelvin-Helmholtz-like instability

Near linear threshold, strong zonal flows can suppress primary ITG instability → low time-averaged transport

- Leads to nonlinear upshift of effective threshold

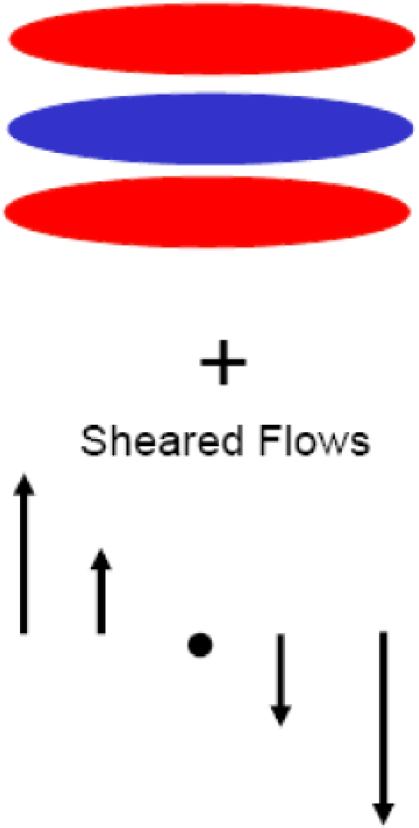


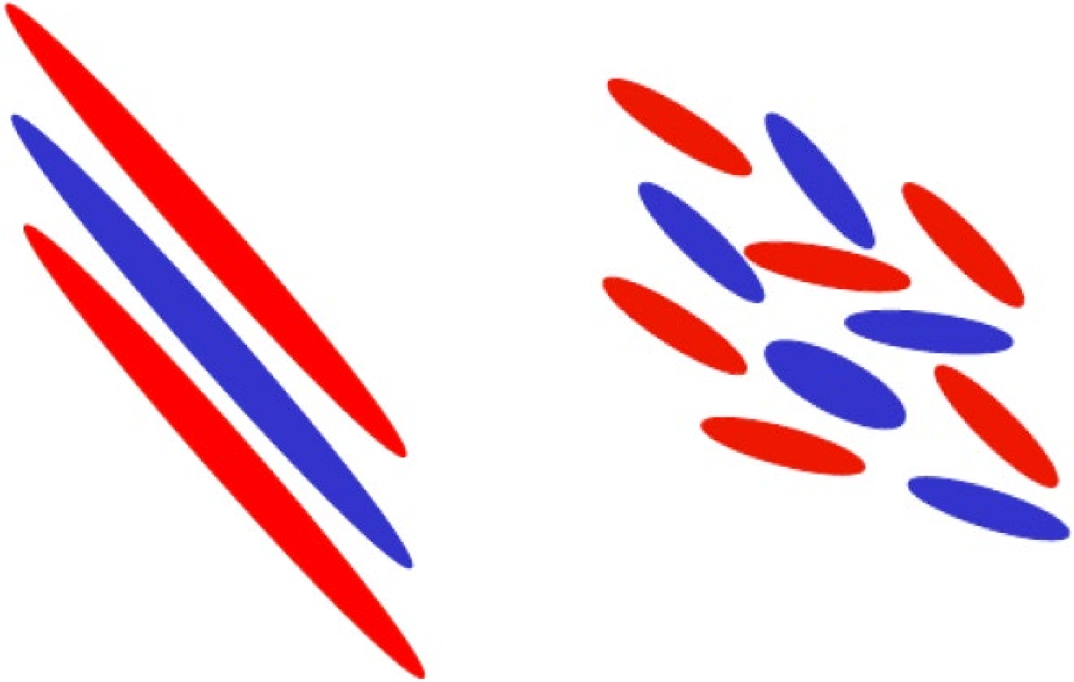


Dorland (2000)

Large scale equilibrium sheared flows also influence saturation

- Large scale background flow shear distorts eddies → reduces radial correlation length, fluctuation strength, cross-phases and transport





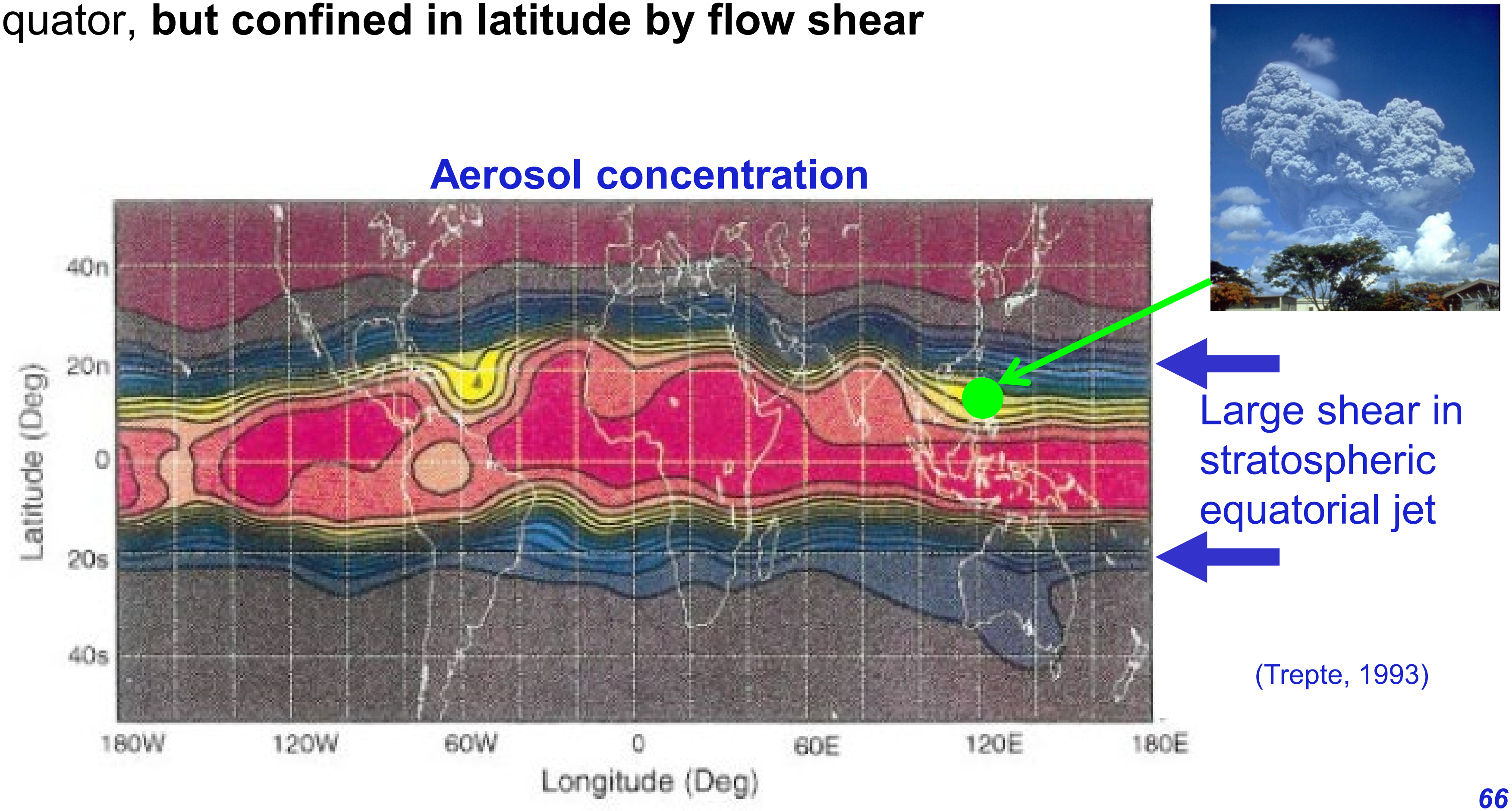
Turbulent transport expected to be reduced as the mean flow shear rate ($\omega_s \sim dU_0/dy$) approaches the turbulence decorrelation rate ($\Delta\omega_D$)

