Fall 2010
Applied Physics E4990
Kinetic Theory and Simulation of Plasmas
Tuesday 4:10 - 6:40 pm; Room 327, Mudd

Instructor: Wei-li Lee [wwl2113@columbia.edu; 609-243-2647]

Prerequisite: Introduction to Plasma Physics

1. Basics of Kinetic Theory
2. Kinetic Plasma Simulation
3. Simulation Models and Properties
4. Perturbative Particle Simulation
5. Gyrokinetic Theory and Simulation
6. Microinstabilities: linear and nonlinear properties
7. Shear-Alfven Waves: finite-β effects on microturbulence
8. Gyrokinetic MHD
9. High Frequency Gyrokinetics
10. Integrated Simulation
11. Applications to (Magnetic and Beam) Fusion and Space Physics

1. Basics of Plasma Kinetics

- The Klimontovich-Dupree-Vlasov Model
- Conservation Properties
- Linearization
- Landau Damping
- Dispersion Relation
- Normal Modes
- Fluctuation-Dissipation Theorem
- Debye shielding
- Collisional damping

The Klimotovich-Dupree-Vlasov Model

[D. C. Montegomery, Theory of Unmagnetized Plasmas, 1971]

- Equations of Motion:

$$\frac{dx_j}{dt} = v_j \quad , \quad \frac{dv_j}{dt} = \frac{q}{m} E(x_j)$$

- Klimontovich-Dupree discrete representation:

$$F(x, v, t) = \sum_{j=1}^{N} \delta(x - x_j)\delta(v - v_j)$$

- Partial Differentiation in time and the chain rule

$$\frac{\partial}{\partial t} = \frac{d\mathbf{x}_j}{dt} \cdot \frac{\partial}{\partial \mathbf{x}_j} + \frac{d\mathbf{v}_j}{dt} \cdot \frac{\partial}{\partial \mathbf{v_j}}$$

- Smoothing the distribution - the Vlasov Model: incompressible phase space fluid

$$\frac{dF}{dt} \equiv \frac{\partial F}{\partial t} + \frac{dx}{dt}\frac{\partial F}{\partial x} + \frac{dv}{dt}\frac{\partial F}{\partial v} = 0$$

- Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi e \int (F_i - F_e) dv \qquad E = -\frac{\partial \phi}{\partial x}$$

Conservation Properties of the Vlasov-Poisson System

$$\frac{dF}{dt} \equiv \frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} + \frac{q}{m}E\frac{\partial F}{\partial v} = 0\ , \quad \frac{\partial^2 \phi}{\partial x^2} = -4\pi e\int (F_i - F_e)dv \quad , \quad E = -\frac{\partial \phi}{\partial x}$$

$$< \int v^{(n)}\frac{dF}{dt}dv >_x = 0\ , \qquad < \cdots >_x \equiv \frac{1}{L}\int \cdots dx$$

- Number Density Conservation: $\frac{d}{dt}\left\langle \int F dv \right\rangle_x = 0 \qquad n = \int F dv$

- Momentum Conservation: $\frac{d}{dt}\left\langle m_i \int vF_i dv + m_e \int vF_e dv \right\rangle_x = 0 \qquad \Gamma \equiv nV = \int vF dv$

- Energy Conservation: $\frac{d}{dt}\left\langle \frac{m_e}{2}\int v^2 F_e dv + \frac{m_i}{2}\int v^2 F_i dv + \frac{1}{8\pi}\left|\frac{\partial \phi}{\partial x}\right|^2 \right\rangle_x = 0$

$$p \equiv nT = \frac{m}{2}\int v^2 F dv$$

- Entropy Conservation: $\frac{d}{dt}\left\langle \int F ln F dv \right\rangle_x = 0 \qquad S = -\int F ln F dv$

$$\left[\text{From} \quad \frac{d}{dt}F ln F = 0 \quad \text{and} \quad \left\langle \int \frac{d}{dt}(F ln F) dv \right\rangle_x = 0\right]$$

Linear Properties of the Vlasov-Poisson System

- Linearization

$$F = F_0 + \delta f$$

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} + \frac{q}{m}E\frac{\partial F_0}{\partial v} = 0$$

$$\frac{\partial E}{\partial x} = 4\pi e \int (\delta f_i - \delta f_e) dv \qquad E = -\frac{\partial \phi}{\partial x}$$

$$\boxed{\begin{aligned} F(x) &= \sum_k \tilde{F}(k) e^{ikx} \\ \tilde{F}(k) &= \frac{1}{N}\sum_x F(x) e^{-ikx} \end{aligned}}$$

