Darwin Electromagnetic (finite-β) Gyrokinetic Equations

- $$\frac{dF_\alpha}{dt} \equiv \frac{\partial F_\alpha}{\partial t} + v_\parallel \hat{\mathbf{b}} \cdot \frac{\partial F_\alpha}{\partial \mathbf{x}} + (\mathbf{E}^L + \mathbf{E}^T_\perp) \times \hat{\mathbf{b}}_0 \cdot \frac{\partial F_\alpha}{\partial \mathbf{x}} + s_\alpha v_{t\alpha}^2 (\mathbf{E}^L \cdot \hat{\mathbf{b}} + E^T_\parallel) \frac{\partial F_\alpha}{\partial v_\parallel} = 0$$

  $$n = n_{gc} + n_p \qquad \frac{\partial n_p}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \int \mathbf{v}_p^L F_{igc} d\mathbf{v} = 0 \qquad \mathbf{v}_p^L = -\frac{mc^2}{eB^2} \frac{\partial \nabla_\perp \phi}{\partial t}$$

- $$\nabla^2 \phi + \frac{\omega_{pi}^2}{\Omega_i^2} \nabla_\perp^2 \phi = -4\pi e (n_{igc} - n_{egc})$$

- $$\mathbf{J} = \mathbf{J}_{gc} + e \int \mathbf{v}_p^T F_{igc} d\mathbf{v} \qquad \mathbf{v}_p^T = -\frac{mc^2}{eB^2} \frac{1}{c} \frac{\partial \mathbf{A}_\perp}{\partial t}$$

- $$\nabla^2 \mathbf{A}_\perp - \frac{1}{v_A^2} \frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} = -\frac{4\pi}{c} \mathbf{J}_{\perp gc}$$

- $$\nabla^2 A_\parallel = -\frac{4\pi}{c} J_{\parallel gc} \qquad J_{\parallel gc} = \sum_\alpha q_\alpha \int v_\parallel F_{\alpha gc} dv_\parallel d\mu.$$

- $$\mathbf{E}^T_\perp = -\frac{\partial \mathbf{A}_\perp}{\partial t} \qquad E^T_\parallel = -\frac{\partial A_\parallel}{\partial t}$$

- $$\mathbf{J}_{\perp gc}(\mathbf{x}) = \sum_\alpha q_\alpha \langle \int F_{\alpha gc}(\mathbf{R}) \mathbf{v}_\perp \delta(\mathbf{R} - \mathbf{x} + \rho) d\mathbf{R} dv_\parallel d\mu \rangle_\varphi$$

  $$\downarrow$$

- $$\mathbf{J}_{\perp gc}(\mathbf{x}) = \sum_\alpha q_\alpha \sum_{\mathbf{k}} \int F_{\alpha gc}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \langle \mathbf{v}_\perp e^{i\mathbf{k}\cdot\rho} \rangle_\varphi dv_\parallel d\mu$$

  Diamagnetic Current

Diamagnetic Current and Pressure Balance Equation

$$\mathbf{J}_{\perp gc}(\mathbf{x}) = \sum_\alpha q_\alpha \sum_{\mathbf{k}} \int F_{\alpha gc}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \langle \mathbf{v}_\perp e^{i\mathbf{k}\cdot\rho} \rangle_\varphi dv_\| d\mu$$

$$\mathbf{k} = k_\perp cos\theta \hat{\mathbf{e}}_1 + k_\perp sin\theta \hat{\mathbf{e}}_2 + k_\| \hat{\mathbf{b}} \qquad \mathbf{v} = v_\perp cos\varphi \hat{\mathbf{e}}_1 + v_\perp sin\varphi \hat{\mathbf{e}}_2 + v_\| \hat{\mathbf{b}}$$

$$exp(\mp i\mathbf{k}\cdot\rho) = exp[\pm i \frac{k_\perp v_\perp}{\Omega} sin(\varphi - \theta)] = \sum_{n=-\infty}^{+\infty} J_n(\frac{k_\perp v_\perp}{\Omega}) e^{\pm in(\varphi-\theta)}$$

$$\langle \mathbf{v}_\perp e^{-i\mathbf{k}\cdot\rho} \rangle_\varphi = i v_\perp J_1(\frac{k_\perp v_\perp}{\Omega_\alpha}) \frac{\hat{\mathbf{b}} \times \mathbf{k}_\perp}{k_\perp}$$

$$J_1(k_\perp v_\perp/\Omega_\alpha) \approx \frac{k_\perp v_\perp}{2\Omega_\alpha}$$

$$\boxed{\mathbf{J}_{\perp gc}(\mathbf{x}) = \frac{c}{B} \hat{\mathbf{b}} \times \nabla_\perp \sum_\alpha p_{\alpha\perp}}$$

$$p_{\alpha\perp} = m_\alpha \int (v_\perp^2/2) F_{\alpha gc}(\mathbf{x}) dv_\| d\mu$$