Biglari, Diamond, Terry 1990

In neutral fluids, sheared flows are often a source of free energy to drive turbulence

- Thin (quasi-2D) atmosphere in axisymmetric geometry of rotating planets similar to tokamak plasma turbulence → can also suppress transport

- Stratospheric ash from Mt. Pinatubo eruption (1991) spread rapidly around equator, **but confined in latitude by flow shear**


Aerosol concentration

Large shear in stratospheric equatorial jet

(Trepte, 1993)

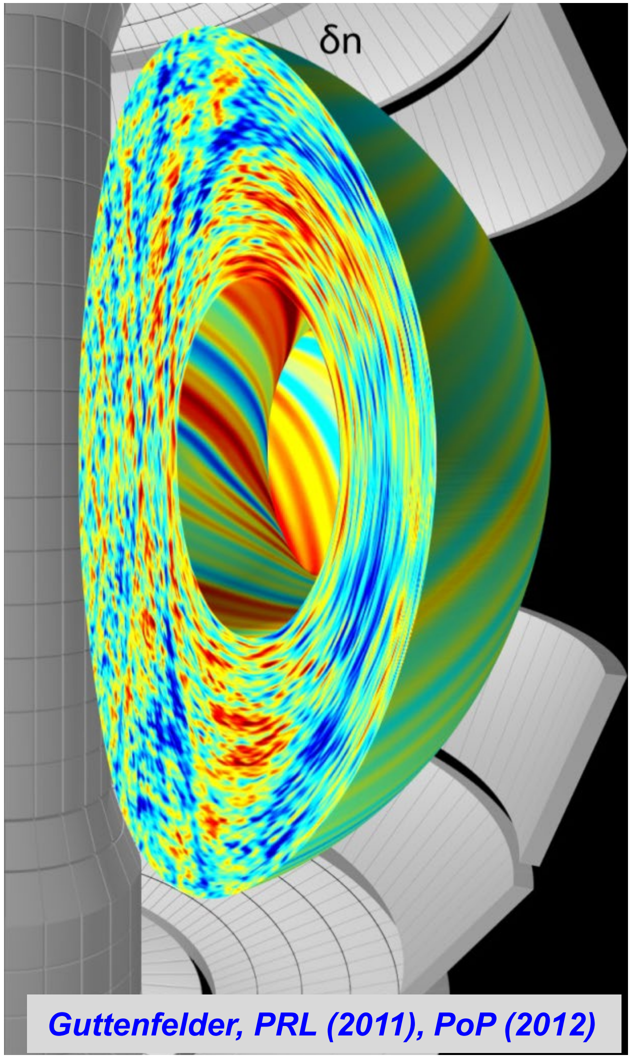
Beyond general characteristics, there are many theoretical "flavors" of drift waves possible in tokamak core & edge

- Usually think of drift waves as gradient driven (∇T_i, ∇T_e, ∇n)
  - Often exhibit threshold in one or more of these parameters

- Different theoretical "flavors" exhibit different parametric dependencies, predicted in various limits, depending on gradients, T_e/T_i, ν, β, geometry, location in plasma…
  - Electrostatic, ion scale ($k_\theta \rho_i \leq 1$)
    - Ion temperature gradient (ITG) – driven by ∇T_i, weakened by ∇n
    - Trapped electron mode (TEM) – driven by ∇T_e & ∇n_e, weakened by ν_e
    - Parallel velocity gradient (PVG) – driven by $R\nabla\Omega$ (like Kelvin-Helmholtz)
  - Electrostatic, electron scale ($k_\theta \rho_e \leq 1$)
    - Electron temperature gradient (ETG) - driven by ∇T_e, weakened by ∇n
  - Electromagnetic, ion scale ($k_\theta \rho_i \leq 1$)
    - Kinetic ballooning mode (KBM) - driven by $\nabla \beta_{pol} \sim \alpha_{MHD}$
    - Microtearing mode (MTM) – driven by ∇T_e, at sufficient β_e

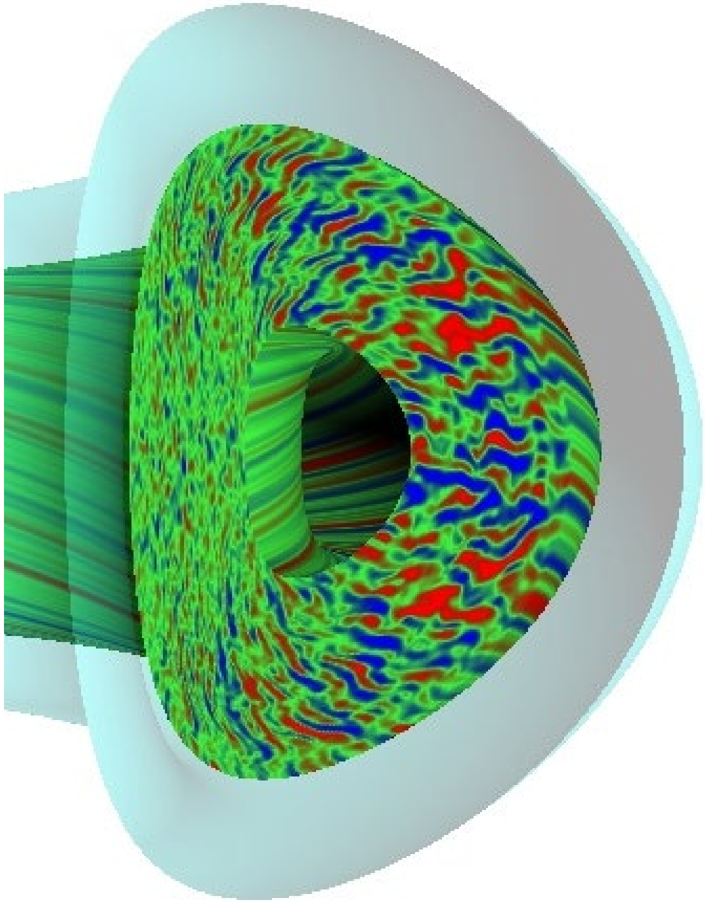
MTM density fluctuations distinct from ballooning modes like ITG (simulations)

NSTX MTM turbulence



Guttenfelder, PRL (2011), PoP (2012)

DIII-D ITG turbulence



Candy, Waltz (GA)