- Fourier Transforms $\qquad \delta f \propto e^{ikx - i\omega t} \qquad \phi \propto e^{ikx}$

$$\delta f = -\frac{q}{T}\left[1 + \frac{\omega/k}{v - \omega/k}\right]\phi F_0 \qquad F_0 = \frac{n_0}{\sqrt{2\pi}v_t} exp(-\frac{v^2}{2v_t^2})$$

- Linear Dispersion Relation: $\quad \epsilon \equiv 1 + \sum_\alpha \frac{1}{k^2\lambda_{D\alpha}^2}\int_{-\infty}^{+\infty}\left(1 + \frac{\omega/k}{v - \omega/k}\right) F_{0\alpha} dv = 0$

$$\longrightarrow \qquad \epsilon \equiv 1 + [1 + \xi_e Z(\xi_e) + \tau + \tau\xi_i Z(\xi_i)]/(k\lambda_{De})^2 = 0,$$

$$\xi_\alpha \equiv \omega/\sqrt{2}kv_{t\alpha} \ , \qquad v_{t\alpha} \equiv \sqrt{T_\alpha/m_\alpha} \ , \qquad \tau \equiv T_e/T_i \ , \qquad \lambda_{De} \equiv \sqrt{T_e/4\pi n_0 e^2}$$

$$Z(\zeta) \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\exp(-t^2)}{t - \zeta}dt,$$

Z -- Plasma Dispersion Function

Landau Damping - collisionless dissipation: $\frac{\partial F}{\partial v} < 0$

- Weakly damped modes $\quad \omega \equiv \omega_R + i\omega_I \qquad |\omega_R| \gg |\omega_I|$

$$\epsilon(k,\omega) = \epsilon_R(k,\omega) + i\epsilon_I(k,\omega)$$

$$= \epsilon_R(k,\omega_R + i\omega_I) + i\epsilon_I(k,\omega_R)$$

$$= \epsilon_R(k,\omega_R) + i\epsilon_I(k,\omega_R) + i\,\omega_I \frac{\partial \epsilon_R(k,\omega)}{\partial \omega}\bigg|_{\omega=\omega_R} = 0.$$ Taylor's expansion

$$\epsilon_R(k,\omega_R) = 0 \qquad\qquad \omega_I = -\frac{\epsilon_I(k,\omega_R)}{\partial\epsilon_R/\partial\omega|_{\omega=\omega_R}}$$

- Cold response: $\quad \frac{\omega}{k} \gg v_t$

$v_{ph} \equiv \frac{\omega}{k}$ -- phase velocity of the wave

- Warm response: $\quad \frac{\omega}{k} \ll v_t$

- Resonant denominator:

$$\frac{1}{x-a} = \frac{1}{x - a_R - ia_I} \approx \frac{1}{x-a_R} + i\pi\delta(x-a_R) \qquad |a_I| \ll |a_R|$$

- Dirac delta function:

$$\pi\delta(x) = lim_{\sigma\to 0}\frac{\sigma}{x^2+\sigma^2}$$

- Landau contour (1945)

Normal Modes in a One-Dimensional Plasma

$$\delta f = -\frac{q}{T}\left[1 + \frac{\omega/k}{v - \omega/k} + i\pi\frac{\omega}{k}\delta(v - \omega/k)\right]\phi F_0$$ -- weakly damped modes

- Cold plasma limit, $|\omega/k| \gg v_{t\alpha}$

$$\omega \approx \omega_{pe}\sqrt{1 + 3k^2\lambda_{De}^2}\left[\pm 1 - i\sqrt{\frac{\pi}{8}}\sqrt{1 + 3k^2\lambda_{De}^2}\frac{\exp(-1/2k^2\lambda_{De}^2)}{(k\lambda_{De})^3}\right]$$

$$\omega_{pe} \equiv \pm\sqrt{4\pi n_o e^2/m_e}$$ -- plasma waves contributed by electrons only

- Cold ions, $|\omega/kv_{ti}| >> 1$ and warm electrons, $|\omega/kv_{te}| << 1$

$$\omega \approx \frac{kc_s}{(1 + k^2\lambda_{De}^2)^{1/2}}\left[\pm 1 - i\sqrt{\frac{\pi}{8}}\sqrt{\frac{m_e}{m_i}}\frac{1}{(1 + k^2\lambda_{De}^2)^{3/2}}\right] \qquad c_s \equiv \sqrt{T_e/m_i}$$

-- ion acoustic waves contributed by both electrons and ions

- These are the damped normal modes based on the linear dispersion relation
- These oscillations can be observed in an one dimensional particle code.
- This is the first step in verifying the code: frequencies and damping rates of these normal modes.

