Gyrokinetic Theory and Simulation

- Drift Kinetic Vlasov-Poisson equations
  - Guiding center motion
  - ExB drift and Polarization Drift
  - Lowest order gyrokinetic-Poisson equations

- Gyrokinetic Vlasov-Poisson equations
  - Gyrokinetic ordering
  - Gyrocenter coordinates
  - Gyrophase averaging

- Gyrokinetic particle pushing
  - Coordinates transformation

- Gyrokinetic field solver
  - Integral equation
  - Pede approximation

Magnetized Vlason-Poisson Equations

$$\frac{dF}{dt} \equiv \frac{\partial F}{\partial t} + \mathbf{v} \cdot \frac{\partial F}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial F}{\partial \mathbf{v}} = 0$$

$$\nabla^2 \phi = -4\pi e \int (F_i - F_e) d\mathbf{v} \qquad \mathbf{E} = -\nabla \phi$$

- Linear Properties for a magnetized plasma

$$\epsilon \equiv 1 + \frac{\omega_{pi}^2}{\Omega_i^2 - \omega^2}\frac{k_\perp^2}{k^2} + \frac{\omega_{pe}^2}{\Omega_e^2 - \omega^2}\frac{k_\perp^2}{k^2} - \frac{\omega_{pi}^2}{\omega^2}\frac{k_\parallel^2}{k^2} - \frac{\omega_{pe}^2}{\omega^2}\frac{k_\parallel^2}{k^2} = 0$$

$$\left.\begin{array}{ll} \text{gives the lower hybrid waves,} & \omega_{LH}^2 = \omega_{pi}^2 + \Omega_i^2 \\ \text{the upper hybrid waves,} & \boxed{\omega_{UH}^2 = \omega_{pe}^2 + \Omega_e^2} \end{array}\right\} \text{-- space charge waves}$$

- Linear Properties for a plasma with gyrating ions and drift kinetic electrons,

$$\epsilon \equiv 1 + \frac{\omega_{pi}^2}{\Omega_i^2 - \omega^2}\frac{k_\perp^2}{k^2} - \frac{\omega_{pe}^2}{\omega^2}\frac{k_\parallel^2}{k^2} = 0 \qquad \boxed{\omega_{LH}^2 = \omega_{pi}^2 + \Omega_i^2}$$

-- lower hybrid waves

- Linear Properties for a plasma with gyrokinetic ions and drift kinetic electrons,

$$\epsilon \equiv 1 + \frac{\omega_{pi}^2}{\Omega_i^2}\frac{k_\perp^2}{k^2} - \frac{\omega_{pe}^2}{\omega^2}\frac{k_\parallel^2}{k^2} = 0 \quad \omega_{pi}^2 \gg \Omega_i^2, \quad k_\parallel^2 \ll k_\perp^2 \quad \boxed{\omega_H^2 = \frac{m_i}{m_e}\frac{k_\parallel^2}{k_\perp^2}}$$

-- Quasineutral waves [Lee, '87]

- Linear Properties for a drift kinetic plasma

$$\epsilon \equiv 1 - \frac{\omega_{pi}^2}{\omega^2}\frac{k_\parallel^2}{k^2} - \frac{\omega_{pe}^2}{\omega^2}\frac{k_\parallel^2}{k^2} = 0 \qquad \boxed{\omega^2 = \omega_{pe}^2 \frac{k_\parallel^2}{k^2}}$$

plasma waves with $k_\parallel^2 \ll k^2$

Drift-Kinetic-Poisson Equations

$$\boxed{\frac{d}{dt}F(\mathbf{x}, v_{\parallel}, t) \equiv \frac{\partial F}{\partial t} + (\mathbf{v}_{\parallel} + \frac{c}{B}\mathbf{E}\times\hat{\mathbf{b}})\cdot\frac{\partial F}{\partial \mathbf{x}} + \frac{q}{m}\mathbf{E}_{\parallel}\cdot\frac{\partial F}{\partial \mathbf{v}} = 0}$$

$$\nabla^2\phi = -4\pi e\int(F_i - F_e)dv_{\parallel} \qquad \mathbf{E} = -\nabla\phi$$