Summary

- Turbulence ubiquitous throughout the universe
  - Lots of free energy sources

- Turbulence is deterministic yet unpredictable (chaotic), appears random

- Turbulence causes increased mixing, transport larger than collisional transport
  - Transport is the key application of why we care about turbulence
  - **Understanding and optimizing transport critical for fusion reactors**

- Turbulence spans a wide range of spatial and temporal scales
  - Large Reynolds # (3D neutral fluids) / Dorland # (6D kinetic plasmas)
  - 6D kinetic plasmas lead to additional degrees of freedom for driving and dissipation mechanisms

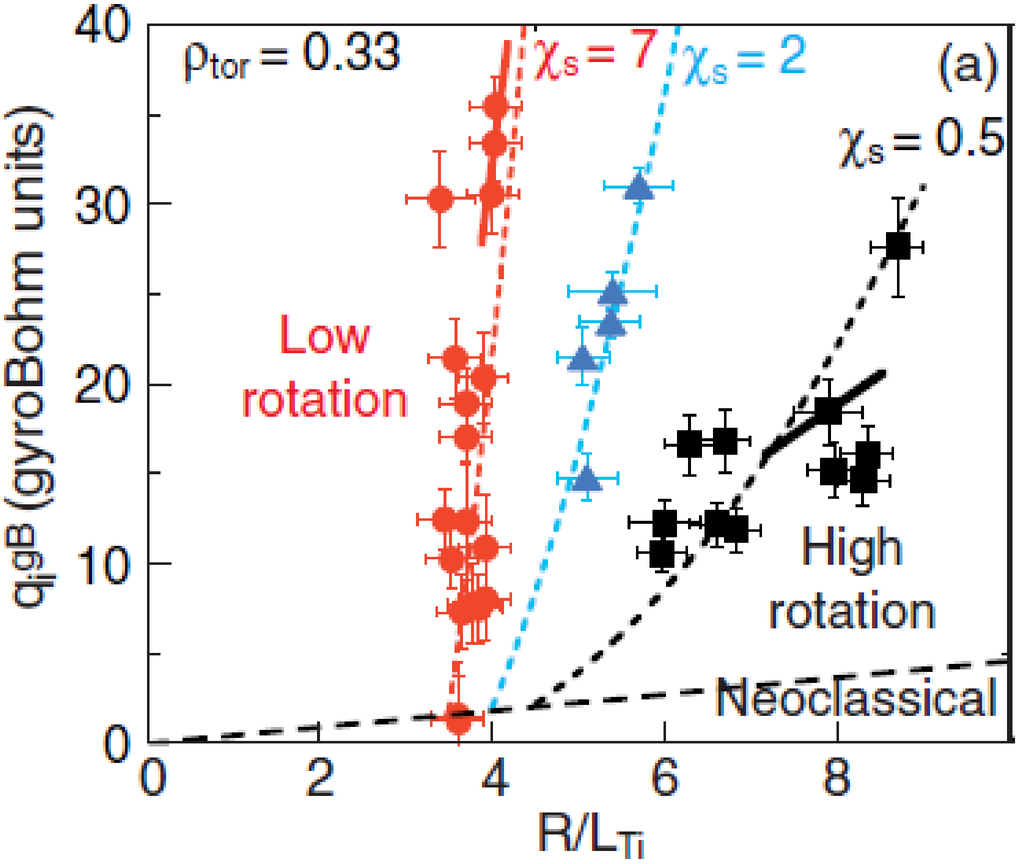
Some review references

- Transport & Turbulence reviews:
  - Liewer, Nuclear Fusion (1985)
  - Wootton, Phys. Fluids B (1990)
  - Carreras, IEEE Trans. Plasma Science (1997)
  - Wolf, PPCF (2003)
  - Tynan, PPCF (2009)
  - ITER Physics Basis (IPB), Nuclear Fusion (1999)
  - Progress in ITER Physics Basis (PIPB), Nuclear Fusion (2007)
- Drift wave reviews:
  - Horton, Rev. Modern Physics (1999)
  - Tang, Nuclear Fusion (1978)
- Gyrokinetic simulation review:
  - Garbet, Nuclear Fusion (2010)
- Zonal flow/GAM reviews:
  - Diamond et al., PPCF (2005)
  - Fujisawa, Nuclear Fusion (2009)
- Measurement techniques:
  - Bretz, RSI (1997)

THE END

Threshold-like behavior observed experimentally

- Experimentally inferred threshold varies with equilibrium, plasma rotation, ...
- Stiffness (~dQ/d∇T above threshold) also varies
- χ = -Q/n∇T highly nonlinear (also use perturbative experiments to probe stiffness)



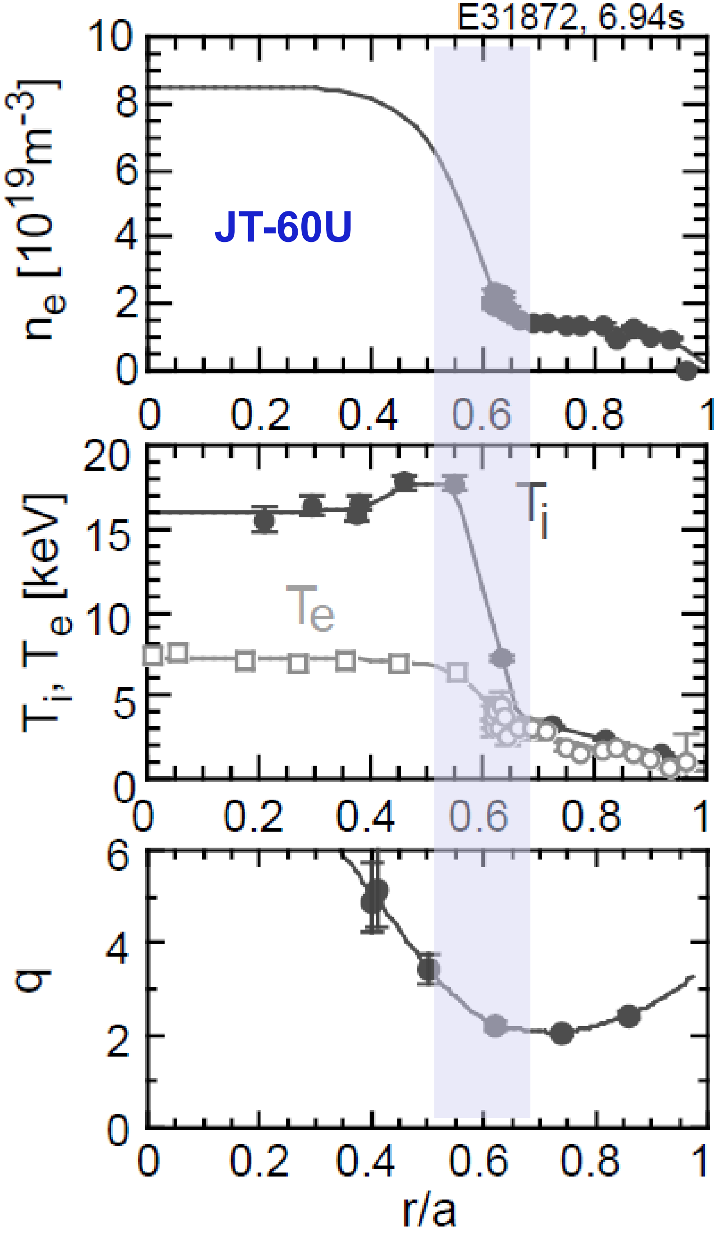
JET
Mantica, PRL (2011)