Landau Contour [L. D. Landau, J. Phys. **9**, 25 (1945)]

- Converting an initial value problem to an eigenmode problem (Cauchy integral formula)

$$\tilde{f}(x,v,p) = \int_0^\infty f(x,v,t)e^{-pt}dt \qquad \tilde{f}(x,v,\omega) = \int_{-\infty}^\infty f(x,v,t)e^{i\omega t}dt$$

$$p = -i\omega \qquad Re(p) > 0 \longrightarrow Im(\omega) > 0$$

$$I(Z) = \int_{-\infty}^\infty \frac{f(u)}{u-z}du \qquad Im\ z > 0$$

$$I(Z) = \int_{-\infty}^\infty \frac{f(u)}{u-z}du + 2\pi i f(z) \qquad Im\ z < 0$$

$$I(Z) = P\int_{-\infty}^\infty \frac{f(u)}{u-Re\ z}du + \pi i f(Re\ z) \qquad Im\ z \rightarrow 0$$

- Plasma Dispersion Function uses the Landau contour.
- For most cases, we are only interested in the weakly damped (marginally stable) modes, *Im z → 0*, i.e., normal modes of the system.
- Marginally unstable modes are the normal modes of the system and particle simulation needs them for code verification.
- Particle simulation also needs all the damped modes for numerical stability.
- However, most of the theoretical work is focussed on unstable modes.

Fluctuation-Dissipation Theorem for a Plasma in Equilibrium

$$\frac{dF_\alpha}{dt} \equiv \frac{\partial F_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial F_\alpha}{\partial \mathbf{x}} + \frac{q_\alpha}{m_\alpha}\mathbf{E}\cdot\frac{\partial F_\alpha}{\partial \mathbf{v}} = 0$$

$$\nabla^2\phi = -4\pi e\int (F_i - F_e)d\mathbf{v}$$

$$\frac{d}{dt}\left\langle \frac{m_e}{2}\int v^2 F_e d\mathbf{v} + \frac{m_i}{2}\int v^2 F_i d\mathbf{v} + \frac{1}{8\pi}|\nabla\phi|^2 \right\rangle_{\mathbf{x}} = 0$$ -- Energy Conservation

$$\frac{\partial \delta f_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial \delta f_\alpha}{\partial \mathbf{x}} + \frac{q_\alpha}{m_\alpha}\mathbf{E}\cdot\frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} = 0$$

$$\nabla^2\phi = -4\pi e\int (\delta f_i - \delta f_e)d\mathbf{v}$$

$$\mathbf{E} = -\nabla\phi$$

$$\delta f_\alpha = -\frac{q_\alpha}{m_\alpha}\frac{1}{i(\mathbf{k}\cdot\mathbf{v} - \omega)}\mathbf{E}\cdot\frac{\partial F_{\alpha 0}}{\partial \mathbf{v}}$$

$$\epsilon k^2\phi \equiv \left[1 + \sum_\alpha \int \frac{1}{k^2\lambda_{D\alpha}^2}\frac{\mathbf{k}\cdot\mathbf{v}}{\mathbf{k}\cdot\mathbf{v} - \omega}F_{\alpha 0}d\mathbf{v}\right]k^2\phi = 0$$

$$\epsilon \equiv 1 + \frac{1}{k^2\lambda_{De}^2}[1 + \tau + \xi_e Z(\xi_e) + \tau\xi_i Z(\xi_i)]$$

$$\delta f_\alpha = \left[\delta f_\alpha(t=0) - \frac{q_\alpha}{m_\alpha}\mathbf{E}\cdot\frac{\partial F_{\alpha 0}}{\partial \mathbf{v}}\right]\frac{1}{i(\mathbf{k}\cdot\mathbf{v}-\omega)}$$

$$\epsilon\phi = -\frac{4\pi e}{k^2}\int\frac{\delta f_e(t=0)}{i(\mathbf{k}\cdot\mathbf{v}-\omega)}d\mathbf{v}$$