- Drift Kinetic Approximation: $\rho_e[\equiv \frac{v_{te}}{\Omega_e}] \to 0 \qquad \rho_i[\equiv \frac{v_{ti}}{\Omega_i}] \to 0$
- Lowest order guiding center motion: $\mathbf{v} = \mathbf{v}_{\parallel} + \frac{c}{B}\mathbf{E}\times\hat{\mathbf{b}}$
- Next order guiding center motion: $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{E\times B} + \mathbf{v}_p^L$

$$\mathbf{v}_{E\times B} = \frac{c}{B}\mathbf{E}\times\hat{\mathbf{b}} \qquad \mathbf{v}_p^L = -\frac{mc^2}{eB^2}\frac{\partial\nabla_\perp\phi}{\partial t}$$ -- polarization drift $\qquad \frac{v_p^L}{v_{E\times B}} \propto \frac{\omega}{\Omega_i} \ll 1$

$$\frac{\partial n_p}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot\int \mathbf{v}_p^L F_{igc} d\mathbf{v} = 0 \qquad \rho_p = \frac{1}{4\pi}\frac{\omega_{pi}^2}{\Omega_i^2}\nabla_\perp^2\phi$$

$$\boxed{\nabla^2\phi + \frac{\omega_{pi}^2}{\Omega_i^2}\nabla_\perp^2\phi = -4\pi e\int(F_i - F_e)dv_{\parallel}}$$

- Lowest order gyrokinetic approximation:

$$\rho_s[\equiv \frac{c_s}{\Omega_i}] \neq 0 \qquad \rho_e[\equiv \frac{v_{te}}{\Omega_e}] \to 0 \qquad \rho_i[\equiv \frac{v_{ti}}{\Omega_i}] \to 0$$

Gyrokinetic Ordering

- Dispersion relation for magnetized plasmas (gyrating ions and drift kinetic electrons)

$$\epsilon \equiv 1 + \boxed{\frac{\omega_{pi}^2}{\Omega_i^2 - \omega^2} \frac{k_\perp^2}{k^2}} - \frac{\omega_{pe}^2}{\omega^2} \frac{k_\parallel^2}{k^2} = 0$$

- Lower hybrid waves, $\omega_{LH}^2 = \omega_{pi}^2 + \Omega_i^2 \approx \omega_{pi}^2$ -- space charge waves

- Electrostatic shear Alfven waves for $\boxed{\omega \ll \Omega_i}$

$$\omega = \pm \omega_H \equiv \pm (\frac{k_\parallel}{k})(\frac{\omega_{pe}}{\omega_{pi}})\Omega_i$$ -- quasineutral waves

- Polarization response is frequency independent

- Gyrokinetic ordering [Rutherford and Frieman '68]

$$\omega/\Omega \sim \rho_i/L_e q \sim k_\parallel \rho_i \sim e\Phi/T_e \sim \delta B/B_0 \sim O(\epsilon)$$

$$k_\perp \rho_i \sim O(1)$$

Nonlinear Gyrokinetic Equations

- In the late seventies, two different routes were taken at PPPL to develop the nonlinear gyrokinetic equation based on the gyrokinetic ordering by Rutherford and Frieman.

  - Frieman and Chen, Phys. Fluids **25**, 505 (1982).
  - Lee, Phys. Fluids **26**, 556 (1983).

- The main philosophical difference is that one is looking for the next order nonlinear modifications to the linear gyrokinetic equations while the other is looking for the nonlinear equations that can be used for simulations.

