Fully Electromagnetic Vlasov-Maxwell System

$$\frac{dF}{dt} \equiv \frac{\partial F}{\partial t} + \mathbf{v} \cdot \frac{\partial F}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial F}{\partial \mathbf{v}} = 0$$ Vlasov equation

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \rho = e\int (F_i - F_e) d\mathbf{v}$$ Coulomb's law

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \qquad \mathbf{J} = e\int \mathbf{v}(F_i - F_e) d\mathbf{v}$$ Ampere's law

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$ Faraday's law

$$0 = 4\pi\nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$ No magnetic monopole

From Vlasov + Poisson: $$\boxed{0 = 4\pi\nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{E}}{\partial t}}$$

Let $\mathbf{Q} = \mathbf{Q}^L + \mathbf{Q}^T$ such that $\nabla \cdot \mathbf{Q}^T = 0$ and $\nabla \times \mathbf{Q}^L = 0$

$$\nabla \cdot \mathbf{E}^L = 4\pi\rho$$

$$\nabla \times \mathbf{B}^T = \frac{4\pi}{c}\mathbf{J}^T + \frac{1}{c}\frac{\partial \mathbf{E}^T}{\partial t} \qquad 0 = \frac{4\pi}{c}\mathbf{J}^L + \frac{1}{c}\frac{\partial \mathbf{E}^L}{\partial t}$$

$$\nabla \times \mathbf{E}^T = -\frac{1}{c}\frac{\partial \mathbf{B}^T}{\partial t}$$

$$\nabla \cdot \mathbf{B}^T = 0$$

Various Physics Models based on Maxwell's Equations

- Electrostatic Model

$$\boxed{\nabla \cdot \mathbf{E}^L = 4\pi\rho}$$

- Darwin Model -- aka Finite-Beta Model

$$\nabla \cdot \mathbf{E}^L = 4\pi\rho$$

$$\nabla \times \mathbf{B}^T = \frac{4\pi}{c}\mathbf{J}^T$$ ----- ignore the transverse displacement current > no light waves

$$\mathbf{J}^L = -\frac{1}{4\pi}\frac{\partial \mathbf{E}^L}{\partial t}$$

$$\nabla \times \mathbf{E}^T = -\frac{1}{c}\frac{\partial \mathbf{B}^T}{\partial t}$$

$$\nabla \cdot \mathbf{B}^T = 0$$

- MHD Model -- only keeping track of transverse quantities

$$\nabla \times \mathbf{B}^T = \frac{4\pi}{c}\mathbf{J}^T$$

$$\nabla \times \mathbf{E}^T = -\frac{1}{c}\frac{\partial \mathbf{B}^T}{\partial t}$$

$$\nabla \cdot \mathbf{B}^T = 0$$

Maxwell Equations in Coulomb Gauge:

$$\mathbf{E}^L = -\nabla\phi$$

$$\mathbf{B}^T = \nabla \times \mathbf{A}$$

$$\boxed{\nabla \cdot \mathbf{A} = 0}$$

- Fully Electromagnetic Maxwell Equations:

$$\nabla^2\phi = -4\pi\rho \qquad \mathbf{E}^T = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}^T \qquad \mathbf{J}^L = -\frac{1}{4\pi}\frac{\partial \mathbf{E}^L}{\partial t}$$

- Electrostatic model:

$$\nabla^2\phi = -4\pi\rho$$

- Darwin model: no light waves

$$\nabla^2\phi = -4\pi\rho \qquad \mathbf{E}^T = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}^T \qquad \mathbf{J}^L = -\frac{1}{4\pi}\frac{\partial \mathbf{E}^L}{\partial t}$$

- MHD model:

$$\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}^T \qquad \mathbf{E}^T = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

On Ampere's Law

- Continuity Equation $$0 = 4\pi\nabla\cdot\mathbf{J} - \frac{\partial\nabla^2\phi}{\partial t} \qquad 4\pi i\mathbf{k}\cdot\mathbf{J} + k^2\frac{\partial\phi}{\partial t} = 0$$

- Ampere's Law (longitudinal)

$$0 = \frac{4\pi}{c}\mathbf{J}^L + \frac{1}{c}\frac{\partial\mathbf{E}^L}{\partial t} \qquad 4\pi i\mathbf{J}^L + \mathbf{k}\frac{\partial\phi}{\partial t} = 0$$