-- Original Landau problem

-- Following Klimontovich '67 (pp. 168 - 172)

$$\mathbf{E}\cdot\mathbf{E} = \frac{1}{|\epsilon(\mathbf{k}^2,\omega)|^2}\sum_{a,b}\frac{(4\pi e)^2}{k^2} lim_{\Delta\to 0} 2\Delta\int\frac{\delta f_e^a(t=0)\delta f_e^b(t=0)d\mathbf{v}d\mathbf{v}'}{(\omega-\mathbf{k}\cdot\mathbf{v}+i\Delta)(\omega-\mathbf{k}\cdot\mathbf{v}'+i\Delta)}$$

$$\frac{|\mathbf{E}(\mathbf{k},\omega)|^2}{8\pi} = \frac{T}{\omega}\frac{\epsilon_I(\mathbf{k},\omega)}{\epsilon_R^2(\mathbf{k},\omega)+\epsilon_I^2(\mathbf{k},\omega)}$$

-- Another form

$$\boxed{\frac{|\mathbf{E}(\mathbf{k},\omega)|^2}{8\pi} = -\frac{T}{\omega} Im\left[\frac{1}{\epsilon(\mathbf{k},\omega)} - 1\right]}$$

-- Valid for a weakly-damped system in thermal equilibrium

- Dispersion relation for the cold ions and kinetic electrons:

$$\epsilon \equiv 1 + \frac{1}{k^2 \lambda_{De}^2}[1 + \xi_e Z(\xi_e)]$$

- Integration over the ω space

$$\frac{\mathbf{E}(\mathbf{k}) \cdot \mathbf{E}(\mathbf{k})}{8\pi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathbf{E}(\mathbf{k},\omega) \cdot \mathbf{E}(\mathbf{k},\omega)}{8\pi} d\omega$$

$$= -\frac{T}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega} Im \left[\frac{1}{\epsilon(\mathbf{k},\omega)} \right] d\omega$$

Kramers-Kronig Relation: connecting real and imaginary parts of a complex function, which is analytic in the upper half plane, through Cauchy's Theorem.

$$= -\frac{T}{2} Re \left[\frac{1}{\epsilon(\omega = 0)} - \frac{1}{\epsilon(\omega = \infty)} \right]$$

$$\epsilon(\omega = 0) = 1 + \frac{1}{k^2 \lambda_{De}^2} \qquad \epsilon(\omega = \infty) = 1$$

- Total thermal fluctuations

$$\boxed{\frac{\mathbf{E}(\mathbf{k}) \cdot \mathbf{E}(\mathbf{k})}{8\pi} = \frac{T}{2} \frac{1}{1 + k^2 \lambda_{De}^2}}$$