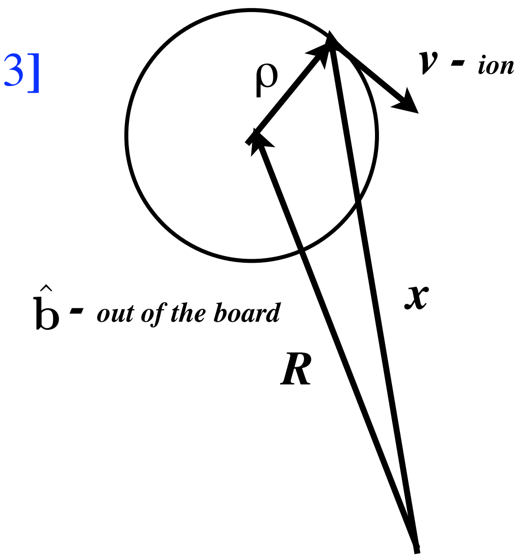
- The starting point is the same

$$\frac{dF}{dt} \equiv \frac{\partial F}{\partial t} + \mathbf{v} \cdot \frac{\partial F}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial F}{\partial \mathbf{v}} = 0$$

$$\nabla^2 \phi = -4\pi e \int (F_i - F_e) d\mathbf{v}$$

$$\mathbf{E} = -\nabla \phi$$

Gyrokinetic Vlasov Equation [Lee, PF '83]



- Gyrokinetic change of variables

$$F(\mathbf{x}, \mathbf{v}, t) \rightarrow F(\mathbf{R}, \mu, U, \varphi, t)$$

$$\mathbf{x} = \mathbf{R} + \rho$$

- Vlasov equation in new variables

$$\frac{\partial F}{\partial t} + (U\hat{\mathbf{b}} + \frac{c}{B}\mathbf{E} \times \hat{\mathbf{b}}) \cdot \frac{\partial F}{\partial \mathbf{R}} + \frac{q}{m}\mathbf{E} \cdot \hat{\mathbf{b}}\frac{\partial F}{\partial U}$$

-- drift-kinetic-like terms

$$+\mathbf{v} \cdot (\frac{\partial \rho}{\partial \mathbf{x}} \cdot \frac{\partial F}{\partial \mathbf{R}} + \frac{\partial \mu}{\partial \mathbf{x}}\frac{\partial F}{\partial \mu} + \frac{\partial U}{\partial \mathbf{x}}\frac{\partial F}{\partial U} + \frac{\partial \varphi}{\partial \mathbf{x}}\frac{\partial F}{\partial \varphi})$$

-- toroidal geometry terms

$$-\Omega\frac{\partial F}{\partial \varphi} + \frac{q}{m}\mathbf{E} \cdot (\frac{\mathbf{v}_\perp}{B}\frac{\partial F}{\partial \mu} + \frac{\hat{\mathbf{b}} \times \mathbf{v}_\perp}{v_\perp^2}\frac{\partial F}{\partial \varphi}) = 0$$

-- fast gyromotion terms

$$\hat{\mathbf{b}} \equiv \mathbf{B}/B, \quad B = |\mathbf{B}|, \quad \mu \equiv \frac{v_\perp^2}{2B}\left(1 - \frac{mc}{e}\frac{U}{B}\hat{\mathbf{b}} \cdot \nabla\times\ \hat{\mathbf{b}}\right),$$

$$\rho = \hat{\mathbf{b}} \times \mathbf{v}_\perp/\Omega, \quad \mathbf{v}_\perp = v_\perp(cos\varphi\hat{\mathbf{e}}_1 + sin\varphi\hat{\mathbf{e}}_2), \quad \hat{\mathbf{b}} = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2, \quad \mathbf{v} = U\hat{\mathbf{b}} + \mathbf{v}_\perp$$

$$\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$$

- Mixed representations between gyrocenter coordinates ***R*** and original particle coordinates ***X.***

Gyrokinetic Vlasov Equation (cont.)