Gyrokinetic MHD

- GK Three-field Equations for $k_\perp \rho_i \ll 1$ w/o geometric simplification [Lee and Qin PP '03]

$$\frac{d}{dt}\nabla_\perp^2\phi + \frac{v_A^2}{c}(\hat{\mathbf{b}}\cdot\nabla)\nabla^2 A_\parallel - 4\pi\frac{v_A^2}{c^2}\nabla_\perp\cdot\mathbf{J}_{\perp gc}^d = 0$$

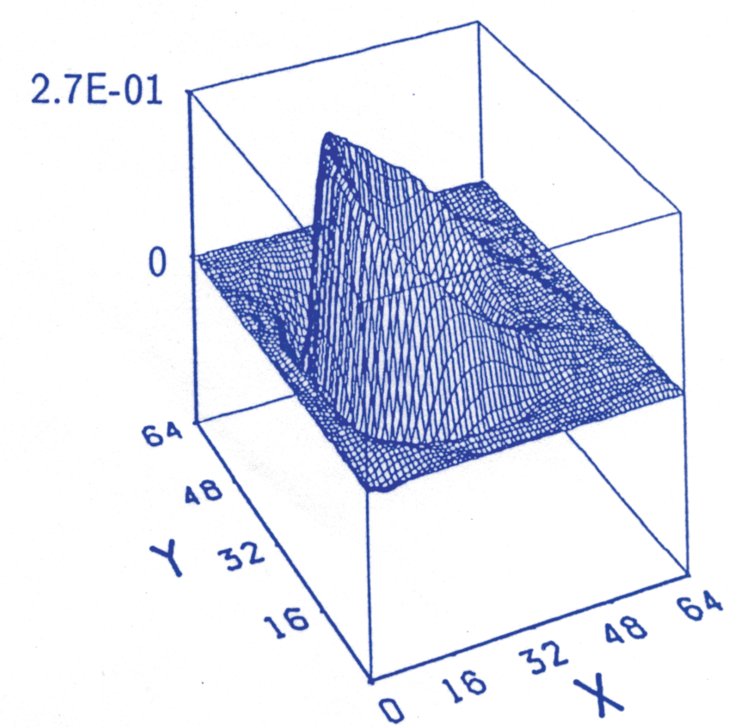
$$\frac{1}{c}\frac{\partial A_\parallel}{\partial t} + \mathbf{b}\cdot\nabla\phi = 0.$$

valid for general geometry but reducible to Strauss' high beta reduced MHD equation

$$\frac{dp_\alpha}{dt} = 0$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} - \frac{c}{B}\nabla\bar{\phi}\times\hat{\mathbf{b}}_0\cdot\nabla$$

$$\mathbf{J}_{\perp gc}^d = \frac{c}{B_0}\sum_\alpha\left[p_\alpha(\nabla\times\hat{\mathbf{b}}_0)_\perp + p_\alpha\hat{\mathbf{b}}_0\times(\nabla ln B_0)\right]$$



- Simulations of MHD modes via global gyrokinetic particle codes:

-- Naitou, Tsuda, Lee and Sydora, "Gyrokinetic simulation of internal kink modes," PoP **2**, 4257 (1995).

MHD Equilibrium

- Longitudinal Ampere's Law: $0 = \frac{4\pi}{c}\mathbf{J}^L + \frac{1}{c}\frac{\partial \mathbf{E}^L}{\partial t}$ $\qquad \frac{\partial}{\partial t} \to 0 \qquad \mathbf{J}^L \to 0$

- Transverse Ampere's Law: $\nabla \times \mathbf{B}^T = \frac{4\pi}{c}\mathbf{J}^T$ $\qquad \nabla^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{J}^T$

- Diamagnetic Current/ Pressure Balance Equation: $\mathbf{J}^T = \frac{c}{B}\mathbf{b} \times \nabla p$ $\qquad \mathbf{b} \equiv \frac{\mathbf{B}}{B}$

- From

$$\nabla \cdot \mathbf{J}^T = 0$$

$$= \frac{c}{B}\nabla p \cdot (\nabla \times \mathbf{b})$$

$$\boxed{\nabla \times \mathbf{b} = \frac{1}{B}\nabla \times \mathbf{B} + \mathbf{b} \times \nabla lnB}$$

- We obtain

$$\mathbf{b} \times \nabla p \cdot \nabla lnB = 0$$

-- OK, unless pressure gradient is zero, then magnetic field is undefined.

-- Can we study MHD equilibrium in the presence of magnetic islands?