Reverse magnetic shear can lead to internal transport barriers (ITBs)

- ITBs established on numerous devices

- Used to achieve “equivalent” $Q_{DT,eq}$~1.25 in JT-60U (in D-D plasma)

- $\chi_i \sim \chi_{i,NC}$ in ITB region (complete suppression of ion scale turbulence)



Ishida, NF (1999)

Very simple growth rate derivation of toroidal ITG cartoon picture

Can identify key terms in “gyrofluid” equations responsible for toroidal ITG instability

- Start with toroidal GK equation in the δf limit ($\delta f/F_M << 1$)
- Take fluid moments
- Apply clever closures that “best” reproduce linear toroidal gyrokinetics (Hammett, Perkins, Beer, Dorland, Waltz, …)

Ion continuity and energy (M. Beer thesis, 1995):

$$\frac{\partial \tilde{n}}{\partial t} + \mathbf{v}_E \cdot \nabla n_0 + \mathbf{v}_E \cdot \nabla \tilde{n} + n_0 \nabla \cdot \mathbf{v}_E + \frac{2}{m\Omega B^2} \mathbf{B} \times \nabla B \cdot \nabla \tilde{p} = 0. \tag{1.5}$$

$$\frac{\partial \tilde{p}}{\partial t} + \mathbf{v}_E \cdot \nabla p_0 + \mathbf{v}_E \cdot \nabla \tilde{p} + p_0 \nabla \cdot \mathbf{v}_E + p_0 B \mathbf{v}_E \cdot \nabla \frac{1}{B} + \frac{p_0}{n_0 m \Omega B^2} \mathbf{B} \times \nabla B \cdot \nabla (6\tilde{p} - 3\tilde{n} T_0) = 0. \tag{1.12}$$

Temperature perturbation (δT) leads to compression ($\nabla\cdot v_{di}$), density perturbation – 90° out-of-phase with δT

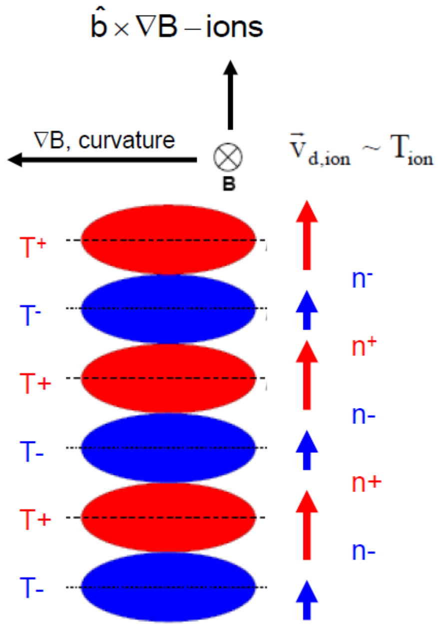
$$\boxed{\frac{\partial \tilde{n}}{\partial t}} + \mathbf{v}_E \cdot \nabla n_0 + \mathbf{v}_E \cdot \nabla \tilde{n} + n_0 \nabla \cdot \mathbf{v}_E + \boxed{\frac{2}{m\Omega B^2}\mathbf{B} \times \nabla B \cdot \nabla \tilde{p}} = 0. \qquad (1.5)$$

$dn/dt + \nabla\cdot(nv)=0$

$-i\omega\delta n$ from $-n_0\nabla\cdot\delta v_d$ $\sim -n_0\nabla\cdot(\delta T_\perp\, b\times\nabla B/B)/B \sim -n_0\, ik_y\delta T / BR$

$-i\omega(\delta n/n_0) \sim -ik_y(\delta T/T_0)\, T/BR \sim -i(k_yV_D)\,(\delta T/T_0) \sim -i\omega_D\,(\delta T/T_0)$

$-i\omega(\delta n/n_0) = -i\omega_D(\delta T/T_0)$



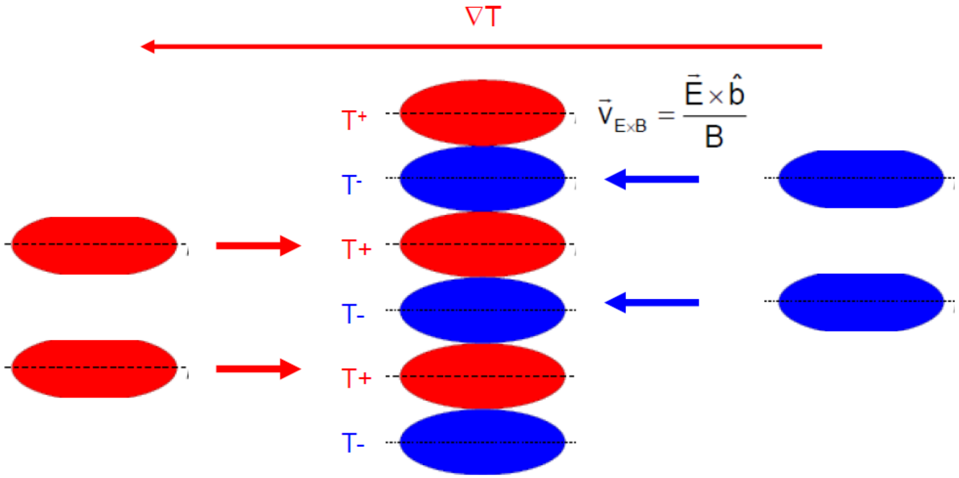
Background Temperature Gradient Reinforces Perturbation ⇒ Instability

$$\boxed{\frac{\partial \tilde{p}}{\partial t}} + \boxed{\mathbf{v}_E \cdot \nabla p_0} + \mathbf{v}_E \cdot \nabla \tilde{p} + p_0 \nabla \cdot \mathbf{v}_E + p_0 B \mathbf{v}_E \cdot \nabla \frac{1}{B} + \frac{p_0}{n_0 m \Omega B^2} \mathbf{B} \times \nabla B \cdot \nabla (6\tilde{p} - 3\tilde{n} T_0) = 0. \qquad (1.12)$$

$-i\omega\delta T$ from $-\delta v_E \cdot \nabla T_0$ $\sim -(b \times \nabla\delta\varphi/B)\nabla T_0 \sim ik_y\delta\varphi/B \cdot \nabla T_0 \sim ik_y\delta\varphi(T/B)/L_T$

$-i\omega(\delta T/T) \sim ik_y(\delta\varphi/T)T/BL_T \sim i(k_y V_{*T})(\delta\varphi/T) \sim i\omega_{*T}(\delta\varphi/T)$

$$\mathbf{-i\omega(\delta T/T) = i\omega_{*T}(\delta\varphi/T)}$$



Simplest dispersion from these 3 terms

(1) Compression from toroidal drifts

$$\omega(\delta n_i/n_0) = \omega_{Di}\,(\delta T_i/T_{i0})$$

(2) Quasi-neutrality + Boltzmann electron response

$$(\delta n_i/n_0) = (\delta n_e/n_0) = (\delta\varphi/T_{e0}) = (\delta\varphi/T_{i0})(T_i/T_e)$$