Boltzmann Collision Operator

[Krall & Trivelpiece, '74] TRANSPORT PHENOMENA IN PLASMA 313

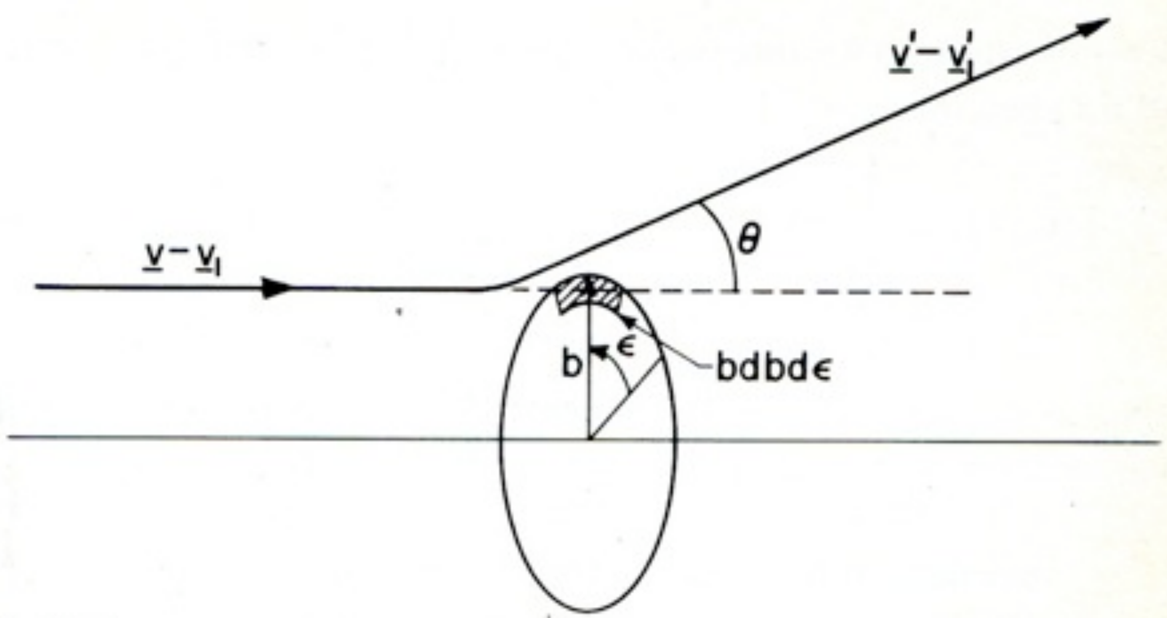

FIGURE 6.6.1
Center-of-mass coordinate system used to calculate the change of the distribution function as a result of collisions.

$$\left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} = N\int d\mathbf{v}_1|\mathbf{v}-\mathbf{v}_1|\int_{b=0}^{\infty} bdb\int_{\epsilon=0}^{2\pi}[f_\alpha(\mathbf{x},\mathbf{v}^{'},t)f_\gamma(\mathbf{x},\mathbf{v}_1^{'},t)-f_\alpha(\mathbf{x},\mathbf{v},t)f_\gamma(\mathbf{x},\mathbf{v}_1,t)]d\epsilon$$

• Fokker-Planck type of equation can be recovered using the cross section for Coulomb scattering and considering only contributions from small angle scattering with cutoffs at the Debye length - strong short range vs. weak long range [pp. 293, K&T '74] .

$$\left(\frac{\delta f}{\delta t}\right)_{coll} = -\Gamma\nabla_{\mathbf{v}}\cdot[\mathbf{A}f]+\frac{1}{2}\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}:[\tilde{\mathbf{B}}f] \qquad g=\frac{1}{n\lambda_D^3}\ll 1$$

• Lorentz model is valid for light particles scattering off the heavy particles.

Lorentz Collision Operator

$$\left(\frac{\delta f}{\delta t}\right)_{coll} \equiv J(f) = Nv\int_{b=0}^{\infty} bdb \int_{\alpha=0}^{2\pi} [f(\mathbf{r},\mathbf{v}',t) - f(\mathbf{r},\mathbf{v},t)]d\alpha$$

$$J(Y_{mn}) = -\nu_n Y_{mn}$$ Y_{mn} -- spherical harmonics

$$\nu_n = 2\pi Nv\int_0^{\infty} [1 - P_n(cos\theta)]bdb$$ P_n -- Legendre polynomial

$$\nu_0 = 0 \qquad \nu_1 = 2\pi Nv\int_0^{\infty} [1 - cos\theta]bdb$$

$$\nabla_{\mathbf{v}}^2 \phi = 0 \qquad \phi = \frac{U(v)}{v} P(\theta)Q(\varphi) \approx P(\theta)$$

$$\longrightarrow \quad \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial P}{\partial\theta}\right) + n(n+1)P = 0$$ -- spherical coordinates

$$P(n=1) \rightarrow f$$

$$\left(\frac{\delta f}{\delta t}\right)_{coll} \equiv J(f) = -\nu_1 f = \frac{\nu_1}{2sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial f}{\partial\theta}\right)$$

Simple Models for Collisions

$$\boxed{C(F_e) = \frac{\nu_{ei}}{2sin\theta}\frac{\partial}{\partial\theta}\left[sin\theta(\frac{\partial F_e}{\partial\theta})\right]}$$

Lorentz Model
- conserve number density and energy

Let $cos\theta = v_\parallel / v$, $sin\theta = v_\perp / v$

$$\frac{\partial}{\partial\theta} = -v_\perp\frac{\partial}{\partial v_\parallel} + v_\parallel\frac{\partial}{\partial v_\perp}$$

Let $F_e = f_{\parallel e} f^M_{\perp e}$

$v_\parallel \to v$

$$\boxed{C(f_e) = \nu_{ei}\frac{\partial}{\partial v}\left(v_{te}^2\frac{\partial f_e}{\partial v} + v f_e\right)}$$

Lenard-Bernstein Model
- conserve number density only

$$\boxed{C(\delta f_e) = -\nu_{ei}\left(\delta f_e - F_{Me}\int \delta f_e d\mu dv_\parallel\right)}$$

Krook Model

- Collisions are strong short range interactions within the Debye sphere with

$$g = \frac{1}{n\lambda_D^3} \ll 1,$$

whereas Poissons's equation describes weak long range interactions in scales much longer than the Debye length.