Take Home Final

APAM 4990 (2010)

In the gyrokinetic units of $\rho_s(\equiv \sqrt{\tau}\rho_i)$ and Ω_i^{-1} for length and time, respectively, the governing gyrokinetic Vlasov equation for a finite-β plasma in the limit of $k_\perp^2\rho_i^2 \ll 1$ can be written as

$$\frac{dF_\alpha}{dt} \equiv \frac{\partial F_\alpha}{\partial t} + v_\parallel \hat{\mathbf{b}} \cdot \frac{\partial F_\alpha}{\partial \mathbf{x}} + \mathbf{E}^L \times \hat{\mathbf{b}}_0 \cdot \frac{\partial F_\alpha}{\partial \mathbf{x}} + s_\alpha v_{t\alpha}^2 (\mathbf{E}^L \cdot \hat{\mathbf{b}} + E_\parallel^T)\frac{\partial F_\alpha}{\partial v_\parallel} = 0, \tag{1}$$

where $\tau \equiv T_e/T_i$, α denotes species, $v_{te}^2 = m_i/m_e$, $v_{ti}^2 = 1/\tau$, $s_e = -1$, $s_i = \tau$,

$$\hat{\mathbf{b}} \equiv \hat{\mathbf{b}}_0 + \frac{\delta \mathbf{B}}{B_0} = \frac{\mathbf{B_0}}{B_0} + \nabla A_\parallel \times \hat{\mathbf{b}}_0, \tag{2}$$

$$\mathbf{E}^L = -\nabla\phi, \qquad E_\parallel^T = -\frac{\partial A_\parallel}{\partial t}, \tag{3}$$

the superscripts L(ongitudinal) and T(ranseverse) denote the vector decomposition relative to the direction of wave propagation and the subscript $\parallel$ indicates the direction parallel to the external magnetic field. The gyrokinetic Poisson's equation for $k_\perp^2\rho_i^2 \ll 1$ can be simplified as

$$\nabla_\perp^2 \phi = \int (F_e - F_i) dv_\parallel d\mu, \tag{4}$$

where the electrostatic potential ϕ is normalized by T_e/e, $\int F_{0\alpha} dv_\parallel d\mu = 1$ and $\mu \equiv v_\perp^2/2$. Ampere's law then becomes

$$\nabla^2 A_\parallel = \beta \int v_\parallel (F_e - F_i) dv_\parallel d\mu, \tag{5}$$

where the vector potential $\mathbf{A}$ is normalized by cT_e/ec_s, $\beta \equiv c_s^2/v_A^2$, $v_A \equiv c\lambda_D/\rho_s$ is the Alfvén speed, and λ_D is the electron Debye length. [Note that the ion acoustic speed $c_s(\equiv \rho_s\Omega_i)$ is unity in the gyrokinetic unit.] Equations (1) - (5) are the so-called electromagnetic Darwin model, when the transverese induction electric field, $\partial \mathbf{E}^T/\partial t$, is neglected in Ampere's law. This is a quasineutral system without the space charge waves By setting $A_\parallel = 0$, one recovers the electrostatic gyrokinetic Vlasov-Poisson system, discussed earlier in the class.

1) Show that the linearized Eq. (1), with the ansartz of $exp(ik_\parallel v_\parallel - i\omega t)$ and the cold plasma response of $k_\parallel v_\parallel/\omega \ll 1$, gives rise to the Alfven normal modes of

$$\omega^2 = \frac{k_\parallel^2 v_A^2}{1 + k^2\delta_e^2} = \frac{\omega_H^2}{1 + 1/k^2\delta_e^2}, \tag{6}$$

where

$$\delta_e = c/\omega_{pe} = \rho_s\sqrt{m_e/m_i\beta} \tag{7}$$

is the electron skin depth and

$$\omega_H \equiv \frac{k_\parallel}{k_\perp}\sqrt{\frac{m_i}{m_e}}$$

is the shear-Alfven mode in the electrostatic limit. For the warm electron response of $\omega/k_\parallel v_\parallel \ll 1$, the Alfven modes take the form of

$$\omega^2 = k_\parallel^2 v_A^2 (1 + k_\perp^2).$$

2) Show also that the energy conservation of this system of equations can be expressed as

$$\frac{1}{2}\frac{d}{dt}\left\langle \int \frac{v_\parallel^2 + v_\perp^2}{v_{te}^2} F_e dv_\parallel d\mu + \int \frac{v_\parallel^2 + v_\perp^2}{\tau v_{ti}^2} F_i dv_\parallel d\mu + |\nabla_\perp \phi|^2 + \frac{1}{\beta}|\nabla A_\parallel|^2 \right\rangle = 0, \tag{8}$$

where $< \cdots >$ is the spatial average.