(3) E×B advection of background gradient

$$-\omega(\delta T_i/T_{i0}) = \omega_{*T}(\delta\varphi/T_i)$$

(1)+(2): $\omega(T_i/T_e)(\delta\varphi/T_{i0}) = \omega_{Di}\,(\delta T_i/T_{i0})$

(+3): $\omega(T_i/T_e) = -\omega_{Di}\,\omega_{*T}\,/\,\omega$

$$\omega^2 = -(k_y\rho_i)^2 v_{Ti}^2\,/\,RL_T \text{ (assume } T_e=T_i\text{)}$$

$$\boxed{\gamma = (k_y\rho_i)\frac{v_{Ti}}{(RL_T)^{1/2}}}$$

“bad curvature”

$\omega_{Di}\cdot\omega_{*T} \sim \nabla B\cdot\nabla T > 0$

Finite gyroradius effects limit characteristic size to ion-gyroradius ($k_\perp \rho_i$~1)

- Drift velocity increases with smaller wavelength (larger $k_\perp \rho_i$)

$$\vec{v}_E = \frac{\hat{b} \times \nabla \varphi}{B} = -ik_\perp \frac{\varphi}{B} = -ik_\perp \left(\frac{\varphi}{T_i}\right)\left(\frac{T_i}{B}\right) = -i(k_\perp \rho_i)\left(\frac{\varphi}{T_i}\right) v_{Ti}$$

- If wavelength approaches ion gyroradius $(k_\perp \rho_i) \geq 1$, average electric field experienced over fast ion-gyromotion is reduced:

$\langle \nabla \varphi \rangle_{\text{gyro-average}} \sim \nabla \varphi$

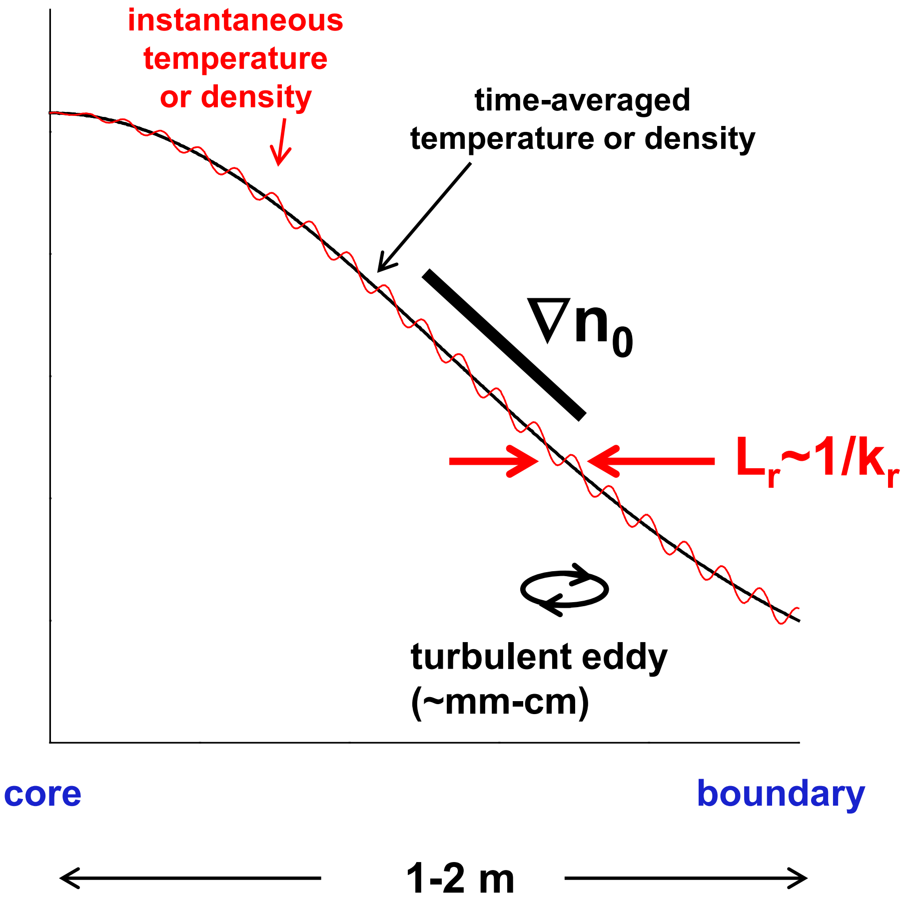
$\langle \nabla \varphi \rangle_{\text{gyro-average}} \sim \nabla \varphi [1-(k_\perp \rho_i)^2]$

⇒ **Maximum growth rates (and typical turbulence scale sizes) occur for $(k_\perp \rho_i) \lesssim 1$**

Mixing length estimate of fluctuation amplitude

Mixing length estimate for fluctuation amplitude

- In the presence of an equilibrium gradient, ∇n_0, turbulence with radial correlation L_r will mix regions of high and low density