• Invoking gyrokinetic ordering to the lowest order to obtain

$$\Omega \frac{\partial F}{\partial \varphi} = 0$$ -- F is independent of phase to the lowest order

• Let $F = f + \epsilon g(\varphi)$ where $f \neq f(\varphi)$

$$\boxed{\begin{aligned} &\frac{\partial f}{\partial t} + (U\hat{\mathbf{b}} + \frac{c\mathbf{E}\times\hat{\mathbf{b}}}{B}) \cdot \frac{\partial f}{\partial \mathbf{R}} + \frac{q}{m}\mathbf{E}\cdot\hat{\mathbf{b}}\frac{\partial f}{\partial U} \\ &+\mathbf{v}\cdot(\frac{\partial \rho}{\partial \mathbf{x}}\cdot\frac{\partial f}{\partial \mathbf{R}} + \frac{\partial \mu}{\partial \mathbf{x}}\frac{\partial f}{\partial \mu} + \frac{\partial U}{\partial \mathbf{x}}\frac{\partial f}{\partial U}) \\ &-\Omega\frac{\partial}{\partial\varphi}(g - \frac{q\Phi}{mB}\frac{\partial f}{\partial \mu}) = 0 \end{aligned}}$$

-- drift-kinetic-like terms

-- toroidal geometry terms

-- fast gyromotion terms

by using

$$\frac{\partial \Phi}{\partial \mathbf{x}} \approx \frac{\partial \Phi}{\partial \mathbf{R}}$$

$$\Omega\frac{\partial \Phi}{\partial \varphi} = -\mathbf{v}_\perp \cdot \frac{\partial \Phi}{\partial \mathbf{R}}$$

$$\partial\Phi/\partial\mathbf{v} = 0$$

Gyrokinetic Vlasov Equation (cont.)

- Taking gyrophase average of the equation for f to obtain

$$\boxed{\frac{\partial f}{\partial t} + \frac{d\mathbf{R}}{dt} \cdot \frac{\partial f}{\partial \mathbf{R}} + \frac{dU}{dt}\frac{\partial f}{\partial U} = 0}$$

$$\frac{d\mathbf{R}}{dt} = U\hat{\mathbf{b}} + \frac{c}{B}\bar{\mathbf{E}} \times \hat{\mathbf{b}} + \hat{\mathbf{b}} \times \left[\frac{U^2}{\Omega}(\hat{\mathbf{b}} \cdot \frac{\partial}{\partial \mathbf{R}})\hat{\mathbf{b}} + \frac{v_\perp^2}{2\Omega}\frac{\partial lnB}{\partial \mathbf{R}}\right]$$

$$\frac{dU}{dt} = (\frac{q}{m}\bar{\mathbf{E}} - \frac{v_\perp^2}{2}\frac{\partial lnB}{\partial \mathbf{R}}) \cdot \left[\hat{\mathbf{b}} + \frac{U}{\Omega}\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \frac{\partial}{\partial \mathbf{R}})\hat{\mathbf{b}}\right]$$

$$\bar{\mathbf{E}}(\mathbf{R}) \equiv \oint \mathbf{E}(\mathbf{x}) d\varphi / 2\pi$$

- Valid for $\boxed{\rho_s / L_{eq} \sim O(\epsilon)}$

Φ^2 terms are neglected - [Dubin et al., PF '83]

Gyrokinetic Poisson's Equation [Lee, PF '83]

- Now, we know how to calculate *f* in $F = f + \epsilon g(\varphi)$, what about *g*?
- To the lowest order, we assume

$$g = \frac{q}{mB}\frac{\partial f}{\partial \mu}[\Phi(\mathbf{x}) - \bar{\Phi}(\mathbf{R})]$$

- Thus, we have

$$\boxed{F = f + \frac{q}{mB}\frac{\partial f}{\partial \mu}(\Phi - \bar{\Phi})}$$

- What is $\bar{\Phi}$?