- We obtain

$$\mathbf{J}^L = \frac{\mathbf{k}\cdot\mathbf{J}}{k^2}\mathbf{k} \qquad \mathbf{J}^T = \mathbf{J} - \frac{\mathbf{k}\cdot\mathbf{J}}{k^2}\mathbf{k} \qquad \nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}^T$$

- For $J_\parallel \gg J_\perp$

$$J_\parallel^T \approx J_\parallel(1 - k_\parallel^2/k^2) \qquad \mathbf{J}_\perp^T \approx -\frac{k_\parallel J_\parallel}{k^2}\mathbf{k}_\perp$$

- For long thin approximation, $k_\parallel \ll k$

$$J_\parallel^T \approx J_\parallel \qquad \mathbf{J}_\perp^T \approx 0$$

- Ampere's Law (transverse) $$\boxed{\nabla^2 A_\parallel = -\frac{4\pi}{c}J_\parallel}$$

Finite-Beta Gyrokinetic Model (in GK units)

$$\frac{dF_\alpha}{dt} \equiv \frac{\partial F_\alpha}{\partial t} + v_\parallel \hat{\mathbf{b}} \cdot \frac{\partial F_\alpha}{\partial \mathbf{x}} + \mathbf{E}^L \times \hat{\mathbf{b}}_0 \cdot \frac{\partial F_\alpha}{\partial \mathbf{x}} + s_\alpha v_{t\alpha}^2 (\mathbf{E}^L \cdot \hat{\mathbf{b}} + E_\parallel^T) \frac{\partial F_\alpha}{\partial v_\parallel} = 0$$

$$\hat{\mathbf{b}} \equiv \hat{\mathbf{b}}_0 + \frac{\delta \mathbf{B}}{B_0} = \frac{\mathbf{B_0}}{B_0} + \nabla A_\parallel \times \hat{\mathbf{b}}_0$$

$$\mathbf{E}^L = -\nabla \phi \qquad E_\parallel^T = -\frac{\partial A_\parallel}{\partial t} \qquad e\phi/T_e \to \phi \qquad \frac{eA_\parallel}{T_e}\frac{c_s}{c} \to A_\parallel$$

$$v_A = c\frac{\lambda_{De}}{\rho_s} \qquad \beta = \frac{c_s^2}{v_A^2}$$

$$\nabla_\perp^2 \phi = \int (F_e - F_i) dv_\parallel \qquad \nabla_\perp^2 A_\parallel = \beta \int v_\parallel (F_e - F_i) dv_\parallel$$

Dispersion: $\epsilon \equiv 1 + \{k_\perp^2 \rho_s^2 + [1 - \beta \frac{\omega^2}{k_\parallel^2}][1 + \xi_e Z(\xi_e) + \tau + \tau \xi_i Z(\xi_i)]\}/(k\lambda_{De})^2 = 0,$

cold species: $\omega = \pm \frac{\omega_H}{\sqrt{1 + \omega_{pe}^2/c^2k^2}} = \pm \frac{k_\parallel v_A}{\sqrt{1 + c^2k^2/\omega_{pe}^2}}$, warm electrons: $\omega = \pm k_\parallel v_A \sqrt{1 + k_\perp^2 \rho_s^2}$

Energy: $\left\langle \frac{1}{2}\int (v_\parallel/v_{te})^2 F_e dv_\parallel + \frac{1}{2}\int (v_\parallel/v_{ti})^2 F_i dv_\parallel + \frac{1}{2}|\nabla_\perp \phi|^2 + \frac{1}{2\beta}|\nabla_\perp A_\parallel|^2 \right\rangle = cons.$

- Electrostatic model: $A_\parallel \to 0 \qquad \beta \to 0$

$$\epsilon \equiv 1 + [k_\perp^2 \rho_s^2 + 1 + \xi_e Z(\xi_e) + \tau + \tau \xi_i Z(\xi_i)]/(k\lambda_{De})^2 = 0, \;\; \omega = \pm \omega_H = \pm \frac{k_\parallel}{k_\perp}\sqrt{\frac{m_i}{m_e}}\Omega_i$$

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