Mixing length estimate for fluctuation amplitude

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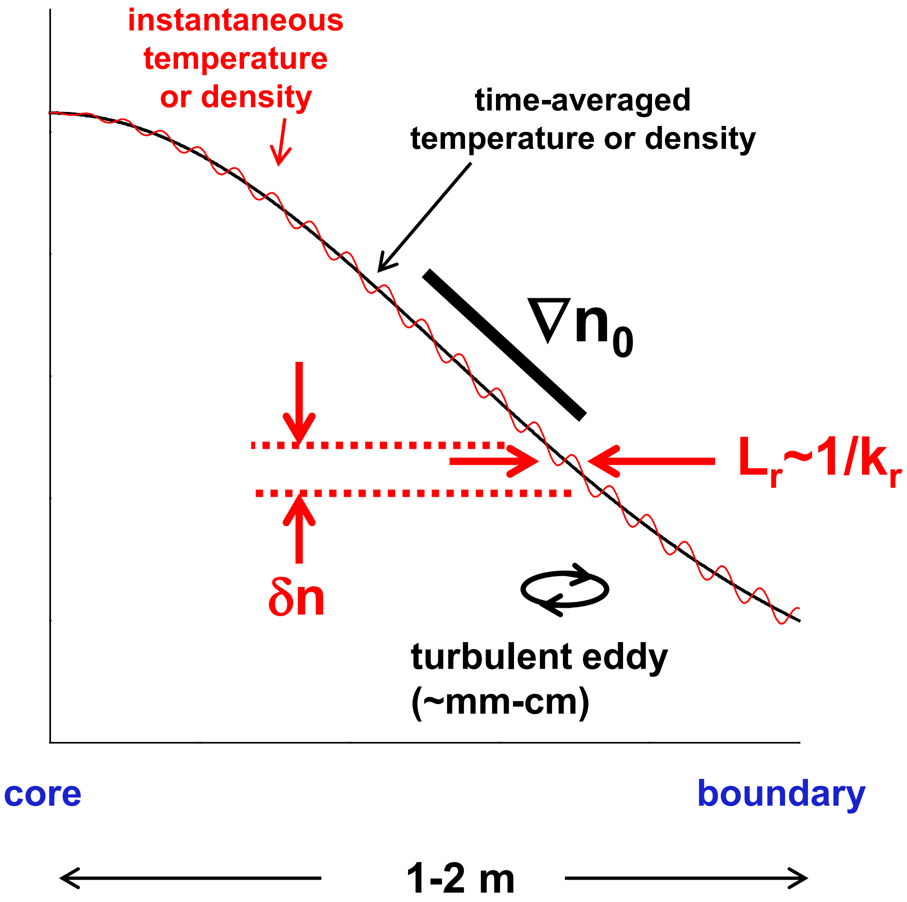
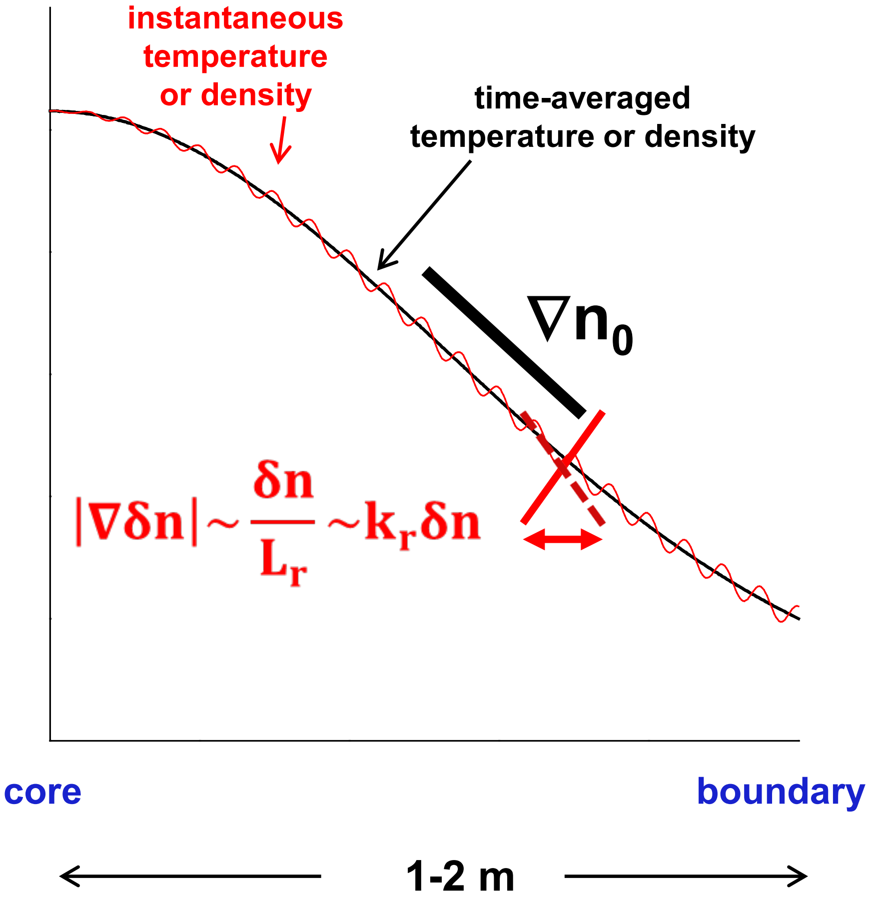
- Leads to fluctuation δn



Mixing length estimate for fluctuation amplitude

- In the presence of an equilibrium gradient, ∇n_0, turbulence with radial correlation L_r will mix regions of high and low density
- Leads to fluctuation δn
- Another interpretation: local, instantaneous gradient limited to equilibrium gradient



$$|\nabla \delta n| \sim \frac{\delta n}{L_r} \sim k_r \delta n$$

Mixing length estimate for fluctuation amplitude

$$\delta n \approx \nabla n_0 \cdot L_r$$

$$\frac{\delta n}{n_0} \approx \frac{\nabla n_0}{n_0} \cdot L_r \approx \frac{L_r}{L_n} \qquad \left(1/L_n = \nabla n_0 / n_0\right)$$

$$\frac{\delta n}{n_0} \sim \frac{1}{k_\perp L_n} \sim \frac{\rho_s}{L_n} \qquad \left(k_\perp^{-1} \sim L_r ; k_\perp \rho_s \sim \mathrm{cons\,tan\,t}\right)$$

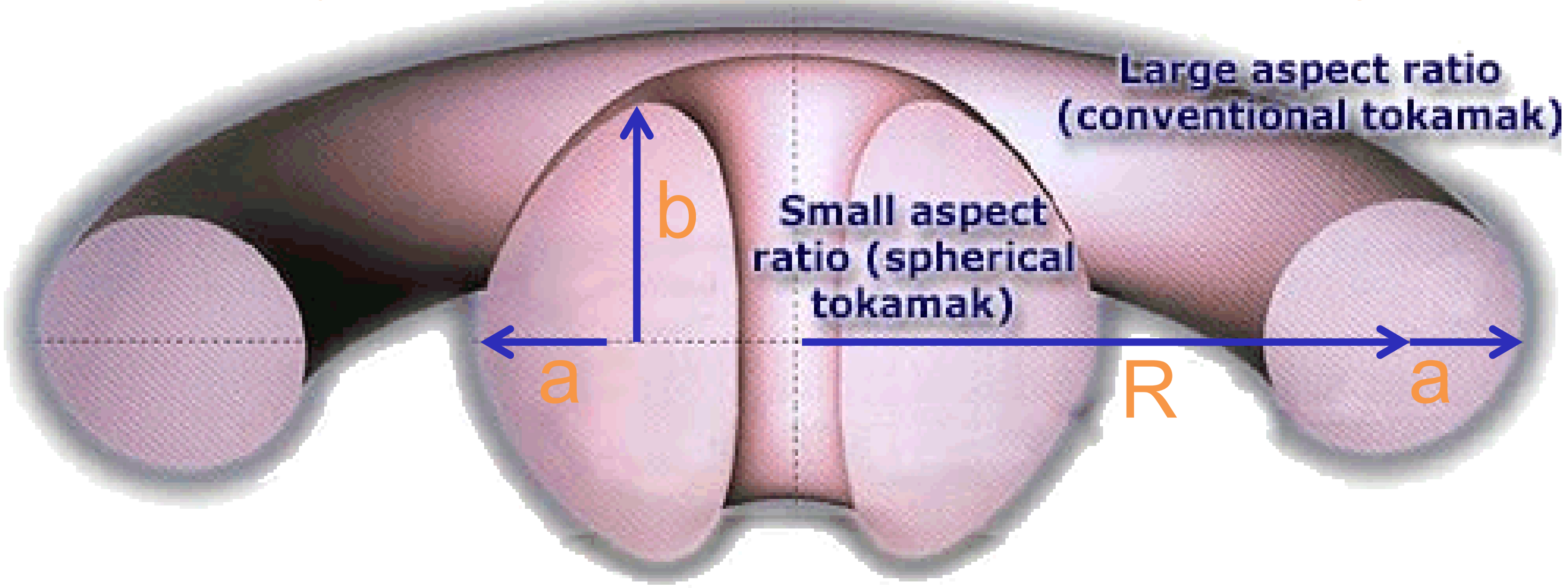
Expect $\delta n/n_0 \sim \rho_s/L \sim \rho_*$