$$\Phi(x) = \sum_{\mathbf{k}} \Phi(\mathbf{k}) exp(i\mathbf{k}\cdot\mathbf{x}) = \sum_{\mathbf{k}} \Phi(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{R} + i\mathbf{k}\cdot\rho)$$

$$\bar{\Phi}(\mathbf{R}) = \frac{1}{2\pi}\oint d\varphi \int \Phi(\mathbf{x})\delta(\mathbf{x} - \mathbf{R} - \rho)d\mathbf{x} = \sum_{\mathbf{k}} \Phi(\mathbf{k}) J_0(\frac{k_\perp v_\perp}{\Omega}) exp(i\mathbf{k}\cdot\mathbf{R})$$

$$\langle e^{\pm i\mathbf{k}\cdot\rho_{\alpha j}}\rangle_\varphi \equiv \frac{1}{2\pi}\oint d\varphi \exp(\pm i\mathbf{k}\cdot\rho) = J_0(\frac{k_\perp v_\perp}{\Omega})$$

$$\bar{\mathbf{E}} = -\partial\bar{\Phi}/\partial\mathbf{R}$$

Gyrokinetic Poisson's Equation [Lee, PF '83]

- Let $\dfrac{\partial f}{\partial \mu} \approx \dfrac{\partial F_0}{\partial \mu}$

$$F = f + \frac{q}{mB}\frac{\partial f}{\partial \mu}(\Phi - \bar{\Phi}) \longrightarrow F = f - \frac{q}{T}[\Phi(\mathbf{x}) - \bar{\Phi}]F_0$$

- Gyrokinetic Poisson's equation

$$\boxed{\underline{\nabla^2\Phi} - \underline{\tau\frac{n_{i0}}{n_0\lambda_{De}^2}(\Phi - \tilde{\Phi})} = -4\pi e(\bar{n}_i - n_e)}$$

≈ 0 Polarization density

Quasineutral response

$$\bar{n}_i(\mathbf{x}) = \frac{1}{2\pi}\oint d\varphi \int f(\mathbf{R})\delta(\mathbf{R} - \mathbf{x} + \rho)Bd\mathbf{R}dUd\mu = \sum_{\mathbf{k}}\int f(\mathbf{k})J_0(\frac{k_\perp v_\perp}{\Omega})exp(i\mathbf{k}\cdot\mathbf{x})BdUd\mu$$

$$n_e(\mathbf{x}) = \int f_e(\mathbf{R})\delta(\mathbf{R} - \mathbf{x})Bd\mathbf{R}dUd\mu$$

$$n_{i0}(\mathbf{x}) = \int F_{0i}(\mathbf{x})dUd\mu$$

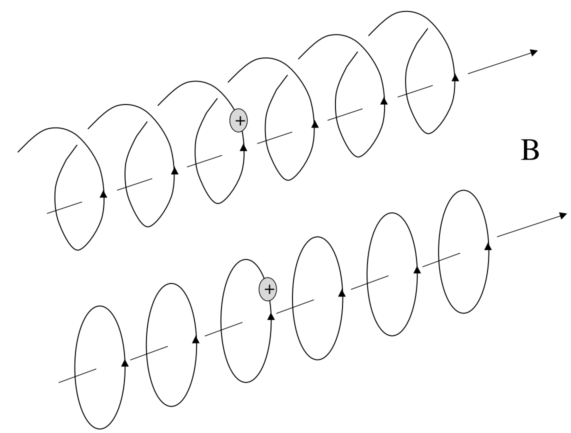
$$\tilde{\Phi}(\mathbf{x}) = \frac{1}{2\pi}\oint d\varphi \int \bar{\Phi}(\mathbf{R})F_{0i}(\mathbf{x})\delta(\mathbf{R} - \mathbf{x} + \rho)Bd\mathbf{R}dUd\mu = \sum_k \Phi(\mathbf{k})\Gamma_0(k_\perp^2\rho_i^2)exp(i\mathbf{k}\cdot\mathbf{x})$$

$$\Gamma_0(b) \equiv I_0(b)\exp(-b) = \frac{1}{v_t^2}\int_0^\infty \exp(-v_\perp^2/2v_t^2)J_0^2(k_\perp v_\perp/\Omega)v_\perp dv_\perp$$

Gyrokinetic Particle Simulation [Lee, JCP '87]

- Particle pushing is carried out in the gyrocenter coordinates based on the gyrokinetic Vlasov equation

  -- gyromotion of a particle is approximated by a charged ring moving along the field line



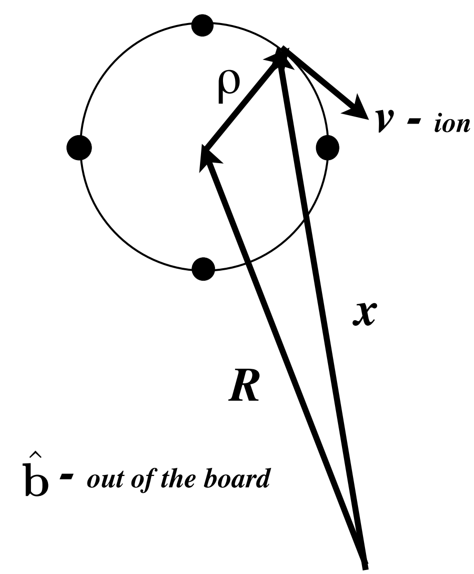
- Field is solved in the laboratory (original particle) coordinates based on the gyrokinetic Poisson equation

- Efficient coordinates transformation between ***R*** and ***x*** can be achieved through 4-point average.

$$\langle e^{\pm i\mathbf{k}\cdot\rho_{\alpha j}}\rangle_{\varphi} = \sum_{n=-\infty}^{\infty} J_n(k_{\perp}\rho_{\alpha j})\frac{1}{L}\sum_{l=1}^{L} exp(\frac{i2\pi nl}{L})$$

$$= \sum_{n=-\infty}^{\infty} J_n(k_{\perp}\rho_{\alpha j})\frac{1}{2L} sin2\pi n\frac{cos\frac{n\pi}{L}}{sin\frac{n\pi}{L}}$$

$$= J_0 + O(J_{\pm mL}), m = 1, 2, 3, \cdots$$



L → ∞, we recover *Jo*

L = 4 is good for $k_{\perp}\rho_i \leq 2$

Gyrokinetic Particle Simulation (cont.)

- Gyrokinetic Poisson's equation in ***k***-space: $b \equiv k_\perp^2 \rho_i^2$

$$k^2\lambda_D^2 \frac{e\Phi}{T_e}(\mathbf{k}) + \tau[1-\Gamma_0(b)]\frac{e\Phi}{T_e}(\mathbf{k}) = \frac{\bar{n}_i(\mathbf{k}) - n_e(\mathbf{k})}{n_0}$$

- Gyrokinetic Poisson's equation in the limit of $k_\perp^2 \rho_i^2 \ll 1$

$$\rho_s^2 \nabla_\perp^2 \frac{e\Phi}{T_e} = -\frac{\bar{n}_i - n_e}{n_0}$$

- Pade approximation: $\Gamma_0 \approx \frac{1}{1+b}$

$$\rho_s^2 \nabla_\perp^2 \frac{e\Phi}{T_e} = -(1-\rho_i^2\nabla_\perp^2)\frac{\bar{n}_i - n_e}{n_0}$$

- Energy Conservation

$$\sum_{\alpha=e,i} m_\alpha \langle \int (\frac{v_\perp^2}{2} + \frac{U^2}{2}) F_\alpha d\frac{v_\perp^2}{2} dU \rangle + \frac{1}{8\pi}\sum_{\mathbf{k}} k^2 |\Phi(\mathbf{k})|^2 + \frac{\tau}{8\pi\lambda_D^2}\sum_{\mathbf{k}}[1-\Gamma_0(b)]|\Phi(\mathbf{k})|^2 = const.$$