Low aspect ratio equilibrium can stabilize electrostatic drift waves, minimize transport (i.e. one motivation for NSTX-U at PPPL)

Aspect ratio is an important free parameter, can try to make more compact devices (i.e. hopefully cheaper)

Aspect ratio A = R / a
Elongation κ = b / a

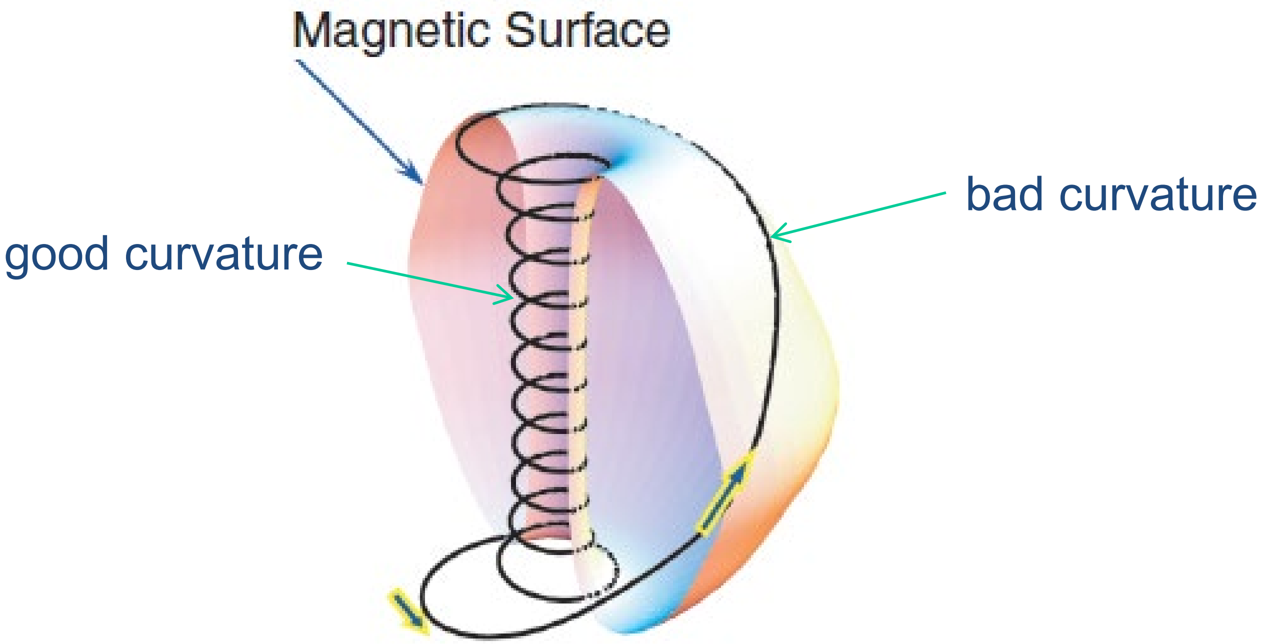
R = major radius, a = minor radius, b = vertical ½ height



But smaller R = larger curvature, ∇B (~1/R) -- isn't this terrible for "bad curvature" driven instabilities?!?!?!

Many elements of ST are stabilizing to toroidal, electrostatic ITG/TEM drift waves

- Short connection length → **smaller average bad curvature**



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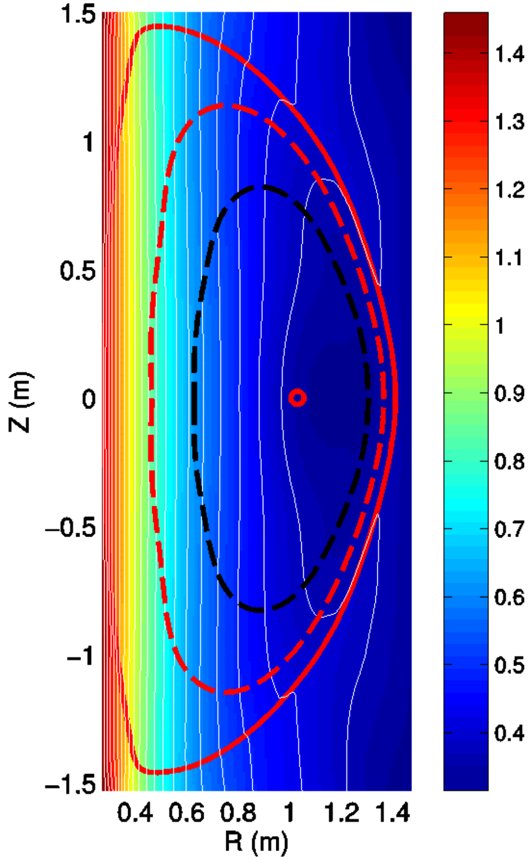
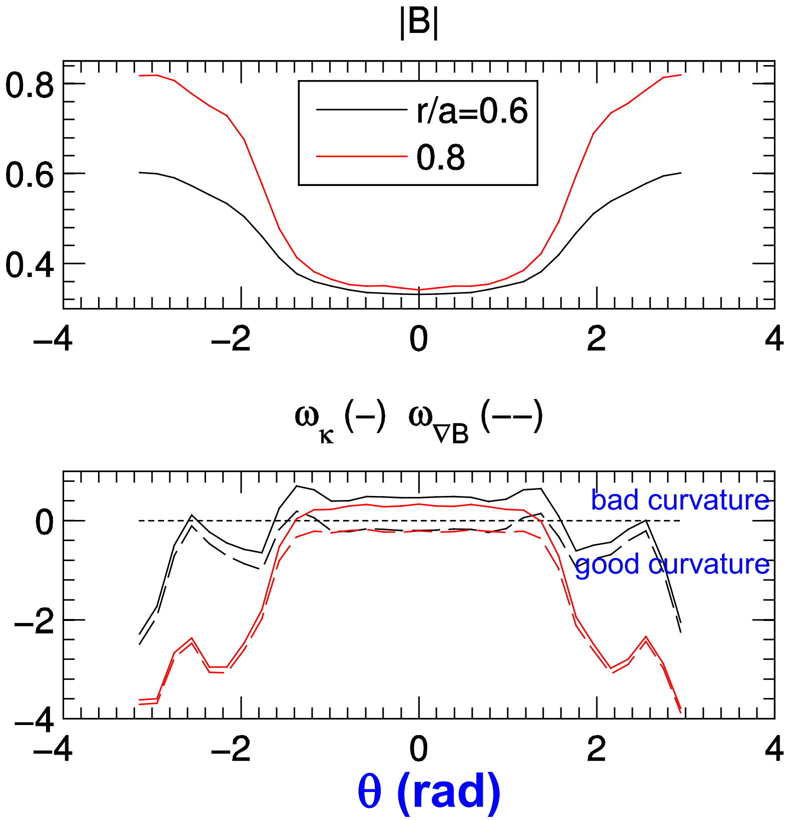
- Quasi-isodynamic (~constant B) at high β → **grad-B drifts stabilizing [Peng & Strickler, NF 1986]**





$$\vec{v}_\kappa = m v_{||}^2 \frac{\hat{b}\times\vec{\kappa}}{qB}$$

$$\vec{v}_{\nabla B} = \frac{m v_\perp^2}{2}\,\frac{\hat{b}\times\nabla B/B}{qB}$$

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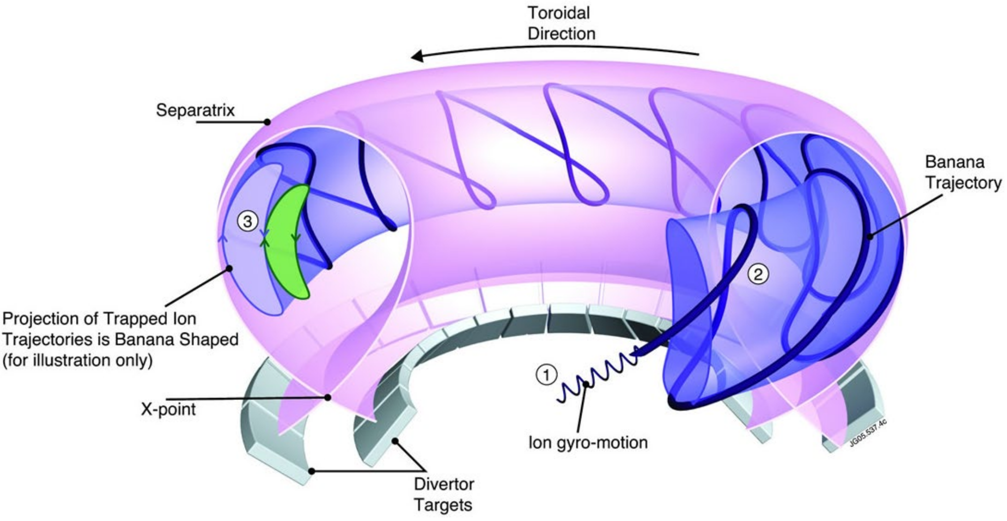
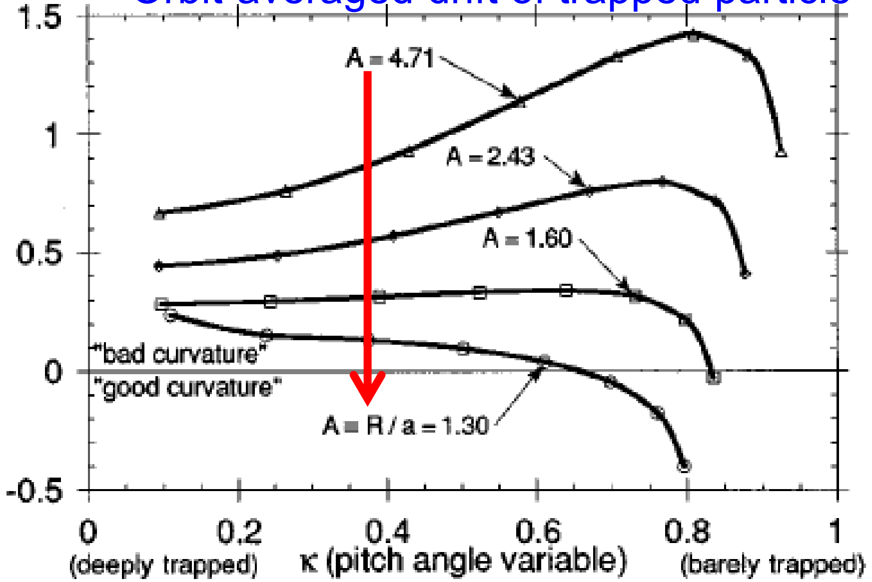


Orbit-averaged drift of trapped particle



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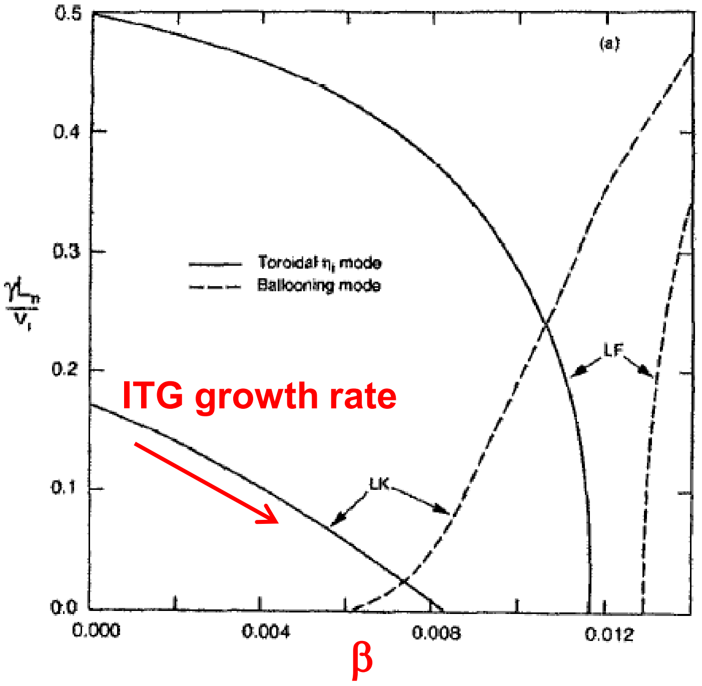

Kim, Horton, Dong, PoFB (1993)

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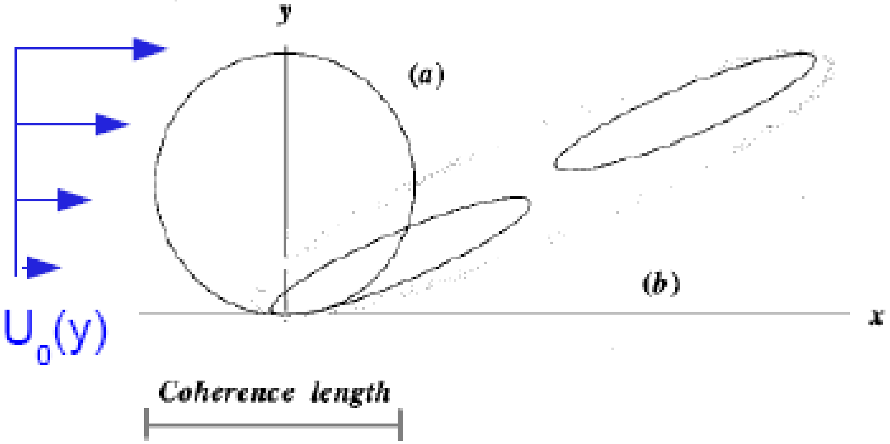

Biglari, Diamond, Terry, PoFB (1990)

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⇒ **Not expecting strong ES ITG/TEM instability (much higher thresholds)**

- BUT
- High beta drives EM instabilities: microtearing modes (**MTM**) ~ $\beta_e \cdot \nabla T_e$, kinetic ballooning modes (**KBM**) ~ $\alpha_{MHD} \sim q^2 \nabla P/B^2$
- Large shear in parallel velocity can drive Kelvin-Helmholtz-like instability $\sim dv_{||}/dr$